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THE FIXED POINT INDEX FOR ACCRETIVE MAPPING* WITH K—SET CONTRACTION PERTURBATION IN CONES

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ABSTRACT: Let P be a cone in Banach space E, A, K are two mappings in P, A is accretive, K is k—set contraction, then a fixed point index is defined for mapping $-A+K$, some fixed point theorems are also deduced.

KEY WORDS AND PHRAESE: accretive mapping, k—set contraction, cone, fixed point index.
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1. INTRODUCTION

The fixed point index is a important tool in solving positive solutions of nonlinear equations in ordered Banach space. So what nonlinear mapping could be defined a index theory becomes a very interesting problem, many authors have studied this problem. see[1], [2], [8], [10], [12], [13]. In this paper, E is a Banach space, $P\subset E$ is a closed cone, i. e P is closed convex, and

$$\lambda P\subset P, \forall\ \lambda\geqslant 0,\ P\cap(-P)=\{0\};$$

$\Omega\subset E$ is a nonempty open bounded subset. Let $A: D(A)\subset P\to 2^P$ be a multivalued accretive mapping. i, e

$$\| x-y \| \leqslant \| x-y+\lambda(a_1-a_2) \|,\ x,y\in D(A),\ a_1\in Ax,\ a_2\in Ay;$$

$K:\bar{\Omega}\cap P\to P$ is a strict k—set contraction, i, e $0\leqslant k<1$; If

$$(I+A)(D(A))=P,\ and\ x\notin -Ax+Kx, \forall\ x\in\partial\Omega\cap D(A),$$

then a fixed point index is defined for $-A+K$, when K is compact, such type mapping were studied by [4], [5], [14], [15].

2. MAIN RESULTS

Let E be a Banach space, $P\subset E$ is a closed cone, "$\leqslant$" is the order induced by P in E, i, e $x\leqslant y$ if and only if $y-x\in P$.

PROPOSITION 1: $A: D(A)=P\to P$ is a continuous accretive mapping, for each $x\in P$, there exists $\beta(x)>0$, such that $Ax\leqslant\beta(x).\ x$, then $(\lambda I+A)P=P, \forall\ \lambda>0$;

PROOF. : For each $z\in P$, consider the following differential equation

$$\begin{cases} x'(t)=-(\lambda I+A)x(t)+z,\ t\in[0,+\infty) \\ x(0)=u\in P \end{cases} \qquad (2\cdot 1)$$

For each $x\in P$, since $Ax\leqslant\beta(x)\cdot x$, so there exists $W(x)\in P$, such that $\beta(x)\cdot x=Ax+W(x)$

So we have $x+\varepsilon(-\lambda x-Ax+z)=(1-\varepsilon\lambda-\varepsilon\beta(x))x+\varepsilon W(x)+\varepsilon z$

For sufficient small $\varepsilon>0$, such that $1-\varepsilon\lambda-\varepsilon\beta(x)>0$, then $(1-\varepsilon\lambda-\varepsilon\beta(x))x+\varepsilon W(x)+\varepsilon z\in P$

Hence

$$\lim_{\varepsilon\to 0^-}\frac{1}{\varepsilon}\rho(x+\varepsilon(-\lambda x-Ax+z),P)=0, \forall\ x\in P;$$

by[6], we know (E1) has only one solution. Let $x(t,u)$ be the unique solution of (E1) with $x(0)=u$.
Now, define a mapping $B_t:P\to P$ as following

$$B_t u=x(T,u),u\in P,\ T>0 \text{ is a constant};$$

For u, $v\in P$. Let $\varnothing(t)=\|x(t,u)-x(t,v)\|$, then

$$\varnothing(t)D\ \varnothing(t)\leqslant(x'(t,u)-x'(t,v),x(t,u)-x(t,v)) .$$

where $D\ \varnothing(t)=\overline{\lim_{h\to 0^-}}\dfrac{\varnothing(t)-\varnothing(t-h)}{h}$; see ([6]P.36)

$$\varnothing(t)D\ \varnothing(t)\leqslant(-\lambda x(t,u)-Ax(t,u)+\lambda x(t,v)+Ax(t,v),\ x(t,u)-x(t,v))$$

A is accertive, so

$$(-Ax(t,u)+Ax(t,v),x(t,v)-x(t,v))\ =-(Ax(t,u)-Ax(t,v),x(t,u)-x(t,v))_-\leqslant 0$$

Therefore

$$\varnothing(t)D\ \varnothing(t)\leqslant-\lambda\varnothing^2(t)$$

$$\varnothing(t)\leqslant e^{-\lambda T}\varnothing(0)$$

So we have $\|B_t u-B_T v\|\leqslant e^{-\lambda T}\|u-v\|$

Hence, B_T has a unique fixed point $u_0\in P$, i, e $B_T u_0=u_0$. This implies $x'(t,u_0)=0, t>0$,
So $0=-\lambda u_0-Au_0+z, z\in(A+\lambda I)(P)$.
This complete the proof.

In the following, we assume $A:D(A)\subset P\to 2^P$ is a multivalued accritive mapping, $(A+I)(D(A))=P$, it's well known $(I+A)^{-1}$ is nonexpansive(see[4]).

Let Ω be a open bounded subset of E, $K:\bar{\Omega}\cap P\to P$ is a strict k-set contraction, i, e $k\in[0,1)$;

Suppose $D(A)\cap\bar{\Omega}\neq\varnothing$, and $x\notin -Ax+Kx, \forall\ x\in\partial\Omega\cap D(A)$, then

$$x\neq(I+A)^{-1}Kx, \forall\ x\in\partial\Omega\cap P;$$

$(I+A)^{-1}K$ is also a strict k-set contraction, so the fixed point index $i((I+A)^{-1}K,\ \Omega\cap P)$ is well defined, see [1], [8]. Now, we define

$$i(-A+K,\Omega\cap D(A))=i((I+A)^{-1}K,\Omega\cap P)$$

THEOREM 1: (a) If $\Omega=B(0,r)$, $Kx=x_0\in B(0,r)\cap P, \forall\ x\in B(0,r)\cap P$, then

$$i(-A+K,B(0,r)\cap D(A))=1$$

(b) Suppose $\Omega=\Omega_1\cup\Omega_2, \Omega_1\cap\Omega_2=\varnothing$, then

$$i(-A+K,\Omega\cap D(A))=i(-A+K,\Omega_1\cap D(A))+i(-A+K,\Omega_2\cap D(A))$$

(c)Let $H(t,x):[0,1]\times(\bar{\Omega}\cap P)\to P$. if H(t,x)is uniformly continuous in x for each t, and for each $t\in[0,1]$. $H(t,\cdot):\bar{\Omega}\cap P\to P$ is a strict k-set contraction, k doesn't depend on t, suppose

$$x\notin -Ax+H(t,x), \forall\ x\in\partial\Omega\cap D(A), t\in[0.1];$$

then $i(-A+H(t,x),\Omega\cap D(A))$ doesn't depend on t.

(d) If $i(-A+K,\Omega\cap D(A))\neq 0$, then $x\in -Ax+Kx$ has a solution in $\Omega\cap D(A)$. *i, e* $-A+K$ has a fixed point.

PROOF: by the definition, (b), (c), (d) is obvious. (see[1]or [8])

Now, we prove(a). First, we have

$$0\in D(A), and\ 0\in A0 \tag{2.2}$$

In fact, $(A+I)D(A)=P$, so there exists $x\in D(A), a\in Ax$, such that $x+a=0$.

Since $x\geqslant 0, a\geqslant 0$, So we must have $x=0, a=0\in A0$. Hence

$$(A+I)^{-1}0=0 \tag{2.3}$$

by the definition, we need to prove

$$i((I+A)^{-1}K,\Omega\cap P)=1, \Omega=B(0,r) \tag{2.4}$$

Since $(I+A)^{-1}Kx=(I+A)^{-1}x_0, \forall\ x\in\bar{\Omega}\cap P$, and

$$\|(I+A)^{-1}x_0-(I+A)^{-1}0\|\leqslant\|x_0\|<r$$

So $(I+A)^{-1}x_0\in\Omega\cap P=B(0,r)\cap P$, by [1](see also[8]).

$$i((I+A)^{-1}K,B(0.r)\cap P)=1$$

So $i=(-A+K,B(0,r)\cap D(A))=1$.
In the following, K, A, Ω, are same as above.

LEMMA 1: *If* $Kx\ngeq x$. $\forall\ x\in\partial\Omega\cap P$; and $0\in\Omega$, then

$$i(-A+K.\Omega\cap D(A))=1$$

PROOF: Let $H(t,x)=tKx, t\in[0,1], x\in\bar{\Omega}\cap P$. If $x\in -Ax+tKx$ for some $x\in\partial\Omega\cap\ D(A)$ and $t\in[0,1]$, then $t\neq 0$(otherwise, we get $x=0\in\partial\Omega$, a contradiction)

So $Kx\geqslant\frac{x}{t}\geqslant x$, a contradiction to $Kx\not\geqq x$.

Hence, $H(t,x)$ satisfy all the conditions of (c) in theorem 1.

So

$$i(-A+K,\Omega\cap D(A))=i(-A+0,\Omega\cap D(A))$$

by (2.3), we have $(I+A)^{-1}0=0\in\Omega\cap P$

So $i((I+A)^{-1}0,\Omega\cap P)=1$, and we get

$$i(-A+0,\Omega\cap D(A))=1 \tag{2.5}$$

Hence

$$i(-A+K,\Omega\cap D(A))=1$$

COROLLARY 1: If $0\in\Omega$, and $Kx<x$, $\forall\ x\in\partial\Omega\cap P$, then $-A+K$ has a fixed point in $\Omega\cap D(A)$

PROOF: It's obvious $Kx\not\geqq x$, $\forall\ x\in\partial\Omega\cap P$. By lemma 1,

$$i(-A+K,\Omega\cap D(A))=1$$

Theorem 1,(d) implies $-A+K$ has a fixed point in $\Omega\cap D(A)$.

LEMMA 2: Let $u_0\neq 0$, $u_0\in P$, suppose $x-tu_0\notin -A(x-tu_0)+Kx$, if $x\in\partial\Omega\cap P$, and $x-tu_0\in D(A)$, for $t\geqslant 0$; Then

$$i(-A+K,\Omega\cap D(A))=0$$

PROOF: Suppose $i((I+A)^{-1}K,\Omega\cap D(A))\neq 0$

For each $\tau>0$, Let $H(t,x)=(I+A)^{-1}K+t\tau u_0$, $\forall\ x\in\Omega\cap P$, $t\in[0,1]$;

It's obvious H(t,x) is uniformly continuous in x for each t, and H(t, •)is strict k−set contraction for each t. By[1]. (see also [8]). We get

$$i((I+A)^{-1}K+\tau u_0,\Omega\cap P)=i((I+A)^{-1}K,\Omega\cap P)\neq 0$$

So there exists $x_\tau\in\Omega\cap P$, such that

$$x_\tau-(I+A)^{-1}Kx_\tau=\tau u_0 \tag{2 • 6}$$

Letting $\tau\to\infty$, the left side of (2.6) is bounded, but the right side of (2,6) is unbounded, a contradiction.

We must have $i(-A+K,\Omega\cap D(A))=0$

THEOREM 2: If A: $D(A)\subset P\to 2^P$ is an accretive mapping, $(I+A)D(A)=P$, Ω_1, Ω_2 are two open bounded subsets of E. $0\in\Omega_1\subset\Omega_2$, $K:\bar{\Omega}\cap P\to$ is a strict k−set contraction mapping, $0\neq u_0\in P$

(i) For each $x\in\partial\Omega_2$, $x\not\leqq Kx$; for each

$$x\in\partial\Omega_1\cap P,x-tu_0\in D(A),t\geqslant 0,x-tu_0\notin -A(x-tu_0)+Kx;$$

(ii) For each $x\in\partial\Omega_1$; $x\not\leqq Kx$, for each

$$x\in\partial\Omega_2\cap P,x-tu_0\in D(A),t\geqslant 0,x-tu_0\notin -A(x-tu_0)+Kx;$$

Suppose either (i)or (ii)is satisfied, then −A+K has a fixed point in $(\Omega_2-\bar{\Omega}_1)\cap D(A)$

PROOF: Suppose condition (i) is satisfied by, Lemma 1, we have

$$i(-A+K,\Omega_2\cap D(A))=1 \tag{2.7}$$

by Lemma 2, we have

$$i(-A+K,\Omega_1\cap D(A))=1 \tag{2.8}$$

by (b) of Theorem 1, and (6),(7). We get

$$i(-A+K,(\Omega_2-\bar{\Omega}_1)\cap D(A))=1$$

by (d) of Theorem 1, we know $-A+K$ has a fixed point in $(\Omega_2-\bar{\Omega}_1)\cap D(A)$.

If (ii) is satisfied, the proof is similar. We complete the proof.

THEOREM 3: For each $x\in\partial\Omega\cap D(A)$, $\|Kx\|\leqslant\|x\|$, and $0\in\Omega$, then $-A+K$ has a fixed point in $\bar{\Omega}\cap D(A)$

PROOF: we may suppose

$$x\notin -Ax+Kx, \forall\ x\in\partial\Omega\in D(A) \tag{2.9}$$

Let $H(t,x)=tKx$, $\forall\ x\in\partial\Omega\cap P$, $t\in[0,1]$;

It's obvious H(t,x) is uniformly continuous in x, and H(t, •)is strict k−set contraction for each t.

We show that

$$x\notin -Ax+H(t,x),\forall\ x\in\partial\Omega\cap D(A),t\in[0,1] \tag{2.10}$$

If $x\in -Ax+H(t,x)$ for some $x\in\partial\Omega\cap D(A)$, $t\in[0,1]$, then $x=(I+A)^{-1}H(t,x)$

Since $(I+A)^{-1}$ is nonexpansive and $(I+A)^{-1}0=0$. So

$$\| x \| \leqslant \| H(t,x) \| = \| tKx \| \leqslant t \| x \|$$

Therefore t=1, contradict to (8), by (c) of Theorem1,

$$i(-A+K,\Omega\cap D(A))=i(-A+0,\Omega\cap D(A))$$

and (2.5) implies $i(-A+K,\Omega\cap D(A))=1$.

by (d) of Theorem 1, $-A+K$ has a fixed point in $\Omega\cap D(A)$.

THEOREM 4: If $0\in\Omega$, $\| Kx \| \leqslant \| x+a \|$, $\forall\ x\in\partial\Omega\cap D(A)$, $a\in Ax$; then $-A+K$ has a fixed point in $\bar{\Omega}\cap D(A)$.

PROOF: We may assume $x\notin -Ax+Kx$, $\forall\ x\in\partial\Omega\cap D(A)$;

Let $H(t,x)=tKx$, $t\in[0,1]$, $x\in\bar{\Omega}\cap P$;

If $x\in -Ax+tKx$ for some $t\in[0,1]$, $x\in\partial\Omega\cap D(A)$, then $tKx\in x+Ax$

So there exists $a\in Ax$, such that $tKx=x+a$. We have $\| Kx \| \leqslant t \| Kx \|$

By the assumption (2.11), $t\neq1$, we must have $Kx=0$, $x+a=0$

By (2.3), $x=0\in\partial\Omega$, a contradiction to $0\in\Omega$.

So we have $x\notin -Ax+H(t,x)$, $\forall\ x\in\partial\Omega\cap D(A)$, $t\in[0,1]$.

The following proof is similar to that of Theorem 3. This end the proof.

REFERENCES

1. **H. AMMAN**, On the number of solutions of nonlinear equations in ordered Banach spaces, J. Funct. Anal. 11 (172). 346−384
2. **H. AMMAN**, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM, Rev, 18 (1976) 620−709
3. **F, E. BROWEDR**, Nonlinear mappings of nonexpansive and accretive type in Banach spaces. Bull. Amer. Math. Soc. 73(1967)875−882
4. **F, E. BROWDER**, Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure. Math. V18, part2. AMS 1976
5. **Y, Z. CHEN**. The generalized degree for compact perturbations of m−accretive operators and applications. Non. Anal. 13(1989)393−403
6. **K. DEIMLING**. Ordinary differential equations in Banach spaces. Lect, Notes. Math. 596. 1977
7. **M. A. KRASNOSELSKII**. Positive solutions of operator equations. Noordhoff. 1964
8. **G. Z. LI**. The fixed point index and the fixed point theorems of 1−set−contraction mappings. Proc. Amer. Math. Soc 104(1988). 393−1170
9. **T. C. LIN**. Approximation theorems and fixed point theorems in cones. Proc. Amer. Math. Soc. 102(1988). 502−506
10. **P. M. FITZPATRICK, W. V. PETRYSHYN**. Fixed Point theorems and the fixed point index for multivalued mappings in cones. J. L. Math. Soc. 12(1975)75−85
11. **I. MASSABO, C, A. STUART**. Positive eigenvectors of k−set contractions. Non. Anal. 3(179). 35−44
12. **R. D. NAUSSBAUM**. The fixed point index and asymptotic fixed point theorems for k−set contractions. Bull. Amer. Math. Soc. 75(1969). 490−495.
13. **R. D. NAUSSBAUM**. The fixed point index for local condensing maps. Ann. Mat. Pura. Appl. 89(1992). 1−9
14. **C. MORALES**, on the range of sums of accretive and continuors operatous in Banach spaces. Non. Anal. 19 (1992). 1−9
15. **A. G. KARTSATOS**, Mapping theorems involving ranges of sums on nonlinear operators. Non. Anal. 6(1982). 271−278