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STABILITY OF Ψ-ADDITIVE MAPPINGS: APPLICATIONS TO NONLINEAR ANALYSIS

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ABSTRACT. The Hyers-Ulam stability of mappings is in development and several authors have remarked interesting applications of this theory to various mathematical problems. In this paper some applications in nonlinear analysis are presented, especially in fixed point theory. These kinds of applications seem not to have ever been remarked before by other authors.

KEY WORDS AND PHRASES: Stability, nonlinear analysis, cone, homomorphism, eigenvalue, bifurcation, Hammerstein equation, completely continuous, operator, Banach spaces
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INTRODUCTION

In 1940 S.M. Ulam posed the following problem Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a $\delta > 0$ such that if $f:G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x,y \in G_1$, then a homomorphism $T:G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x,y \in G_1$? (cf [26], [27]).

The first affirmative answer was given by D.H. Hyers [9] in 1941 and a generalization of Hyers' result was obtained by Th.M Rassias [24] in 1978. Several papers have been published on this subject and some interesting variants of Ulam's problem have been also investigated by a number of mathematicians (cf. [11],[13],[10],[12],[15],[16],[8],[6],[23]). The concept of the **Hyers-Ulam stability of mappings** was thus created and it is now currently used in the spirit of Ulam's problem. We note that till now the Hyers-Ulam stability has been mainly used to study problems concerning approximate isometries or quasi-isometries, the stability of Lorentz and conformal mappings, the stability of stationary points, the stability of convex mappings, or of homogeneous mappings, etc. (cf. [11], [12], [23], [6]).

In this paper we intend to introduce another way for future applications of this theory. We will apply some stability results, obtained recently, to the study of some important problems in nonlinear analysis For example, the existence of fixed points on cones for nonlinear mappings, the study of eigenvalues for a couple of nonlinear operators and the study of bifurcations to the infinity, with respect to a convex cone, of solutions of the Hammerstein equation.

In nonlinear analysis it is well known that finding the expression of the asymptotic derivative of a nonlinear operator can be a difficult problem In this sense, in this paper it is explained how the Hyers-Ulam stability theory can be used to evaluate the asymptotic derivative of some nonlinear operators.

Since, by the Hyers-Ulam stability we come to find the expression at every point of the asymptotic derivative of some nonlinear operators, we succeed in finding interesting assumptions for new fixed point theorems and for some new existence theorems

The nonlinear problems considered in this paper have been much studied by several mathematicians (cf. [1],[2],[3],[5],[7],[17-22],[25],[28],[29]) but the idea to use the Hyers-Ulam stability theory for the study of these problems had not appeared earlier in the mathematical literature.

PRELIMINARIES

Let E be a Banach space. We denote by S the set $S=\{x\in E \mid \|x\|=1\}$. A subset $K\subset E$ is said to be a **cone** if it is closed and satisfies the following properties:

1) $K+K\subseteq K$, 2) $\lambda K\subseteq K$ for all $\lambda\in R_+$ and 3) $K\cap(-K)=\{0\}$. We denote by K^* the dual of K, i e. $K^*=\{\phi\in E^* \mid \phi(x)\geq 0 \text{ for all } x\in K\}$. Each cone $K\subset E$ induces an ordering on E by $x\leq y \Leftrightarrow y-x\in K$. If in E a closed cone is defined, we say that E is an ordered Banach space. A cone $K\subset E$ is said to be **generating** if $E=K-K$ and it is said to be **normal** if there exists $\delta\geq 1$ such that for every pair $x,y\in K, \|x\|\leq\delta\|x+y\|$. We say that a cone $K\subset E$ is **solid** if its topological interior is non-empty. The cone R_+^n is solid. Let $E=C_R(\Omega)$ be the space of all continuous functions from Ω into R with the max-norm, where Ω is a topological compact space. The cone $C_R^+(\Omega)=\{x \mid x(t)\geq 0 \text{ on } \Omega\}$ is solid.

The cones $L_+^p(\Omega)=\{x\in L^p(\Omega) \mid x(t)\geq 0 \text{ almost everywhere}\}$ and $l_+^p=\{x\in l^p \mid x_i\geq 0 \text{ for all } i\}$ considered in $L^p(\Omega)$ and in l^p respectively are not solid if $1\leq p<\infty$. We call a point $x_0\in K$ a **quasi-interior point** if $\phi(x_0)>0$ for any non-zero $\phi\in K^*$. If the cone K is solid, then the quasi-interior points of K coincide precisely with its interior points. There exist non-solid cones but with quasi-interior points. For example, if K is the cone of nonnegative functions in $L^p(\Omega)(1\leq p<\infty)$, its quasi-interior points are the functions, which are positive almost everywhere; similarly in the space $l^p(1\leq p<\infty)$ the quasi-interior points of the cone K of all nonnegative sequences are these sequences with only positive components. We denote by $L(E,E)$ the set of linear bounded operators from E into E. It is well known that for every $T\in L(E,E)$ the spectral radius $r(T)$ is well defined, where $r(T)=\max\{|\lambda| \mid \lambda\in\sigma(T)\}$ and $\sigma(T)$ is the spectrum of T. We say that $T\in L(E,E)$ **is strictly monotone increasing** if for all $x,y\in E$ such that $x<y$ (i.e., $x\leq y$ and $x\neq y$) we have $T(y)-T(x)\in\operatorname{int}(K)$. Let $D\subset E$ be a bounded set. Then we define $\gamma(D)$ **the measure of noncompactness** of D to be the minimum of all positive numbers δ such that D can be covered by finitely many sets of diameter less than δ. A mapping

$f\ E \to E$ is said to be a **k-set-contraction** if it is continuous and there exists $k \in R_+$ such that for every bounded set $D \subset dom(f), \gamma(f(D)) \le k\gamma(D)$ A mapping $f\ E \to E$ is said to be a **strict-set-contraction** if it is a k-set-contraction for some $k < 1$ An appropriate reference for the measure of noncompactness is [4] A mapping $f\ E \to E$ is said to be **compact** if it maps bounded subsets of *dom(f)* onto relatively compact subsets of E and f is said to be **completely continuous** if it is continuous and compact Every completely continuous mapping is a strict-set-contraction

Let K be a generating $\left(\text{or total i e } E = \overline{K - K}\right)$ cone in E The mapping $f\ K \to E$ is said to be **asymptotically differentiable along K** if there exists $T \in L(E,E)$ such that

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \frac{\|f(x) - T(x)\|}{\|x\|} = 0$$

In this case T is the unique such mapping and we call it the **derivative at infinity along** K of f. We say that a mapping $f\ E \to E$ is **asymptotically close to zero along K** if

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \frac{\|f(x)\|}{\|x\|} = 0.$$

Let $\phi. R_+ \to R_+$ be a function such that $\phi(t) > 0$ for all $t \ge \gamma$, where $\gamma \in R_+$ We say that $f: K \to E$ is **φ-asymptotically bounded** along K if there exist $b, c \in R_+ \setminus \{0\}$ such that for all $x \in K$, with $\|x\| \ge b$, we have

$$\|f(x)\| \le c\phi(\|x\|).$$

An interesting spectral analysis of φ-asymptotically bounded mappings can be found in [28]. Every φ-asymptotically bounded mapping (along K) such that $\lim_{t \to +\infty} \frac{\phi(t)}{t} = 0$ is asymptotically close to zero.

If K is a generating (or total) cone in E, then a mapping $f: K \to E$ is said to be **differentiable** at $x_0 \in K$ along K if there exists $f'(x_0) \in L(E,E)$ such that

$$\lim_{\substack{x \in K \\ x \to 0}} \frac{\|f(x_0 + x) - f(x_0) - f'(x_0)x\|}{\|x\|} = 0.$$

In this case $f'(x_0)$ is the **derivative** at x_0 along K of f and it is uniquely determined

THE MAIN RESULTS

Let E_1, E_2 be real normed vector spaces The following definition was introduced in [15].

DEFINITION 1. *For a given function $\psi: R_+ \to R_+$, we say that a mapping $f\ E_1 \to E_2$ is ψ-additive if and only if there exists a constant $\theta \ge 0$ such that $\|f(x+y) - f(x) - f(y)\| \le \theta[\psi(\|x\|) + \psi(\|y\|)]$ for all $x, y \in E_1$*

If the function ψ satisfies the following assumptions:

i_0) $\lim_{t\to\infty}\frac{\psi(t)}{t}=0$

i_1) $\psi(ts)\le\psi(t)\psi(s)$ for all $t,s\in R_+$,

i_2) $\psi(t)<t$ for all $t>1$

then we have the following result proved in [15]

THEOREM 1. *Consider E_1 to be a real normed vector space, E_2 a real Banach space and $f:E_1\to E_2$ a mapping such that $f(tx)$ is continuous in t for each fixed x. If f is ψ-additive and ψ satisfies the assumptions i_0), i_1) and i_2), then there exists a unique linear mapping $T:E_1\to E_2$ such that $\|f(x)-T(x)\|\le\frac{2\theta}{2-\psi(2)}\psi(\|x\|)$, for all $x\in E_1$. Moreover the expression of T at every point $x\in E_1$ is $T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$.*

The class of functions which satisfy the assumptions i_0), i_1) and i_2) is not empty. In this sense we can cite the following functions:

1°) $\psi(t)=t^p$ with $p\in[0,1)$,

2°) $\psi(t)=\begin{cases}0 \text{ if } t=0\\ t^p \text{ if } t>0, \text{ where } p<0.\end{cases}$

To enlarge the class of functions ψ such that the conclusion of Theorem 1 remains valid we consider the following:

Let F_ψ be the set of all functions ψ from R_+ into R_+ satisfying the assumptions i_1) and i_2) and such that $\lim_{t\to+\infty}\frac{\psi(t)}{t}=0$. Let $P(\Psi)$ be the convex cone generated by the set F_ψ. We remark that a function $\psi\in P(\Psi)$ satisfies the assumption i_0) but generally does not satisfy the assumptions i_1) and i_2). However, we will show that Theorem 1 remains valid for ψ-additive functions with $\psi\in P(\Psi)$. Let E_1 and E_2 be normed vector spaces and $f:E_1\to E_2$ a mapping. The following result is a consequence of the principal theorem proved by P. Gavruta in [8].

LEMMA. *If $\phi:E_1\times E_1\to[0,+\infty)$ is a mapping such that $\tilde{\phi}(x,y):=\sum_{k=0}^{\infty}2^{-k}\phi(2^k x,2^k y)<+\infty$ for all $x,y\in E_1$ and $f:E_1\to E_2$ is a continuous mapping such that $\|f(x+y)-f(x)-f(y)\|\le\phi(x,y)$, for all $x,y\in E_1$, then there exists a unique linear mapping $T:E_1\to E_2$ such that $\|f(x)-T(x)\|\le\frac{1}{2}\tilde{\phi}(x,x)$ for all $x\in E$. Moreover $T(x)=\lim_{n\to\infty}\frac{f(2^n x)}{2^n}$ for all $x\in E$.*

A consequence of the Lemma is the following result.

THEOREM 2. *Let E_1 be a real normed vector space, E_2 a real Banach space and $f:E_1\to E_2$ a continuous mapping. If f is ψ-additive with $\psi\in P(\Psi)$ then there exists a unique linear mapping $T:E_1\to E_2$ such that $\|f(x)-T(x)\|\le\frac{2\theta}{2-\psi(2)}\psi(\|x\|)$, for all $x\in E_1$. Moreover the* *expression of T at every point $x \in E_1$ is given by* $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$

PROOF. We apply the Lemma for the function $\phi(x,y) = \theta[\psi(\|x\|) + \psi(\|y\|)]$ In this case, using the properties of the functions $\psi \in P(\Psi)$, we can show that $\tilde{\phi}(x,y) < +\infty$ for all $x,y \in E_1$ and the conclusion of the Theorem follows Q E D

REMARKS. The last Theorem is significant since the class of ψ-additive functions with $\psi \in P(\Psi)$ is strictly larger than the class of functions defined in Theorem 1. In this sense we remark the following results

1) If $f.E_1 \to E_2$ is a ψ-additive mapping with $\psi \in P(\Psi)$ and $T_0 \in L(E_1,E_2)$ then $T_0 + f$ is a ψ-additive mapping with respect to the same function ψ

2) If $f.E_1 \to E_2$ is a ψ-additive mapping with $\psi \in P(\Psi)$ and $T_* \in L(E_2,E_3)$ then $T_* \circ f$ is a ψ-additive mapping with respect to the same function ψ and the constant θ replaced by $\theta\|T_*\|$.

3) If $f_1, f_2 . E_1 \to E_2$ are mappings such that f_1 is ψ_1-additive and f_2 is ψ_2-additive with $\psi_1, \psi_2 \in P(\Psi)$ then for every $a_1, a_2 \in R$ we have that $a_1 f_1 + a_2 f_2$ is a ψ-additive mapping. In this case the function ψ is defined by $\psi(t) = \psi_1(t) + \psi_2(t)$ for all $t \in R_+$ and $\theta = \max(a_1\theta_1, a_2\theta_2)$.

THEOREM 3. *Let E be a Banach space ordered by a generating cone K and let $f{:}E \to E, g{:}K \to K$ be two mappings such that:*

(f_1). f is completely continuous, positive and ψ-additive with respect to a function $\psi \in P(\Psi)$ and to a constant $\theta > 0$ $(i.e.\ f(K) \subseteq K)$;

(f_2): there exists a quasi-interior point $x_0 \in K$ and $0 < \lambda_0 < 1$ such that $\lim_{n\to\infty} \frac{f(2^n x_0)}{2^n} \le \lambda_0 x_0$;

(g_1): g is asymptotically close to zero along K;

(h_1): $h = f + g$ is a strict set-contraction from K to K.

Then $h = f + g$ has a fixed point in K.

PROOF. By assumption (f_1) and Theorem 2 we have that $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ is well defined for every $x \in E$ and T is the unique linear operator satisfying the inequality

$$(\tilde{\alpha}) \quad \|f(x) - T(x)\| \le \frac{2\theta}{2-\psi(2)}\psi(\|x\|) \text{ for all } x \in E$$

Since f is compact, we have that $f(S)$ is bounded, which implies that T is continuous. Indeed, the continuity of T is a consequence of the following inequalities

$$\|T(x)\| \le \|f(x)\| + \|T(x) - f(x)\| \le \|f(x)\| + \frac{2\theta}{2-\psi(2)}\psi(\|x\|) \le \|f(x)\| + \frac{2\theta}{2-\psi(2)}\psi(1) \text{ for all } x \in S.$$

From the definition of T and the fact that $f(K) \subseteq K$ we deduce that T is positive (i.e $T(K) \subseteq K$) From inequality $(\tilde{\alpha})$ and the properties of ψ we have that

$$\lim_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\|f(x)-T(x)\|}{\|x\|}=0,$$

that is T is the asymptotic derivative of f along K. Also, from the principal theorem of [14] or from [1] we have that T is completely continuous (and also it is a strict-set-contraction). From assumption (g_1) we have

$$\lim_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\|h(x)-T(x)\|}{\|x\|} \le \lim_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\|g(x)\|}{\|x\|} + \lim_{\substack{x\in K \\ \|x\|\to\infty}} \frac{\|f(x)-T(x)\|}{\|x\|}=0,$$

that is T is also the asymptotic derivative of h along K.

Since T is completely continuous its spectrum consists of eigenvalues and zero. Suppose that $r(t)>0$. From assumption (f_2) we have that $T(x_0)\le \lambda_0 x_0$ and using the Krein-Rutman Theorem ([29], Proposition 7.26, p.290) we have that there exists $\phi_0\in K^*\setminus\{0\}$ such that $T^*(\phi_0)=r(T)\phi_0$ and $\phi_0(x_0)>0$ (since x_0 is a quasi-interior point of K). (We denote by T^* the adjoint of T). Hence we deduce

$$r(T)=\frac{[T^*(\phi_0)](x_0)}{\phi_0(x_0)}=\frac{\phi_0(T(x_0))}{\phi_0(x_0)}\le\frac{\phi_0(\lambda_0 x_0)}{\phi_0(x_0)}=\lambda_0,$$

that is, $r(T)<1$.

Now, all the assumptions of Theorem 1 of [1] are satisfied and therefore $h=f+g$ has a fixed point in K. Q.E.D.

REMARK. If $f_1,\dots,f_m:E_1\to E_2$ are continuous ψ-additive mappings respectively to $\psi_1,\psi_2,\dots,\psi_m\in P(\Psi)$ and $\sum_{i=1}^{n} a_i f_i(S)$ is bounded (where $a_1,\dots,a_m\in R$) then using the properties of functions $\psi_1,\dots,\psi_m$ and Theorem 2 we can show that the asymptotic derivative of the mapping

$$\sum_{i=1}^{n} a_i f_i \text{ is exactly } T(x)=\lim_{n\to\infty}\frac{\sum_{i=1}^{m} a_i f_i(2^n x)}{2^n}, \text{ for all } x\in E_1.$$

COROLLARY. *Let E be a Banach space ordered by a generating cone K and let* $f:E\to E$ *be a mapping satisfying conditions* f_1) *and* f_2). *Then f has a fixed point in K.*

REMARK. In Theorem 3 and its Corollary we can replace assumption (f_2) by the following:

(f_3): for all $\lambda\ge 1$ and $x\in K\setminus\{0\}$ there follows $\|f(x)-\lambda x\|>\dfrac{2\theta}{2-\psi(2)}\psi(\|x\|)$.

Hereafter we investigate the existence of non-zero positive fixed points.

THEOREM 4. *Let E be a Banach space ordered by a generating cone K and let* $f:E\to E, g:K\to K$ *be mappings satisfying* (f_1), (g_1), (h_1) *and:*

(f_4): *there exist* $\lambda_0>1$ *and* $x_0\notin -K$ *such that* $\lim_{n\to\infty}\dfrac{f(2^n x_0)}{2^n}\ge\lambda_0 x_0$;

(f_5): $\|f(x)-x\|>\dfrac{2\theta}{2-\psi(2)}\psi(\|x\|)$, *for all* $x\in K\setminus\{0\}$;

(h_2). h is differentiable at 0 along K and $h(0) = 0$,

(h_3). $h'(0)$ does not have a positive eigenvector belonging to an eigenvalue $\lambda \geq 1$

Then $h = f + g$ has a fixed point in $K \setminus \{0\}$.

PROOF. As in the proof of Theorem 3 we have that h is asymptotically differentiable along K and its derivative at infinity along K is $T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, for every $x \in E$. Moreover T is completely continuous and the inequality $(\tilde{\alpha})$ is also satisfied. From assumption (f_5) we have that 1 is not an eigenvalue with corresponding positive eigenvector of T. From assumption (f_4) we obtain that $r(T) \geq \lambda_0$. Indeed, if $r(T) < \lambda_0$ and since $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ (the Gelfand's formula), $\lambda_0^{-n}\|T^n\| \leq k^n$ for sufficiently large n and some $k < 1$. Because $T(x_0) \geq \lambda_0 x_0$ we deduce $\lambda_0^{-n} T^n(x_0) \geq x_0$ (since T is positive) and if we pass to the limit in the last relation we obtain $x_0 \leq 0$, i.e. $x_0 \in -K$, which is a contradiction. Using again the Krein-Rutman Theorem we have that $r(T)$ is an eigenvalue of T with an eigenvector in K. Thus all the assumptions of Theorem 1 of [5] are satisfied and we conclude that h has a fixed point $x_* \in K \setminus \{0\}$.

Q.E.D.

APPLICATIONS

We will apply now Theorem 4 to the following nonlinear eigenvalue problem. Let E be a Banach space ordered by a cone K. Given the mappings L, f and A from E into E (possibly nonlinear) find $x_* \in E \setminus \{0\}$ and $\lambda \in R_+ \setminus \{0\}$ such that (b) $L(x_*) + f(x_*) = x_* + \lambda A(x_*)$.

DEFINITION 2. *We say that $\lambda \in R_+ \setminus \{0\}$ is an asymptotic characteristic value of (L,A) if L and λA are asymptotically equivalent (with respect to K) i.e.,*

$$\lim_{\substack{\|x\|\to 0 \\ x\in K}} \frac{\|L(x) - \lambda T(x)\|}{\|x\|} = 0.$$

An asymptotic characteristic value of (L,A) is a characteristic value of (L,A) in the sense of the definition given by Mininni in [22], i.e., λ is a characteristic value of (L,A) in Mininni's sense, if there exists a sequence $\{x_n\}_{n\in N}$ of elements of K such that

$$\lim_{n\to\infty} \|x_n\| = +\infty \text{ and } \lim_{n\to\infty} \frac{\|L(x_n) - \lambda A(x_n)\|}{\|x_n\|} = 0.$$

The following result is a consequence of Theorem 4.

COROLLARY. *Let E be a Banach space ordered by a generating cone $K \subset E$, $f: E \to E$ a mapping such that $f(K) \subseteq K$ and L,A mappings from E into E. If the following assumptions are satisfied:*

(1): f satisfies conditions (f_1), (f_4) and (f_5);

(2): λ is an asymptotic characteristic value of (L,A);

(3): $h = f + L - \lambda A$ satisfies conditions (h_1) - (h_3).

Then the nonlinear eigenvalue problem (b) has a solution.

It is well known that the study of the nonlinear integral equation

$$(\alpha) \qquad x(u) = \int_\Omega G(u,v) f(v, x(u)) du,$$

known as the Hammerstein equation, is of central importance in the study of several boundary-value problems (cf. [17], [18], [19], [29]).

Also some special interest is focused to the eigenvalue problem

$(\beta) \qquad x(u) = \lambda \int_\Omega G(u,v) f(v, x(u)) du$, (cf. [29] and its references).

If we denote by G the linear integral operator defined by the kernel $G(u,v)$ and by f the Nemyckii's nonlinear operator defined by $f(v,x(v))$, i.e. $f(u)(v) = f(v,x(v))$, then the equation (β) takes the abstract form

$$(\gamma) \qquad x = \lambda G f(x).$$

For the equation (γ) we consider the following hypotheses:

H_1): (E,K) and (F,P) are real ordered Banach spaces. The cone K is normal with non-empty interior.

H_2): The mapping $f: E \to F$ is continuous and the operator $G: F \to E$ is linear, compact and positive.

H_3): G is strongly positive, i.e., $x < y$ implies $G(y) - G(x) \in \operatorname{int}(K)$.

We recall that $(\lambda_*, +\infty)$ is a bifurcation from infinity of equation (α) if $\lambda_* > 0$ and there is a sequence of solutions (λ_n, x_n) of (γ) such that $\lambda_n \to \lambda_*$ and $\|x_n\| \to \infty$ as $n \to \infty$.

THEOREM 5. *Consider equation (γ) and suppose that H_1), H_2) and H_3) are satisfied. If in addition the following assumptions hold:*

1°) *$f(K) \subseteq K$ and $f(S)$ is bounded,*

2°) *f is ψ-additive with $\psi \in P(\Psi)$,*

3°) $\displaystyle\lim_{n\to\infty} \frac{f(2^n x)}{2^n} > 0$, *for every $x \in K \setminus \{0\}$,*

then setting $T(x) = \displaystyle\lim_{n\to\infty} \frac{f(2^n x)}{2^n}$, for all $x \in E$ and $\lambda_* = r(GT)^{-1}$ we have that $(\lambda_*, +\infty)$ is the only bifurcation from infinity of equation (γ).

PROOF. First we note that T is the asymptotic derivative of f along K. Since by assumption 3°) we have that T is strictly positive on K we remark that $r(GT) > 0$. We set

$$g(x) = \begin{cases} \|x^2\| f\left(\dfrac{x}{\|x\|}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

We know that $g'(0) = T$ and $(\lambda, +\infty)$ is a bifurcation point of $x = \lambda G f(x)$ if and only if $(\lambda, 0)$ is a bifurcation point of $x = \lambda G g(x)$. The Theorem now follows from Theorem 7.*H* of [29]. Q.E.D.

REMARK Concerning Theorem 5, the two-sided estimates for the spectral radius that have been obtained recently by Stetsenko in [25], are very essential Stetsenko showed that if $A\, E \to E$ is a completely continuous operator and E is a Banach space ordered by a generating closed cone K with quasi-interior points and if some special assumptions are satisfied, then we can define the numbers λ_0, ρ, the vectors u_0, v_0 and a functional ϕ_0 such that

$$(\delta) \qquad \lambda_0 - \rho \frac{\phi_0(v_0)}{\phi_0(u_0)} \le r(A) \le \lambda_0 - \frac{1}{\rho} \frac{\phi_0(v_0)}{\phi_0(u_0)}.$$

OPEN PROBLEM. It seems to be an interesting problem to find new and more efficient two-sided estimates for the spectral radius of the operator GT where

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n},$$

when f is a ψ-additive mapping with $\psi \in P(\Psi)$.

Such an estimate of $r(GT)$, similar to the estimate (δ), is important for the approximation of the bifurcation point $(\lambda_*, +\infty)$, of the equation (γ)

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