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RESEARCH NOTES

ON HYPER-REFLEXIVITY OF SOME OPERATOR SPACES

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ABSTRACT. In the present note, we define operator spaces with n-hyper-reflexive property, and prove n-hyper-reflexivity of some operator spaces

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1. INTRODUCTION

Let H be a Hilbert space, and $B(H)$ be the algebra of all bounded linear operators on H It is well known that $B(H)$ is the dual space of the Banach space of trace class operators If $T \in B(H)$, $R \subset B(H)$, and n is a positive integer, then $H^{(n)}$ denotes the direct sum of n copies of H, $T^{(n)}$ denotes the direct sum of n copies of T acting on $H^{(n)}$ and $R^{(n)} = \{T^{(n)} | T \in R\}$ Let $P(H)$ be the set of all orthogonal projections in $B(H)$ For any subspace $R \subset B(H)$, we will denote by $l(R)$ the collection of all maximal elements of the set

$$\{(Q,P) | (Q,P) \in P(H) \times P(H), QRT = 0\}$$

with respect to the natural order It can be seen that if R is a unital subalgebra of $B(H)$, then

$$l(R) = \{1-P,P) | P \in lat\ R\}$$

where $lat\ R$ is lattice of all invariant subspace of R Recall that an algebra $R \subset B(H)$ is transitive if $lat\ R\{0,1\}$, and reflexive if the only operators that leave invariant all of the invariant subspaces of R are the operators belonging to R Generalizing this notion, we say that an operator space $R \subset B(H)$ is transitive if $l(R) = \{(0,1),(1,0)\}$ (this is equivalent to $\overline{Rx} = H$ for any $x \in H - \{0\}$), and is reflexive if

$$R = \{T \in B(H) | QTP = 0 \text{ for every } (Q,P) \in l(R)\}.$$

In other words, R is reflexive if the seminorms $d(T,R)$ and $\sup\{\|QTP\| \mid (Q,P) \in P(R)\}$ vanish on R simultaneously, where $d(T,R)$ is the distance from T to R It can be seen that

$$d(T,R) \geq \sup\{\|QTP\| \mid (Q,P) \in l(R)\}$$

for any $T \in B(H)$.

Reflexive operator space $R \subset B(H)$ is called hyper-reflexive if there exists some constant $C \geq 1$ such that

$$d(T,R) \leq C \sup\{\|QTP\| \mid (Q,P) \in l(R)\}$$

for any $T \in B(H)$, (see [1-5]).

In [4], an example of non hyper-reflexive operator algebras is constructed

In the present note, we define operator spaces with n-hyper-reflexive property, and prove n-hyper-reflexivity of some operator spaces

The operator space $R \subset B(H)$ is called n-reflexive if $R^{(n)}$ is reflexive It can be shown that

$$d(T,R) \geq \sup\{\|QT^{(n)}P\| \mid (Q,P) \in l(R^{(n)})\}$$

for any $T \in B(H)$ and $n \in N$

We say that the n-reflexive operator space $R \subset B(H)$ is n-hyper-reflexive if there exists some constant $C \geq 1$ such that

$$d(T,R) \leq C \sup\{\|QT^{(n)}P\| \mid (Q,P) \in l(R^{(n)})\}$$

for any $T \in B(H)$

It is easily seen that if R is n-reflexive (n-hyper-reflexive) then it is k-reflexive (k-hyper-reflexive) for every $k > n$

2. MAIN RESULT

Let us consider in $B(H)$ the following operator equation

$$\sum_{i=1}^{n} A_i X B_i = X . \qquad (2\ 1)$$

The space of all solutions of the equation (2 1) will be denoted by R

PROPOSITION 1. R is $(n+1)$-reflexive

PROOF. For given any $x,\, y \in H - \{0\}$, put

$$x = (B_1x, ..., B_nx, x) \in H^{(n+1)} \quad \text{and} \quad y = (A_1^*y, ..., A_n^*y,\, -y) \in H^{(n+1)} .$$

Let P_x and Q_y be the one-dimensional projections on one-dimensional subspaces $\{C_x\}$ and $\{C_y\}$ respectively From (2.1), we have $(Q_y, P_x) \in l(R^{(n+1)})$ On the other hand, it is easy to see that any $T \in B(H)$ is a solution of equation (2.1) if and only if $Q_yT^{(n+1)}P_x = 0$ This completes the proof.

We will assume that, in case $n > 1$, the coefficients of equation (2 1) satisfy the following conditions

$$\|A_i\| \leq 1\,, \quad \|B_i\| \leq 1\,, \quad A_iA_j = B_iB_j = 0 \ \ (1 \leq i < j \leq n) . \qquad (2\ 2)$$

The purpose of this note is to prove the following.

THEOREM 2. The space R of all solutions of (2.1) and (2 2) is $(n+1)$-hyper-reflexive.

To prove Theorem 2 we need some preliminary results.

Let Y be a Banach space with $Y^* = X$ and S be a weak* continuous linear operator on X with uniformly bounded degree, $\|S^n\| \leq C (n \in N)$ Denote by E the space of all fixed points of S, $E = \{x \in X | Sx = x\}$ If $x_0 \in E$, then for any $x \in X$ we have

$$\|S^n x - x\| = \|S^n(x - x_0) - (x - x_0)\| \leq (C+1)\|x - x_0\|$$

and consequently

$$d(x,E) \geq \frac{1}{C+1} \sup_n \|S^n x - x\|$$

PROPOSITION 3. Under the above assumptions,

$$d(x,E) \leq \sup_n \|S^n x - x\|$$

for any $x \in X$

PROOF. Since E is a weak* closed subspace of X, there exists a subspace $M \subset Y$ such that $M^{\perp} = E$, where $M^{\perp}$ is the annihilator of M It can be seen that the set $\{Ty - y | y \in Y\}$ weak* generates M, where T is the preadjoint of S, that is, $T^* = S$. Let $x \in X$ and let $K(x)$ be the weak* closure of the convex hull of the set $\{S^n x | n \in N\}$ By Alaoğlu's theorem, $K(x)$ is weak* compact We will show that $K(x) \cap E \neq \oslash$ for any $x \in X$ Suppose that $K(x) \cap E = \oslash$. By Hahn-Banach separating theorem, there exists $y_0 \in M$ such that

$$\inf_{a \in K(x)} |\langle a, y_o \rangle| = \sigma > 0$$

where $\langle\ ,\ \rangle$ is the duality between X and Y

Put

$$x_n = \frac{1}{n} \sum_{k=1}^{n} S^k x \,.$$

Then $x_n \in K(x)$ and $\|x_n\| \leq C\|x\|$ Now, we will prove that

$$\lim_n |\langle x_n, y_0 \rangle| = 0 \,. \tag{2 3}$$

Since (x_n) is a bounded set, it is sufficient to prove the equality (2 3) in case $y_0 = Ty - y, (y \in Y)$ In that case

$$\langle x_n, Ty - y \rangle = \langle Sx_n - x_n, y \rangle = \frac{1}{n} \langle S^{n+1}x - Sx, y \rangle \to 0 \,.$$

Now, suppose that $\|S^n x - x\| \leq \delta$ for some $\delta > 0$ and any $n \in N$ It is easy to see that $\|a - x\| \leq \delta$ for any $a \in K(x)$ Let $a_0 \in K(x) \cap E$ Then $\|a_0 - x\| \leq \delta$ and consequently $d(x, E) \leq \delta$

PROOF. OF THEOREM 2. For any $A \in B(H)$ we denote by L_A and R_A the left and right multiplication operators $L_A : X \to AX, R_A : X \to XA$ on $B(H)$ respectively Then we may write equation (2 1) as

$$\left(\sum_{i=1}^{n} L_{A_i} R_{B_i} \right) X = X \,.$$

Thus, the solution space R of (2 1) coincide with the set of all fixed points of the operator

$$S = \sum_{i=1}^{n} L_{A_i} R_{B_i} \,.$$

It is easily seen that S is a weak* continuous linear operator on $B(H)$ Moreover, under assumption (2 2), it can be shown (by induction) that

$$S^k = \sum_{i=1}^{n} L_{A_i^k} R_{B_i^k} \,.$$

and consequently $\|S^K\| \leq n$

By Proposition 3, for any $T \in B(H)$ we have

$$\begin{aligned} d(T, R) &\leq \sup_k \|S^k(T) - T\| = \sup_k \left\| \sum_{i=1}^{n} A_i^k T B_i^k - T \right\| \\ &= \sup_k \sup_{\|x\| \leq 1, \|y\| \leq 1} \left| \sum_{i=1}^{n} (TB_i^k x, A_i^{*k} y) - (Tx, y) \right| . \end{aligned}$$

For $\|x\| \leq 1$ and $\|y\| \leq 1$, let $x_k = (B_1^k x, ..., B_n^k x, x), y_k = (A_1^{*k} y, ..., A_n^{*k} y, -y)$ It can be seen that

$$(R^{(n+1)} x_k, y_k) = 0 \quad \text{and} \quad \|x_k\|^2 \leq n + 1, \, \|y_k\|^2 \leq n + 1 \, (k \in N) \,.$$

Therefore

$$d(T, R) \leq (n+1) \sup \left\{ |(T^{(n+1)} x, y)| \,\middle|\, (R^{(n+1)} x, y) = 0, \|x\| = \|y\| = 1 \right\} .$$

Let P_x, Q_y be the one-dimensional projections (as in the proof of Proposition 1) Then we obtain

$$d(T,R) \le (n+1)\sup\left\{ \|Q_y T^{(n+1)} P_x\| \,\middle|\, Q_y R^{(n+1)} P_x = 0 \right\}$$
$$\le (n+1)\sup\left\{ \|Q T^{(n+1)} P\| \,\middle|\, (Q,P) \in l(R^{(n+1)}) \right\}.$$

This completes the proof

COROLLARY 4. Let $A, B \in B(H)$ with $\|A\| \le 1$, $\|B\| \le 1$ Then, the solution space R of the equation

$$AXB = X \tag{2 4}$$

is 2-hyper-reflexive with constant $C = 2$

Generally speaking, the solution space of equation (2 4) may be reflexive For example, if $Q, P \in P(H)$, then the solution space of equation

$$QXP = X \tag{2 5}$$

is reflexive Hyper-reflexivity (with constant $C = 1$) of the solution space of equation (2 5) was proved in [3]

Note that the space of all Toeplitz operators τ coincide with the solution space of (2 4) in case $A = U^*$ and $B = U$, where U is a unilateral shift operator on Hardy space H^2 [6]

Consequently, τ is a 2-reflexive by Proposition 1 Using Theorem 2, we can deduce even more

COROLLARY 5. The space of all Toeplitz operators τ is 2-hyper-reflexive, with constant $C = 2$ In other words

$$d(T,\tau) \le 2\sup\left\{ \|Q T^{(2)} P\| \,\middle|\, (Q,P) \in l(\tau^{(2)}) \right\}$$

for any $T \in B(H^2)$

On the other hand we have the following

PROPOSITION 6. The space of all Toeplitz operators τ is transitive (consequently τ is not reflexive)

PROOF. Suppose that τ is nontransitive Then there exists $f, g \in H^2 - \{0\}$ such that $(Tf, g) = 0$ for every $T \in \tau$ If we put in last equality $T = U^n$ and $T = U^{*n}$ ($n = 0, 1, 2, \ldots$), then we obtain that the Fourier coefficients of the function $f\bar{g}$ are zero Since $f\bar{g} = 0$ a e , one of these functions vanishes a e on some subset of the unit circle with positive Lebesque measure By F and M Riesz uniqueness theorem [6], one of these functions is zero

Hyper-reflexivity of algebras of analytic Toeplitz operators was proved in [5]

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