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THE Θ-TRANSFORMATION OF CERTAIN POSITIVE LINEAR OPERATORS

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ABSTRACT. The intention of this paper is to describe a construction method for a new sequence of linear positive operators, which enables us to get a pointwise order of approximation regarding the polynomial summator operators which have "best" properties of approximation.

KEY WORDS AND PHRASES. Approximation by positive linear operators, discrete linear operators, $(C,1)$ - means of Chebyshev series.

1. The aim of this paper can be described in the following way: Starting with a sequence $A=(A_n)$ of approximation operators, we construct - by means of the so called Θ - transformation - a new sequence of operators $B=(B_n)=\Theta(A)$.
With the known properties of A we get the corresponding properties of the sequence $B=\Theta(A)$. We also prove, that if A is the sequence of $(C,1)$ - means of Chebyshev series, the polynomials (B_nf), $f\in C(I)$, furnish a pointwise order of approximation similar to the best order of approximation.

Let Π_n, $n\in \mathbb{N}_0$, be the linear space of all algebraic polynomials with real coefficients of degree $\leq n$ and $T_n(t)=cos(n \arccos t)$ the n - th Chebyshev polynomial, $n\in\mathbb{N}_0$.
We denote by X the normed linear spaces $C(I), I:=[-1,1]$ or $L^p_\omega(I)$, $1\leq p<\infty$, endowed with norms $\|f\|_{C(I)} = \|f\| := \max_{t\in I}|f(t)|$ for $f\in C(I)$, respectively $\|f\|_p = \left[\int\limits_{-1}^{1}|f(t)|^p\,\omega(t)dt\right]^{\frac{1}{p}}$, where f is an element of the Lebesgue space $L^p_\omega(I)$ with the weight $\omega(t)=\frac{1}{\sqrt{1-t^2}}$.

Further for $f\in X$ and a polynomial g we use the inner product

$$\langle f,g\rangle = \int\limits_{-1}^{1} f(t)\,g(t)\,\omega(t)dt.$$

The translation operator $\tau_x : X\to X$, $x\in I$, defined by

$$(\tau_x f)(t) = \frac{1}{2}\left[f(xt+\sqrt{1-x^2}\sqrt{1-t^2})+f(xt-\sqrt{1-x^2}\sqrt{1-t^2})\right],\quad (t,x)\in I\times I,$$

has the property

$$(\tau_x T_k)(t) = T_k(t)\,T_k(x),\quad k\in\mathbb{N}$$

(see [3]).
If we use the convolution product $\star : L^1_\omega(I)\times L^1_\omega(I)\to L^1_\omega(I)$

$$(f\star g)(x) = \int\limits_{-1}^{1} f(t)\,(\tau_x g)(t)\,\omega(t)dt\,,$$

then our aim is to construct some approximation operators $A_n : X \to \Pi_n$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \|f - A_n f\|_X = 0$, $f \in X$.

A sequence $a = (a_n)_{n\in\mathbb{N}_0}$, $a_n \in \Pi_n$, with degree $a_n = n$ for all $n \in \mathbb{N}_0$, is called a polynomial sequence. If $\mathcal{P}^+$ denotes the set of all polynomial sequences $a = (a_n)_{n\in\mathbb{N}_0}$ with the properties

$$i.)\quad a_n(x) \geq 0\,, \quad x \in I \qquad ii.)\quad \langle 1, a_n\rangle = 1\,, \quad n \in \mathbb{N}_0,$$

then for

$$a_n(x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x)\,, \qquad \text{where} \qquad \omega_0 = \frac{1}{\pi}, \qquad \omega_k = \frac{2}{\pi},\ k \geq 1, \tag{1.1}$$

$$a_n(x,t) = (\tau_x a_n)(t) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x) T_k(t)\,,$$

we consider the sequence $A := A(a) = (A_n)_{n\in\mathbb{N}_0}$, $A_n : X \to \Pi_n$, of linear positive operators, defined by $A_n f = f \star a_n = a_n \star f$ that is

$$(A_n f)(x) = A_n(f;x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} \langle f, T_k\rangle T_k(x) = \int_{-1}^{1} a_n(x,t) f(t) \omega(t) dt, \quad x \in I\,. \tag{1.2}$$

In this case $a = (a_n)_{n\in\mathbb{N}_0}$ is called the **generating sequence of** $A = (A_n)$.

If $A(a) = (A_n)$ is defined as in (1.1) and (1.2), then $|\alpha_{k,n}| = |\langle T_k, a_n\rangle| \leq 1$ and let us define the functionals $r_n : \mathcal{P}^+ \to \mathbb{R}$, $n \in \mathbb{N}$,

$$r_n(A) := 1 - \alpha_{1,n} = 1 - \langle T_1, a_n\rangle, \qquad n \in \mathbb{N}\,.$$

An important polynomial sequence $\varphi = (\varphi_n)_{n\in\mathbb{N}_0}$, $\varphi \in \mathcal{P}^+$, was considered by L.Fejér, namely

$$\varphi_n(x) = \frac{1 - T_{n+1}}{\pi(n+1)(1-x)} = \sum_{k=0}^{n} \omega_k \left(1 - \frac{k}{n+1}\right) T_k(x)\,. \tag{1.3}$$

The corresponding linear positive operators $F = (F_n)_{n\in\mathbb{N}_0}$, $F_n = f \star \varphi_n$ are the (C,1) – means of Chebyshev series, i.e. the Fejér operators $F_n : X \to \Pi_n$, $n \in \mathbb{N}_0$,

$$(F_n f)(x) = \sum_{k=0}^{n} \omega_k \left(1 - \frac{k}{n+1}\right) \langle f, T_k\rangle T_k(x)\,, \quad f \in X\,. \tag{1.4}$$

There exists a connection between the operators defined in (1.2) and those from (1.4). Indeed, using the equalities $A_n T_k = \alpha_{k,n} T_k$, $k \in \mathbb{N}_0$, we get with

$$a_n = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k = (n+1)\, A_n \varphi_n - n\, A_n \varphi_{n-1}$$

the identity

$$A_n f = (n+1) a_n \star F_n f - n a_n \star F_{n-1} f\,.$$

2. Let $b = (b_n)_{n\in\mathbb{N}_0}$ be an element from $\mathcal{P}^+$ with

$$b_n(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x)\,, \tag{2.1}$$

and $B = B(b) = (B_n)_{n\in\mathbb{N}}$, $B_n : X \to \Pi_n$, the operators with the "generating polynomial sequence b", defined by

$$(B_n f)(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} \langle f, T_k\rangle T_k(x), \quad x \in I\,. \tag{2.2}$$

Suppose that

$$\int_{-1}^{1} h(t)\omega(t)dt = \sum_{k=1}^{m(n)} c_k(n)h(z_k), \tag{2.3}$$

with $c_k(n) \geq 0$, $z_k \in [-1,1]$, $k = 1, 2, \cdots, m(n)$, is a quadrature formula which is exact for all polynomials $h \in \Pi_{s(n)}$ with $s(n) \geq n+2$, $n \in \mathbb{N}$.

For $b = (b_n) \in \mathcal{P}^+$ and $B = (B_n)$ as in (2.1) - (2.2) we consider the linear positive operators $B_n^\star$, $\tilde{B}_n$, $n \in \mathbb{N}_0$, where for $f \in X$

$$(B_n^\star f)(x) = \sum_{k=1}^{m(n)} c_k(n)(\tau_x b_n)(z_k) f(z_k) \tag{2.4}$$

and

$$\left(\tilde{B}_n f\right)(x) = \sum_{k=1}^{m(n)} c_k(n)(\tau_x f b_n)(z_k). \tag{2.5}$$

The sequence $B^\star = (B_n^\star)$ is called **"the discrete form" of** $B = (B_n)$, with respect to (2.3). The operator $\tilde{B}_n$ appears to be useful for the connection between B_n and $B_n^\star$.

Lemma 2.1 *If $\tilde{B}_n$ is defined as in (2.5), then for $j \in \{1,2\}$*

$$\tilde{B}_n(1-t^j;x) = \frac{1}{2^{j-1}}(1-\beta_{j,n}). \tag{2.6}$$

Proof: Let us observe that

$$\int_{-1}^{1}(1-t^j)b_n(t)T_k(t)\omega(t)dt = B_n((1-t^j)T_k(t);1).$$

Therefore

$$\tau_x((1-t^j)b_n(t))(z) = \sum_{k=0}^{n+j}\omega_k B_n((1-t^j)T_k(t);1)T_k(x)T_k(z)$$

and using (2.3) for $j \in \{1,2\}$ we have

$$\begin{aligned}\tilde{B}_n(1-t^j;x) &= \sum_{k=1}^{m(n)} c_k(n)\tau_x((1-t^j)b_n(t))(z_k)\\ &= \int_{-1}^{1}\tau_x((1-t^j)b_n(t))(z)\omega(z)dz = B_n(1-t^j;1).\end{aligned}$$

Finally

$$B_n(1-t;1) = 1-\beta_{1,n}, \qquad B_n(1-t^2;1) = \frac{1}{2}(1-\beta_{2,n}), \tag{2.7}$$

which completes the proof. □

Theorem 2.2 *Suppose that B_n is defined by means of (2.2) with $b \in \mathcal{P}^+$. Let $B_n^\star$ be the discrete operator from (2.4) and*

$$\delta_n(x) \quad \textit{one of the functions} \quad B_n(|x-t|;x) \quad \textit{or} \quad B_n^\star(|x-t|;x).$$

Then for $x \in I$

$$|x|r_n(B) \leq \delta_n(x) \leq \sqrt{1-x^2}\sqrt{\frac{1-\beta_{2,n}}{2}} + |x|r_n(B). \tag{2.8}$$

Proof: With $e_k(t) = t^k, k \in \mathbb{N}_0$, it is known that for convex functions $\gamma \in C(I)$ we have

$$\gamma(Le_1) \leq L\gamma \quad \text{on} \quad I, \tag{2.9}$$

where L is a linear positive operator $C(I) \to C(I)$ with $Le_0 = e_0$ (see [8]).
If we select $\gamma(t) = |x-t|$, $L = B_n$, we have by using the inequality (2.9)

$$|x - x\beta_{1,n}| \leq B_n(|x-t|;x);$$

or on the other hand for $L = B_n^*$

$$|x - (B_n^* e_1)(x)| \leq B_n^*(|x-t|;x).$$

For $h \in \Pi_2$ it is $B_n^* h = B_n h$ and so we obtain the lower bound in (2.8).
Further let us denote

$$\psi_1(x,t) = xt + \sqrt{1-x^2}\sqrt{1-t^2}$$

$$\psi_2(x,t) = xt - \sqrt{1-x^2}\sqrt{1-t^2}.$$

Then for $x, t \in I$, $j \in \{1,2\}$

$$|x - \psi_j(x,t)| \leq \sqrt{1-x^2}\sqrt{1-t^2} + |x|(1-t) \tag{2.10}$$

and

$$|x - t| \leq \sqrt{1-x^2}\sqrt{1-\psi_j^2(x,t)} + |x|(1-\psi_j(x,t)). \tag{2.11}$$

Define the linear positive functionals $J_n : C(I) \to \mathbb{R}$, $n \in \mathbb{N}_0$, by $J_n(f) = \langle f, b_n \rangle$.
We have

$$J_n(1-t^j) = B_n(1-t^j;1)$$

more precisely (see (2.7))

$$J_n(1-t) = 1 - \beta_{1,n}$$

$$J_n(\sqrt{1-t^2}) \leq \sqrt{J_n(1-t^2)} = \sqrt{\frac{1-\beta_{2,n}}{2}}.$$

Because

$$(B_n f)(x) = \int_{-1}^{1} b_n(t)(\tau_x f)(t)\omega(t)dt$$

and (2.10) enables us to write

$$\tau_x(|x-.|;t) = \left| x - \frac{\psi_1(x,t)+\psi_2(x,t)}{2} \right| \leq \sqrt{1-x^2}\sqrt{1-t^2} + |x|(1-t)$$

one finds

$$\begin{aligned} B_n(|x-t|;x) &\leq \sqrt{1-x^2}\, J_n(\sqrt{1-t^2}) + |x|\, J_n(1-t) \\ &\leq \sqrt{1-x^2}\sqrt{\frac{1-\beta_{2,n}}{2}} + |x|(1-\beta_{1,n}), \end{aligned}$$

i.e. the upper bound in (2.8). Regarding the discrete operators (B_n^*), we have from (2.4) and (2.11)

$$\begin{aligned}
B_n^\star(|x-t|;x) &= \sum_{k=1}^{m(n)} c_k(n)|x-z_k|\frac{b_n(\psi_1(x,z_k))+b_n(\psi_2(x,z_k))}{2} \\
&\le \sum_{k=1}^{m(n)} c_k(n)\frac{\sqrt{1-x^2}\sqrt{1-\psi_1^2(x,z_k))}+|x|(1-\psi_1(x,z_k))}{2}\, b_n(\psi_1(x,z_k)) \\
&\quad + \sum_{k=1}^{m(n)} c_k(n)\frac{\sqrt{1-x^2}\sqrt{1-\psi_2^2(x,z_k))}+|x|(1-\psi_2(x,z_k))}{2}\, b_n(\psi_2(x,z_k)) \\
&= \sqrt{1-x^2}\sum_{k=1}^{m(n)} c_k(n)\tau_x(\sqrt{1-t^2}b_n(t))(z_k) \\
&\quad + |x|\sum_{k=1}^{m(n)} c_k(n)\tau_x((1-t)b_n(t))(z_k) \\
&= \sqrt{1-x^2}\hat{B}_n(\sqrt{1-t^2};x)+|x|\hat{B}_n(1-t;x)\,.
\end{aligned}$$

From (2.6) using Schwarz inequality we complete the proof. □

Other upper bounds for δ_n were obtained by J.D.Cao and H.H.Gonska [5].

Theorem 2.3 *Let $b = (b_n)$ be an arbitrary polynomial sequence from $\mathcal{P}^+$. Suppose that $B = (B_n)$, $B^\star = (B_n^\star)$ are defined as in (2.2) respectively (2.4). Then for $f \in C(I)$, $x \in I$,*

$$|f(x)-(B_nf)(x)| \le 2\omega(f;\nabla_n^B(x)) \le 4\omega(f;\Delta_n^B(x)) \tag{2.12}$$

$$|f(x)-(B_n^\star f)(x)| \le 2\omega(f;\nabla_n^B(x)) \le 4\omega(f;\Delta_n^B(x)) \tag{2.13}$$

where $\omega(f;\delta) := \sup\{|f(t+h)-f(t)|;\ |h|\le\delta,\ t,t+h\in I\}$ and

$$\begin{aligned}
\nabla_n^B(x) &= \sqrt{1-x^2}\sqrt{\frac{1-\beta_{2,n}}{2}}+|x|(1-\beta_{1,n}) \\
\Delta_n^B(x) &= \sqrt{(1-x^2)(1-\beta_{1,n})}+|x|(1-\beta_{1,n}),
\end{aligned}$$

with

$$\beta_{1,n} = (B_ne_1)(1)\,.$$

Proof: It is known that an arbitrary linear positive operator $L_n : C(I)\to C(I)$ with $L_ne_0 = e_0$ satisfies the inequality

$$|f(x)-(L_nf)(x)| \le 2\omega(f;L_n(|x-t|;x))\,.$$

The upper - estimate from theorem 2.2 enables us to write

$$|f(x)-(L_nf)(x)| \le 2\omega(f;\nabla_n^B(x)),\quad x\in I,$$

where L_n is one of the operators B_n or $B_n^\star$.

Let $q_m(t) = (1-t)^m$, $m \in \mathbb{N}$, and observe that q_j, q_m are monotone on I in the same sense. By means of Chebyshev inequality we have $(B_n q_j)(x)(B_n q_m)(x) \le (B_n q_{j+m})(x)$, $j, m \in \mathbb{N}_0$, where we find with $j = m = 1$ and $x = 1$

$$0 \le 1 - \beta_{2,n} \le 2(1+\beta_{1,n}) r_n(B) \le 4 r_n(B). \tag{2.14}$$

Therefore

$$\omega(f; \nabla_n^B(x)) \le \omega(f; \sqrt{2}\Delta_n^B(x)) \le 2\omega(f; \Delta_n^B(x)), \quad x \in I,$$

which proves this theorem. □

Remark: One knows that for $(b_n) \in \mathcal{P}^+$ the Fejér inequality [6] holds

$$\beta_{1,n} \le \cos\frac{\pi}{n+2}, \quad n \in \mathbb{N}.$$

In the case of Jacobi polynomials $R_n^{(\alpha,\beta)}$, $\alpha \ge \beta \ge -\frac{1}{2}$, for an arbitrary n a similar extremal problem is solved in [8]. For an even n the problem is considered in ([1], p.68).
However, for all linear positive operators $B = (B_n)$ generated by polynomial sequences $b = (b_n) \in \mathcal{P}^+$ one has

$$r_n(B) \ge 2\sin^2\frac{\pi}{2(n+2)}. \tag{2.15}$$

Let us present a short proof of Fejér's inequality (2.15). If $h \in \Pi_{n+1}$, then it is easy to observe that

$$\langle 1, h\rangle = \sum_{k=0}^{s} c_k h(x_{k,n}), \quad s = \left[\frac{n}{2}\right] + 1,$$

$x_{0,n} = -1$, $x_{k,n} = \cos\frac{2k-1}{n+2}\pi$, $k \ge 1$, $c_0 = \frac{2\pi}{n+2}\frac{1-(-1)^n}{4}$, $c_1 = \cdots = c_s = \frac{2\pi}{n+2}$.

If $h_0(t) = (1-t)b_n(t)$ then

$$r_n(B) = \langle 1, h_0\rangle = \sum_{k=0}^{s} c_k(1-x_{k,n})b_n(x_{k,n}) \ge c_1(1-x_{1,n})b_n(x_{1,n}) \ge 1 - x_{1,n} = 2\sin^2\frac{\pi}{2(n+2)}.$$

Therefore the equality holds if and only if

$$b_n(x) = b_n^*(x) = \lambda_n(x+1)^d \prod_{k=2}^{s}(x-x_{k,n})^2, \quad d = \left[\frac{n+1}{2}\right] - \left[\frac{n}{2}\right],$$

where λ_n is selected such that $b_n(x_{1,n}) = \frac{n+2}{2\pi}$. It may be shown that [9]

$$b_n^*(x) = \kappa_n \frac{1+T_{n+2}(x)}{(x-\cos\frac{\pi}{n+2})^2}, \qquad \kappa_n = \frac{1}{\pi(n+2)}\sin^2\frac{\pi}{n+2}.$$

3. A polynomial sequence $a = (a_n)$ belongs to the class $\mathcal{P}^1$ if and only if

i) $a \in \mathcal{P}^+$ and
ii) for each $n \in \mathbb{N}$ there exists at least a root $z_0(n)$ of a_n in I.

We denote $z_0 = z_0(n+1)$ and remind that $a_n(x,t) = (\tau_x a_n)(t)$, $a_{n+1}(z_0) = a_{n+1}(1, z_0) = 0$. Define $b = (b_n)$ to be the sequence of polynomials

$$b_n(x) = \frac{1}{c_n}\frac{a_{n+1}(x, z_0)}{1-x}, \tag{3.1}$$

where

$$c_n = \int_{-1}^{1} \frac{a_{n+1}(t, z_0)}{1-t} \omega(t)dt .$$

It is clear that the positivity of the translation operator certifies the fact that $b = (b_n) \in \mathcal{P}^+$. If $l : \mathcal{P}^1 \to \mathcal{P}^+$ is the mapping $(a_n) \to (b_n)$, b_n being as in (3.1), we write $b = l(a)$.

Definition *If $a = (a_n) \in \mathcal{P}^1, b = (b_n) = l(a)$, then the sequence $B = (B_n)$ defined in (2.2) is called the* **Θ - transformation of the sequence** *$A = (A_n)$ from (1.2) and we write $B = \Theta(A)$.*

Lemma 3.1 *Suppose that $a = (a_n) \in \mathcal{P}^1$,*

$$a_n(x) = \sum_{k=0}^{n} \omega_k \alpha_{k,n} T_k(x) , \qquad a_{n+1}(z_0) = 0,\ z_0 = z_0(n+1) \in I ,$$

is the generating polynomial sequence for the operators $A = (A_n)$.

If $b = (b_n) = l(a)$, then

$$b_n(x) = -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \varphi_k(x) , \quad n \in \mathbb{N} ,$$

where φ_k is defined in (1.3).

Proof: Let $d_k(t, x)$ be the Dirichlet kernel

$$d_k(t, x) = \sum_{j=0}^{k} \omega_j T_j(t) T_j(x)$$

and $S_n : X \to \Pi_n$ be the partial - sum of Chebyshev series, i.e.

$$(S_n f)(x) = \sum_{j=0}^{n} \omega_j \langle f, T_j \rangle T_j(x) = \int_{-1}^{1} d_n(t, x) f(t) \omega(t) dt . \tag{3.2}$$

From (3.1) we get

$$\begin{aligned} b_n(x) &= \frac{1}{c_n} \frac{a_{n+1}(x, z_0) - a_{n+1}(1, z_0)}{1-x} = -\frac{1}{c_n} \sum_{k=1}^{n+1} \omega_k \alpha_{k,n+1} \frac{1 - T_k(x)}{1-x} T_k(z_0) \\ &= -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \varphi_k(x) . \end{aligned}$$

Further, we may write □

$$b_n(x) = -\frac{2}{c_n} \sum_{k=0}^{n} (k+1) \alpha_{k+1,n+1} T_{k+1}(z_0) \sum_{j=0}^{k} \omega_j \left(1 - \frac{j}{k+1}\right) T_j(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x)$$

with

$$\begin{aligned} \beta_{k,n} &= -\frac{2}{c_n} \sum_{j=k}^{n} (j+1-k) \alpha_{j+1,n+1} T_{j+1}(z_0) \qquad (3.3) \\ &= -\frac{2}{c_n} \int_{-1}^{1} a_{n+1}(t) \left(\sum_{j=k}^{n} (j+1-k) T_{j+1}(t) T_{j+1}(z_0) \right) \omega(t) dt . \end{aligned}$$

Now, if $\varphi_k(t,x) = (\tau_x\varphi_k)(t)$

$$\sum_{j=k}^{n}(j+1-k)T_{j+1}(t)T_{j+1}(z_0)$$
$$= \frac{\pi}{2}\left((n+2-k)d_{n+1}(t,z_0) - d_k(t,z_0) + (k+1)\varphi_k(t,z_0) - (n+2)\varphi_{n+1}(t,z_0)\right).$$

Using (1.4), (3.2) - (3.3) we conclude with

Lemma 3.2 *Under the hypothesis of lemma 3.1 the coefficients* $\beta_{k,n}$ *in*

$$b_n = \sum_{k=0}^{n}\omega_k\beta_{k,n}T_k$$

are

$$\beta_{k,n} = \frac{\pi}{c_n}\left((n+2)(F_{n+1}a_{n+1})(z_0) - (k+1)(F_ka_{n+1})(z_0) + (S_ka_{n+1})(z_0)\right), \tag{3.4}$$

where

$$c_n = \pi(n+2)(F_{n+1}a_{n+1})(z_0). \tag{3.5}$$

By considering the family of linear operators $I_{k,n}$, $k = 0,1,\cdots,n$, $n \in \mathbb{N}$, defined on $\mathcal{P}^+$ by

$$I_{k,n} := (n+2)F_{n+1} - (k+1)F_k + S_k$$

one finds the operational formula

$$\beta_{k,n} = \frac{(I_{k,n}a_{n+1})(z_0)}{(n+2)(F_{n+1}a_{n+1})(z_0)}, \quad k = 0,1,\cdots,n. \tag{3.6}$$

Let us note that if $B = \Theta(A)$, then

$$r_n(B) = 1 - \beta_{1,n} = \frac{1}{\pi(n+2)(F_{n+1}a_{n+1})(z_0)}.$$

Using the above results one can formulate the following

Theorem 3.3 *Let* $a = (a_n) \in \mathcal{P}^1$, $b = (b_n) = l(a) \in \mathcal{P}^+$ *and* $B = \Theta(A)$. *If*

$$m_{k,n} = -\frac{2}{c_n}(k+1)\alpha_{k+1,n+1}T_{k+1}(z_0), \qquad c_n = \pi(n+2)(F_{n+1}a_{n+1})(z_0)$$

then $B = (B_n)$ *is a* **summability method of Fejér operators** $F = (F_n)$, *more precisely*

$$B_n = \sum_{k=0}^{n} m_{k,n}F_k.$$

Moreover, for all $x \in I$ *and* $f \in C(I)$

$$|f(x) - (B_nf)(x)| \le 4\omega\left(f; \frac{|x|}{c_n} + \sqrt{\frac{1-x^2}{c_n}}\right), \quad n \in \mathbb{N}.$$

4. In this section we will consider the case $a = \varphi = (\varphi_n)$, with φ_n being as in (1.3) and $z_0 = z_0(n+1) = \cos\frac{2\pi}{n+2}$. At first we observe in our case

$$c_n = \pi(n+2)\left(F_{n+1}\varphi_{n+1}\right)(z_0) = \pi(n+2)\sum_{k=0}^{n+1}\omega_k\left(1-\frac{k}{n+2}\right)^2\cos\frac{2k\pi}{n+2}$$

that is

$$\frac{1}{c_n} := r_n(B) = \sin^2 \frac{\pi}{n+2}. \tag{4.1}$$

If we select in (3.3) $\alpha_{k+1,n+1} = 1 - \frac{k+1}{n+2}$, $z_0 = \cos\frac{2\pi}{n+2}$ or in (3.6) $a_{n+1} = \varphi_{n+1}$, one finds the following

Lemma 4.1 *If $b = (b_n) = l(\varphi)$, $\varphi = (\varphi_n)$, then*

$$b_n(x) = \sum_{k=0}^{n} \omega_k \beta_{k,n} T_k(x)$$

with

$$\beta_{k,n} = \frac{n-k+2}{n+2}\cos^2\frac{k\pi}{n+2} + \frac{\cos\frac{\pi}{n+2}}{(n+2)\sin\frac{\pi}{n+2}}\cos\frac{k\pi}{n+2}\sin\frac{k\pi}{n+2}. \tag{4.2}$$

Moreover

$$b_n(x) = \kappa_n \frac{(1 - x\cos\frac{2\pi}{n+2})(1 - T_{n+2}(x))}{(1-x)(x - \cos\frac{2\pi}{n+2})^2} \tag{4.3}$$

where

$$\kappa_n = \frac{1}{\pi(n+2)}\sin^2\frac{\pi}{n+2}. \tag{4.4}$$

Further, let

$$B = (B_n) = \Theta(F),$$

where $F = (F_n)$ is the sequence of Fejér operators.
If $f \in X$ and $\beta_{k,n}$, b_n are as in (4.2) - (4.4), then

$$B_n f = \sum_{k=0}^{n} \omega_k \beta_{k,n} \langle f, T_k\rangle T_k = f \star b_n = b_n \star f \tag{4.5}$$

and also, if $\tilde{m}_{k,n} = 2\pi\kappa_n(k+1)(k-n-1)\cos\frac{2(k+1)\pi}{n+2}$ then

$$B_n f = \sum_{k=0}^{n} \tilde{m}_{k,n} F_k f. \tag{4.6}$$

We note that the coefficients $\tilde{m}_{k,n}$ satisfy $\tilde{m}_{k,n} = \tilde{m}_{n-k,n}$, $k = 0, 1, \cdots, n$.

In order to obtain a **discrete form** of the operators $B = (B_n)$ defined by (4.5) let us observe that the translation of b_n from (4.3) is

$$(\tau_x b_n)(y) = \kappa_n \frac{v_n(x;y)(1 - T_{n+2}(x)T_{n+2}(y)) - w_n(x;y)(1-x^2)(1-y^2)U_{n+1}(x)U_{n+1}(y)}{(x-y)^2\left((x - y\cos\frac{2\pi}{n+2})^2 - (1-y^2)\sin^2\frac{2\pi}{n+2}\right)^2}$$

where

$$v_n(x;y) = (1-xy)(\tau_x p)(y) + (1-x^2)(1-y^2)\cos\frac{2\pi}{n+2}\left[(x-y)^2 - (2xy - 1 - \cos\frac{2\pi}{n+2})^2\right] \tag{4.7}$$

$$w_n(x;y) = (x-y)^2 - (2xy - 1 - \cos\frac{2\pi}{n+2})^2 + \cos^2\frac{2\pi}{n+2}(\tau_x p)(y)$$

with $U_{n+1}(x) = \frac{\sin(n+2)\arccos x}{\sqrt{1-x^2}}$ and $p(x) = (1-x)(x - \cos\frac{2\pi}{n+2})^2$.

If in quadrature formula (2.3) we choose the knots $z_k = z_{k,n}$ such that $U_{n+1}(z_k) = 0$, then the polynomials $(\tau_x b_n)(z_{k,n})$ have a simplier form. Therefore, we will consider the Bouzitat formula of the second kind

$$\int_{-1}^{1} g(t)\omega(t)dt = \sum_{k=0}^{n+2} c_k(n) g(z_{k,n}) - \frac{\pi}{2^{2n+3}(2n+4)!} g^{(2n+4)}(\xi_n), \quad g \in C^{(2n+4)}(I), \quad \xi_n \in I,$$

with $c_0(n) = c_{n+2}(n) = \frac{\pi}{2(n+2)}$, $c_1(n) = \cdots = c_{n+1}(n) = \frac{\pi}{n+2}$, $z_{k,n} = \cos\frac{k\pi}{n+2}$, $k \in \mathbb{Z}$.

In conclusion let $B_n^\star : X \to \Pi_n$, $n \in \mathbb{N}_0$, be the linear positive operators with the images

$$(B_n^\star f)(x) = \frac{\pi}{n+2}\left(\frac{f(-1)b_n(-x) + f(1)b_n(x)}{2} + \right.$$
$$\left. + \kappa_n \sum_{k=1}^{n+1} v_n(x; z_{k,n}) \frac{1-(-1)^k T_{n+2}(x)}{(x-z_{k,n})^2(x-z_{k-2,n})^2(x-z_{k+2,n})^2} f(z_{k,n})\right); \qquad (4.8)$$

the polynomials v_n being explained in (4.7).

Another representation of the operator $B_n^\star$ may be obtained in the following way. Let us consider the bilinear form for $f, g : I \to \mathbb{R}$

$$(f,g)_n = \frac{\pi}{n+2}\left(\frac{f(-1)g(-1) + f(1)g(1)}{2} + \sum_{k=1}^{n+1} f(\cos\frac{k\pi}{n+2}) g(\cos\frac{k\pi}{n+2})\right).$$

It is easy to see that $\langle f,g\rangle = (f,g)_n$ for $fg \in \Pi_{2n+3}$.

Now

$$(B_n^\star f)(x) = \sum_{k=0}^{n+2} c_k(n) f(z_{k,n}) (\tau_x b_n)(z_{k,n}) = \sum_{k=0}^{n+2} c_k(n) f(z_{k,n}) \sum_{j=0}^{n} \omega_j \beta_{j,n} T_j(x) T_j(z_{k,n})$$

implies

$$(B_n^\star f)(x) = \sum_{j=0}^{n} \omega_j \beta_{j,n} (f, T_j)_n T_j(x),$$

which is the discrete version of (4.5). Similar discrete approximation operators were studied by A.K. Varma and T.M. Mills [11]. They obtained such operators as a summability method of Lagrange interpolation.

By using (4.1) in (2.12) - (2.13) we obtain

Theorem 4.2 *Suppose that $B = (B_n)$ is the Θ - transformation of the Fejér operators $F = (F_n)$. Let $B^\star = (B_n^\star)$ be defined as in (4.8). If $f \in C(I)$, $x \in I$, and*

$$\epsilon_n(x) = \sqrt{1-x^2}\sin\frac{\pi}{n+2} + |x|\sin^2\frac{\pi}{n+2}$$

then for $n \in \mathbb{N}$

$$|f(x) - (B_n f)(x)| \le 4\omega(f; \epsilon_n(x))$$

$$|f(x) - (B_n^\star f)(x)| \le 4\omega(f; \epsilon_n(x)).$$

Remarks:

- If $B = (B_n)$ is the Θ - transformation of $F = (F_n)$, then

$$\frac{r_n(B)}{r_n(F)} = (n+1)\sin^2\frac{\pi}{n+2},$$

which means that the linear combination of Fejér operators (4.6) approximates the functions from $C(I)$ better than $F_n f$.

- Note that the inequaliy

$$\epsilon_n(x) < \pi^2\left(\frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right), \quad x \in I,$$

furnishes an estimation of Timan's type

$$|f(x) - (B_n^\star f)(x)| \le \tilde{c}_0\,\omega(f; \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2})$$

$$x \in I,\ f \in C(I),\ n \in \mathbb{N},\ \tilde{c}_0 \in (0, 40].$$

By means of the second order modulus of smoothness

$$\omega_2(f,h) := \sup\{|f(x-\delta) - 2f(x) + f(x+\delta)|;\ x, x\pm\delta \in I,\ 0 \le \delta \le h\}, \quad f \in C(I),$$

one finds

Theorem 4.3 *Let $B = (B_n) = \Theta(F)$ and $B^\star = (B_n^\star)$ as in (4.8). For $f \in C(I)$, $x \in I$, we have*

$$|f(x) - (B_n f)(x)| \le c_0\left(\omega_2(f; \frac{1}{n}) + \frac{|x|}{n}\omega(f; \frac{1}{n})\right),$$

$$|f(x) - (B_n^\star f)(x)| \le c_0\left(\omega_2(f; \frac{1}{n}) + \frac{|x|}{n}\omega(f; \frac{1}{n})\right),$$

where $c_0 = 3 + 2\pi^2$ and $n \in \mathbb{N}$.

Proof: Let $\Omega_{2,x}(t) = (t-x)b^2$ then we get with (4.1)

$$\begin{aligned}(B_n\Omega_{2,x})(x) &= (B_n^\star\Omega_{2,x})(x)\\ &= r_n(B)\left(1 + \frac{n+1}{n+2}(1-2x^2)\cos\frac{2\pi}{n+2}\right)\\ &< 2r_n(B) < \frac{2\pi^2}{n^2}.\end{aligned}$$

If L is a linear positive operator which preserves the constant functions, there is – according to H.H.Gonska ([7] theorem 2.4) – for $h \in (0,2]$ and $x \in I$,

$$|f(x) - (Lf)(x)| \le \left(3 + \frac{1}{h^2}(L\Omega_{2,x})(x)\right)\omega_2(f;h) + \frac{2}{h}|e_1(x) - (Le_1)(x)|\omega(f;h).$$

Therefore, with $h = \frac{1}{n}$ in our case we find the desired inequalities. □

Finally let us suppose that $\delta \in (0,1]$ and $f \in Lip_2(\alpha, C)$, $0 < \alpha \le 2$. Then $\omega_2(f;\delta) \le C\delta^\alpha$, $C := const.$ and

$$\omega(f;\delta) \le \begin{cases} 2\,\|f\|, & \alpha \in (0,1] \\ \delta\,\|f'\|, & \alpha \in (1,2] \end{cases}$$

We get

$$\delta\omega(f;\delta) \le \begin{cases} 2\,\|f\|\delta^{\alpha} & , \quad \alpha \in (0,1] \\ \|f'\|\delta^{\alpha} & , \quad \alpha \in (1,2] \end{cases} ,$$

and so one finds a positive constant $M = M(f)$ such that

$$\omega_2(f;\delta) + |x|\delta\omega(f;\delta) \le M\delta^{\alpha}, \quad \alpha \in (0,2],\ \delta \in (0,1],\ x \in I .$$

If we choose $\delta = \frac{1}{n}$, from Theorem 4.3 we get

$$|\,f(x) - (B_n^{\star}f)(x)\,| \le \frac{M}{n^{\alpha}}, \quad x \in I .$$

In conclusion the linear summator operators $(B_n^{\star})$ have the co-domain in Π_n and satisfy

$$\|\,f - B_n^{\star}f\,\|_{C(I)} = \mathcal{O}(n^{-\alpha})$$

provided $f \in Lip_2(\alpha, C)$, $0 < \alpha \le 2$, i.e. a an answer to a problem proposed by P.L.Butzer [2]. Other solutions for Butzer's problem are presented in [5].
However, some summability methods for Lagrange interpolation (see [11], [10]) furnish us also an affirmative answer to the question raised in [2]. .

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