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A CHARACTERIZATION OF POINT SEMIUNIFORMITIES

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ABSTRACT. The concept of a uniformity was developed by A. Weil and there have been several generalizations. This paper defines a point semiuniformity and gives necessary and sufficient conditions for a topological space to be point semiuniformizable. In addition, just as uniformities are associated with topological groups, a point semiuniformity is naturally associated with a semicontinuous group. This paper shows that a point semiuniformity associated with a semicontinuous group is a uniformity if and only if the group is a topological group.

KEY WORDS AND PHRASES: Uniformity, point semiuniformity, vicinities, point semiuniformizable, homogeneous, topological group, semicontinuous group, semifundamental system, point regular, bihomogeneous.
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1. Introduction. In 1937, A. Weil [1] generalized the concept of a metric space by defining a topology-generating structure called a uniformity. There have been several generalizations of uniformities. For example, a semiuniformity, $\mathbb{U}$, for a set $\mathbb{X}$ is a filter of supersets of the diagonal in $\mathbb{X}\times\mathbb{X}$ such that for each U in $\mathbb{U}$, there is a V in $\mathbb{U}$ such that $V^{-1}=\{(y,x)|(x,y)\in V\}\subseteq U$. As with a uniformity and its other generalizations, there is a natural way to try to construct neighborhoods of points. Namely, for each x in $\mathbb{X}$ and U in $\mathbb{U}$ we define a slice, U[x], to be $\{y|(x,y)\in U\}$. For a semiuniformity, the collection {U[x]} does generate a topology on $\mathbb{X}$ but we are left with the unsatisfactory situation that some of the slices are not neighborhoods in this topology. In [2], W. Page gets around this problem by calling a semiuniformity a t·semiuniformity (for topological semiuniformity) if all the slices turn out to be neighborhoods, and he proves that a space is t·semiuniformizable (there is a t·semiuniformity which generates the topology) if and only if the space satisfies a certain separation property. We take a different approach. We define a point semiuniformity, $\mathbb{P}$, to be a semiuniformity with the added condition that for every $S\in\mathbb{P}$ there is a $T\in\mathbb{P}$ having for each $(x,y)\in T$ a $V\in\mathbb{P}$ such that $(x,y)\circ V$ and $V\circ(x,y)$ are contained in S. We will show that the slices gotten from $\mathbb{P}$ will always be neighborhoods in the topology generated by $\mathbb{P}$ and that a space is point semiuniformizable if and only if it satisfies the same separation property referred to above.

The natural association of uniformities with topological groups, or more exactly, with a fundamental system of a topological group is well known. We show that a point semiuniformity is just as naturally associated with a semicontinuous group [3] (called semitopological groups by Bourbaki [4] and L. Fuchs [5]). A semicontinuous group is a group with a topology making inversion and left and right multiplication by single elements continuous. We show that the point semiuniformity associated with a semicontinuous group is a uniformity if and only if the group is a topological group.

2. Point Semiuniformity

We begin by formalizing our definitions. A *point semiuniformity* for a set $\mathbb{X}$ is a filter $\mathbb{P}$ of subsets of $\mathbb{X}\times\mathbb{X}$ such that $\forall S\in\mathbb{P}$,

1) $\Delta\subseteq S$
2) $\exists T\in\mathbb{P}$ such that $T^{-1}\subseteq S$
3) $\exists T\in\mathbb{P}$ such that for each $(x,y)\in T$ there is a $V\in\mathbb{P}$ with $V\circ(x,y)\subseteq S$ and $(x,y)\circ V\subseteq S$.

A pair $(\mathbb{X},\mathbb{P})$ consisting of a set $\mathbb{X}$ and a point semiuniformity $\mathbb{P}$ on $\mathbb{X}\times\mathbb{X}$ is called a *point semiuniform space.* We call β a *base for a point semiuniformity* $\mathbb{P}$ if and only if the collection of all supersets of elements of β is $\mathbb{P}$. It is clear that any filter base satisfying the three conditions above is a base for a point semiuniformity.

THEOREM 1. Let $\mathbb{P}$ be a point semiuniformity for a set $\mathbb{X}$ and let $\mathbb{P}_x=\{S[x] \mid S\in\mathbb{P}\}$. Then $\{\mathbb{P}_x \mid x\in\mathbb{X}\}$ forms a neighborhood base for a topology τ on $\mathbb{X}$.

PROOF. Clearly, for all $S,T\in\mathbb{P}$, $x\in S[x]$ and $S[x]\cap T[x]=(S\cap T)[x]$. Now let $S[x]\in\mathbb{P}_x$. Since $S\in\mathbb{P}$ then $\exists T\in\mathbb{P}$ with the property that for each $(a,b)\in T$, $\exists U_{(a,b)}\in\mathbb{P}$ with $U_{(a,b)}\circ(a,b)\subseteq S$. In addition, since $(x,x)\in T$ then $\exists U_{(x,x)}\in\mathbb{P}$ with $U_{(x,x)}\circ(x,x)\subseteq S$. Since $U_{(x,x)}\in\mathbb{P}$ then $\exists V\in\mathbb{P}$ with the property given any $(s,t)\in V$ $\exists W_{(s,t)}\in\mathbb{P}$ with $W_{(s,t)}\circ(s,t)\subseteq U_{(x,x)}$. Now $V[x]\in\mathbb{P}_x$ and thus we must show that a neighborhood of each point of $V[x]$ is contained in $S[x]$. Suppose that $y\in V[x]$ then $(x,y)\in V$. Now $\exists W_{(x,y)}\in\mathbb{P}$ such that $W_{(x,y)}\circ(x,y)\subseteq U_{(x,x)}$. It suffices to show $W_{(x,y)}[y]\subseteq S[x]$. Therefore, let $z\in W_{(x,y)}[y]$ and so $(y,z)\in W_{(x,y)}$ which implies that $(x,z)\in W_{(x,y)}\circ(x,y)\subseteq U_{(x,x)}$.
Hence, $(x,z)=(x,z)\circ(x,x)\in U_{(x,x)}\circ(x,x)\subseteq S$. Consequently, $z\in S[x]$.∎

A uniformity $\mathbb{U}$ is a semiuniformity with the property that for each $U\in\mathbb{U}$, there is a $V\in\mathbb{U}$ such that $V\circ V\subseteq U$. Thus, we see that every uniformity is a point semiuniformity and every t·semiuniformity is a point semiuniformity. We now turn our attention to which topologies can be generated by these point semiuniformities. Any topology thus induced is called a *point semiuniform topology* and the space is called a *point semiuniformizable topological space.* The finite complement topology on an infinite space is point semiuniformizable, but since the space is not completely regular, it is not uniformizable.

In [2], Page shows that a space is t·semiuniformizable if and only if $\forall x,y\in\mathcal{X}$, $x\in Cl_\tau\{y\}$ iff $y\in Cl_\tau\{x\}$. We restate this closure condition in an equivalent form.

DEFINITION. A topological space $(\mathcal{X},\tau)$ is *point regular* (or *p·regular*) if and only if for every $V\in\tau$ and for every $x\in V$, $Cl_\tau\{x\}\subseteq V$.

The next proposition states some of the basic topological properties possessed by p·regular spaces.

PROPOSITION 2.

i) Every regular space or T_1-space is a p·regular space.

ii) A p·regular, T_0-space is a T_1-space.

iii) The continuous closed image of a p·regular space is a p·regular space, but the quotient of a p·regular space need not be p·regular.

iv) Products and subspaces of p·regular spaces are p·regular.

v) Although homogeneous spaces need not be p·regular, bihomogeneous spaces are p·regular.

In [2], Page shows that a space $\mathcal{X}$ is t·semiuniformizable if and only if it is p·regular. In his proof, he constructs a t·semiuniformity as follows: For each $x\in\mathcal{X}$, let u_x be a neighborhood of x and let $R=\bigcup(x\times u_x)$ and $S=R\cup R^{-1}$. The collection β of all such $S\subseteq\mathcal{X}\times\mathcal{X}$ forms a base for a t·semiuniformity which induces the original topology τ. However, a t·semiuniformity need not be a point semiuniformity as the following example shows.

EXAMPLE 3. Consider $\mathbb{R}$, the real numbers, with the usual topology. For each $r\in\mathbb{R}$, choose neighborhoods, u_r, as follows: Let $u_1=\mathbb{R}$. For each element of the sequence $\langle 1-1/n\rangle_{n=1}^{\infty}$, choose open interval neighborhoods so that

$1/2\notin u_0$, $0,2/3\notin u_{1/2}$, $1/2,3/4\notin u_{2/3}$, ...

$(k-2)/(k-1)$, $k/(k+1)\notin u_{k-1/k}$, etc.

For $y\in\mathbb{R}-\{1,0,1/2,2/3,3/4...\}$, choose any neighborhood u_y of y. Now, let $R=\bigcup_{r\in\mathbb{R}}(r\times u_r)$ and let $S=R\cup R^{-1}$. Let β be the collection of all such S. Let $B\in\beta$ and we may as well assume $B\subseteq S$. Then there exists x such that $x=1-1/(m+1)$, m a positive integer, and such that u_x, the neighborhood of x, is strictly contained in B[1], the neighborhood of 1, since any neighborhood of 1 contains a tail of the sequence $\langle 1-1/n\rangle_{n=1}^{\infty}$. Then, $(x,1)\notin x\times u_x$ but $(x,1)\in B[1]\times 1$. In order to have a point semiuniformity, we need $D\circ(x,1)\subseteq S$ for some $D\in\beta$. The composition

$$D\circ(x,1)=(\ \{\textstyle\bigcup_{y\in\mathbb{R}}y\times D[y]\}\ \cup\ \{\bigcup_{y\in\mathbb{R}}D[y]\times y\}\ )\circ(x,1)$$
$$=(\ x\times D[1]\ )\cup\{\ (x,y)\mid 1\in D[y]\ \}$$

and recall that $S=(\bigcup(r\times u_r))\ \bigcup\ (\bigcup(u_r\times r))$. Now if $x\times D[1]\subseteq\bigcup_{r\in\mathbb{R}}u_r\times r$ then $\forall z\in D[1]$, $x\in u_z$. But $\hat{x}=1-1/m'\in D[1]$ for some integer $m'>m$ and $x\notin u_{\hat{x}}$. Also, $(x,\hat{x})\notin x\times u_x$. Hence, although $(x,\hat{x})\in x\times D[1]\subseteq D\circ(x,1)$, $(x,\hat{x})\notin(\bigcup_{r\in\mathbb{R}}(r\times u_r))\ \bigcup\ (\bigcup_{r\in\mathbb{R}}(u_r\times r))$. Thus, $D\circ(x,1)\nsubseteq S$.
This example shows that Page's construction of a t·semiuniformity may not be a point semiuniformity. Although t·semiuniformities and point semiuniformities are not the same, they are related as the next theorem shows.

THEOREM 4. A topological space $(\mathfrak{X},\tau)$ is point semiuniformizable if and only if $(\mathfrak{X},\tau)$ is p·regular.

PROOF. Since one direction is trivial, we only need to show that a p·regular space is point semiuniformizable.
Thus, let β be the collection of all neighborhoods of Δ in $\tau\times\tau$. Clearly, β is a filter that satisfies property 1) and 2) of the definition of a point semiuniformity. Therefore, to show property 3), let $B\in\beta$ which implies that there exists $U\in\tau\times\tau$ with $\Delta\subseteq U\subseteq B$. Let $C=U\cap U^{-1}$. Then C is open and symmetric and $\Delta\subseteq C\subseteq B$. Pick $(x,y)\in C$. We need to find $D\in\beta$ with $D\circ(x,y)\subseteq B$ or equivalently, $D[y]\subseteq C[x]\subseteq B[x]$ (The proof of $(x,y)\circ D\subseteq B$ is similar using inverse notation). Let $D=C-\{\ y\times(\mathfrak{X}-C[x])\ \}$.
a) To show that $D[y]\subseteq C[x]$, suppose that $z\notin C[x]$. Then $(y,z)\in y\times(\mathfrak{X}-C[x])$ and so $(y,z)\notin D$. Thus, $z\notin D[y]$.
b) To show that D is a neighborhood of Δ, consider $D'=C-\{Cl_\tau\{y\}\times(\mathfrak{X}-C[x])\ \}$.
Since $y\times(\mathfrak{X}-C[x])\subseteq Cl_\tau\{y\}\times(\mathfrak{X}-C[x])$, we have that $D'\subseteq D$. Thus, since D' is open, we only need to show that $\Delta\subseteq D'$.
Case 1] Suppose $(z,z)\in\Delta$ with $z\in Cl_\tau\{y\}$. Since $(\mathfrak{X},\tau)$ is p·regular, $y\in Cl_\tau\{z\}$ and since $(x,y)\in C$ implies that $y\in C[x]$, then $z\in C[x]$. Thus, $(z,z)\in C-\{\ Cl_\tau\{y\}\times(\mathfrak{X}-C[x])\ \}=D'$.
Case 2] Suppose $(z,z)\in\Delta$ with $z\notin Cl_\tau\{y\}$. Then $(z,z)\in D'$ and hence, $\Delta\subseteq D'$.

Clearly, $B[x]\in\tau$, $\forall B\in\beta$ and $B\in\tau\times\tau$ which implies that the topology on $\mathfrak{X}$ generated by β is contained in τ. Conversely, let $x\in W\in\tau$. Choose a cover of $\mathfrak{X}$ by τ-open neighborhoods $\{V_y\ |\ y\in\mathfrak{X}\}$ such that $V_x\subseteq W$ and $x\in\mathfrak{X}-V_y$ for $y\notin Cl_\tau\{x\}$ and $V_y=V_x$ if $y\in Cl_\tau\{x\}$ (or equivalently, $x\in Cl_\tau\{y\}$). Let $S=\bigcup_{y\in\mathfrak{X}}V_y\times V_y$. Clearly, $\Delta\subseteq S$ and S is open in $\mathfrak{X}\times\mathfrak{X}$. Also, $S[x]=\bigcup_{x\in V_y}V_y=V_x$. Hence, $x\in S[x]=V_x\subseteq W$. Therefore, W is open in the topology on $\mathfrak{X}$ generated by β. Thus, τ is contained in the topology on X generated by β.∎

Although we used neighborhoods of Δ in the product topology in the proof above, we could have used neighborhoods of Δ in $(\tau\times\text{discrete})\cap(\text{discrete}\times\tau)$ which would give us a finer point semiuniformity inducing the same topology..

Theorem 4 proves that t·semiuniformizable and point semiuniformizable are equivalent notions for a topological space;

however, this is their only similarity. From the examples we note that vicinities in a t·semiuniform base are constructed simply by forming "crosses" along the diagonal; whereas, in a point semiuniform base vicinities are more carefully constructed possessing "crosses" at each point.

PROPOSITION 5. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a collection of point semiuniformities for $\mathbb{X}$. Let β be the collection of all finite intersections of elements of $\cup_{\alpha\in\Lambda}U_\alpha$. β is a base for a point semiuniformity which is the join of the family $\{U_\alpha\}_{\alpha\in\Lambda}$.

Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be a family of point semiuniformities for the set $\mathbb{X}$. Let $\mathbb{C}$ be the family of all point semiuniformities coarser than each V_α, $\alpha\in\Lambda$. $\mathbb{C}$ is a nonempty collection since $\{\mathbb{X}\times\mathbb{X}\}$ is a point semiuniformity coarser than each V_α, $\alpha\in\Lambda$. Then the meet of the family $\{V_\alpha\}_{\alpha\in\Lambda}$ is $\vee_{U\in\mathbb{C}}U$.

Thus, if we let $\mathbb{X}$ be a fixed set and we consider the collection of point semiuniformities for $\mathbb{X}$, then this collection forms a complete lattice when ordered by set inclusion.

THEOREM 6. If β is a finite base for a point semiuniformity then β is a base for a uniformity.

PROOF. Let $B\in\beta$. Since β is a base for a point semiuniformity, then there exists $C\in\beta$ such that for each $(x,y)\in C$ there exists $D_{(x,y)}\in\beta$ with $D_{(x,y)}\circ(x,y)\subseteq B$. Now, since β is a finite base, then the set $S = \{ D_{(x,y)} \mid (x,y)\in C \text{ and } D_{(x,y)}\circ(x,y)\subseteq B \}$ is finite. Thus, there exists $E\in\beta$ with $E\subseteq(\cap S)\cap C$. Let $(s,t)\in E\circ E$. Then there exists w such that $(s,w)\in E$ and $(w,t)\in E$. Since $E\subseteq(\cap S)\cap C$, which implies that $(s,w)\in C$, then $(w,t)\in D_{(s,w)}$. Therefore, $(s,t)=(w,t)\circ(s,w)\in D_{(s,w)}\circ(s,w)\subseteq B$.∎

COROLLARY 7. On a finite set, point semiuniformities and uniformities coincide.

3. Semicontinuous Groups

It is well known that group topologies on a group $\mathbb{G}$ are characterized by fundamental systems and fundamental systems give rise to left and right uniformities which give the same topology. A semifundamental system $\mathbb{S}$ for a group $\mathbb{G}$ is a collection of subsets of $\mathbb{G}$ each containing the identity and satisfying the following properties:

1) If $U,V\in\mathbb{S}$ then $\exists W\in\mathbb{S}$ such that $W\subseteq U\cap V$
2) If $U\in\mathbb{S}$ and $a\in U$ then $\exists V\in\mathbb{S}$ such that $Va\subseteq U$
3) If $U\in\mathbb{S}$ then $\exists V\in\mathbb{S}$ such that $V^{-1}\subseteq U$
4) If $U\in\mathbb{S}$ and $x\in\mathbb{G}$ then $\exists V\in\mathbb{S}$ such that $xVx^{-1}\subseteq U$.

E. Clay [3] showed that every semicontinuous group has a semifundamental system. Let U be an element of a semifundamental system for a semicontinuous group. Since inversion is continuous, we can always find a symmetric V such that $V \subseteq U$ by letting $V = U \cap U^{-1}$. Consequently, we can always assume that our semifundamental system is a symmetric semifundamental system. The next theorem shows that every semicontinuous group is point semiuniformizable.

THEOREM 8. Let $\mathbb{S}$ be a symmetric semifundamental system for a semicontinuous group $(\mathbb{G},\tau)$. Define $L_U=\{(x,y) \mid x \in yU\}$ $[R_U=\{(x,y) \mid x \in Uy\}]$ Then $\mathcal{L}=\{L_U \mid U \in \mathbb{S}\}$ $[\mathcal{R}=\{R_U \mid U \in \mathbb{S}\}]$ is a base for a point semiuniformity which induces the original topology τ. The point semiuniformity generated by $\mathcal{L}$ $[\mathcal{R}]$ is called the left [right] point semiuniformity of $(\mathbb{G},\tau)$ and is the unique point semiuniformity for $\mathbb{G}$ that generates τ and has a base of left [right] invariant sets.

PROOF. Clearly, $\mathcal{L}$ is a filter base that satisfies the first two properties of a base for a point semiuniformity.

Let $L_U \in \mathcal{L}$ and $(x,y) \in L_U$. Then $y^{-1}x \in U$. By definition of semifundamental system, we can find $W \in \mathbb{S}$ such that $Wy^{-1}x \subseteq U$. Let $(x,z) \in L_W \circ (x,y)$. Then $(y,z) \in L_W$ which implies that $z^{-1}y \in W$. Therefore, $z^{-1}yy^{-1}x \in Wy^{-1}x \subseteq U$. Thus, $z^{-1}x \in U$ or equivalently, $(x,z) \in L_U$.

Let $L_U \in \mathcal{L}$ and $(x,y) \in L_U$. Then $y^{-1}x \in U$. By definition of semifundamental system, we can find $W \in \mathbb{S}$ such that $Wy^{-1}x \subseteq U$. Also, there exists $V \in \mathbb{S}$ such that $y^{-1}xV(y^{-1}x)^{-1} \subseteq W$. So then $y^{-1}xV \subseteq Wy^{-1}x \subseteq U$. Let $(z,y) \in (x,y) \circ L_V$. Then $(z,x) \in L_V$ which implies that $x^{-1}z \in V$. Therefore, $y^{-1}xx^{-1}z \in y^{-1}xV \subseteq U$. Thus, $y^{-1}z \in U$, or equivalently, $(z,y) \in L_U$.

To show that this left point semiuniformity generates the topology τ, let $U \in \mathbb{S}$ and $x \in \mathbb{G}$. Then $L_U[x] = \{y \mid (x,y) \in L_U\} = \{y \mid x \in yU\} = \{y \mid y \in xU\} = xU$.■

THEOREM 9. Suppose that $\mathbb{S}$ is a semifundamental system for a group $(\mathbb{G},\cdot,\tau)$. If the left or right point semiuniformity is a uniformity, then $\mathbb{S}$ is a fundamental system.

PROOF. Assuming that $\mathbb{S}$ generates a base for the left uniformity $\mathcal{L}$, then picking $U \in \mathbb{S}$ implies that $L_U \in \mathcal{L}$ and so, by definition of $\mathcal{L}$, $\exists V \in \mathbb{S}$ such that $L_V \circ L_V \subseteq L_U$. If $x \in V \cdot V$ then $x = a \cdot b$ where $a \in V$ and $b \in V$. Clearly, $x \in a \cdot V$ and $a \in e \cdot V = V$. Therefore, $(x,a) \in L_V$ and $(a,e) \in L_V$. Combining the above yields $(x,e) \in L_V \circ L_V \subseteq L_U$. Thus, $x \in e \cdot U = U$.■

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