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EQUIVARIANT EMBEDDINGS AND COMPACTIFICATIONS OF FREE G-SPACES

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For a compact Lie group G, we characterize free G-spaces that admit free G-compactifications. For such G-spaces, a universal compact free G-space of given weight and given dimension is constructed. It is shown that if G is finite, the n-dimensional Menger free G-compactum μ^n is universal for all separable, metrizable free G-spaces of dimension less than or equal to n. Some of these results are extended to the case of G-spaces with a single orbit type.

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1. Introduction. By a G-space, we mean a triple (G,X,α), where G is a topological group, X is a topological space, and $\alpha : G \times X \to X$ is a continuous action.

In 1960, Palais proved that every Tychonoff G-space can equivariantly be embedded into a compact Hausdorff G-space provided G is a compact Lie group (see [17, Section 1.5]). This result was extended by de Vries [5] to the case of arbitrary locally compact Hausdorff groups. The local compactness is essential here; it was Megrelishvili who constructed in [14] a continuous action α of a separable, complete metrizable group G on a separable, metrizable space X such that (G,X,α) does not admit an equivariant embedding into a compact G-space. The reader can find other examples of this type in [15].

In this paper, we are mostly interested in *free G-spaces*. Recall that a G-space X is free if, for every $x \in X$, the equality $gx = x$ implies $g = e$, the unity of G. In [2], it is proved that if G is a compact Lie group, then any Tychonoff free G-space can equivariantly be embedded in a locally compact free G-space. In this connection, it is natural to ask the following question.

Question 1.1. Does every free G-space have a G-embedding in a free compact G-space?

One of the purposes of the present paper is to answer this question for G a compact Lie group. Namely, we prove that each finitistic free G-space X has a free G-compactification (Theorem 3.4). In the realm of G-spaces that admit a free G-compactification, we construct a universal, compact, free G-space of given weight and given dimension (Theorem 4.1). This result is extended to the case of the G-spaces with a single orbit type (Theorem 5.2).

2. Preliminaries. Throughout the paper, all topological spaces are assumed to be Tychonoff (i.e., completely regular and Hausdorff). All equivariant or G-maps are assumed to be continuous.

The letter "G" will always denote a compact Lie group.

The basic ideas and facts of the theory of G-spaces or topological transformation groups can be found in Bredon [4] and Palais [17].

For the convenience of the reader, however, we recall some more special definitions and facts below.

By e, we will always denote the unity of the group G.

If X is a G-space, for any $x \in X$, we denote the stabilizer (or stationary subgroup) of x by $G_x = \{g \in G \mid gx = x\}$.

If, for all $x \in X$, $G_x = \{e\}$, then we say that the action of G is *free* and X is a *free G-space*.

For a subset $S \subset X$ and a subgroup $H \subset G$, $H(S)$ denotes the H-saturation of S, that is, $H(S) = \{hs \mid h \in H,\ s \in S\}$. In particular, $G(x)$ denotes the G-orbit $\{gx \in X \mid g \in G\}$ of x. If $H(S) = S$, then S is said to be an H-invariant set. The G-orbit space is denoted by X/G.

By G/H, we will denote the G-space of cosets $\{gH \mid g \in G\}$ under the action induced by left translations.

For each subgroup $H \subseteq G$, the H-fixed point set X^H is defined to be the set $\{x \in X \mid H \subseteq G_x\}$.

The family of all subgroups of G which are conjugate to H is denoted by (H), that is, $(H) = \{gHg^{-1} \mid g \in G\}$. The set (H) is called a G-orbit type (or simply an orbit type). For two orbit types (H_1) and (H_2), we say that $(H_1) \preceq (H_2)$ if and only if $H_1 \subseteq gH_2g^{-1}$ for some $g \in G$. If $(H_1) \preceq (H_2)$ and $(H_1) \neq (H_2)$, then we write $(H_1) \prec (H_2)$. The relation $\preceq$ is a partial ordering on the set of all G-orbit types. Since $G_{gx} = gG_xg^{-1}$, for any $x \in X$, $g \in G$, we have $(G_x) = \{G_{gx} \mid g \in G\}$.

We say that a G-space X is of the orbit type (H), or simply of type (H), if (G_x)=(H) for every $x \in X$.

In this paper, we will consider only G-spaces that have a single orbit type (H).

An equivariant map $f : X \to Y$ of G-spaces is said to be *isovariant* or (G-*isovariant*) if $G_x = G_{f(x)}$ for all $x \in X$.

If X and Y are G-spaces, then $X \times Y$ will always be regarded as a G-space equipped by the diagonal action of G.

A G-compactification of a G-space X is a pair (b_G, b_GX), where $b_G : X \to b_GX$ is a G-homeomorphic embedding into a compact G-space b_GX such that the image $b_G(X)$ is dense in b_GX. Usually, b_GX alone is a sufficient denotation. By β_GX, we will denote the maximal G-compactification of X.

In the sequel, we will need the following lemma.

Lemma 2.1 (see [1]). *Let $f : X \to S$ be an isovariant map of G-spaces. Then, the map $h : X \to S \times (X/G)$, defined by $h(x) = (f(x), p(x))$ where $p : X \to X/G$ is the orbit map, is a G-homeomorphic embedding.*

We also recall the well-known and important definition of a slice [17, page 27].

Definition 2.2. A subset S of a G-space X is called an H-slice in X if
(1) S is H-invariant, that is, $H(S) = S$,
(2) the saturation $G(S)$ is open in X,
(3) if $g \in G \setminus H$, then $gS \cap S = \varnothing$,
(4) S is closed in $G(S)$.
The saturation $G(S)$ will be said to be an H-tube. If, in addition, $G(S) = X$, then we say that S is a global H-slice in X.

If S is a global H-slice in X, then X is G-homeomorphic to the so-called *twisted product* $G \times_H S$. Recall that $G \times_H S$ is just the H-orbit space of the product $G \times S$ on which H acts by the rule $h(g,s) = (gh^{-1}, hs)$, where $h \in H$ and $(g,s) \in G \times S$. In turn, G acts on $G \times_H S$ by the formula $g'[g,s] = [g'g,s]$, where $g' \in G$, $[g,s] \in G \times_H S$ (see [4, Section 4]).

One of the basic results of the theory of topological transformation groups is the Slice theorem, which asserts the following: if X is a G-space and $x \in X$, then there exists a G_x-slice $S \subset X$ containing the point x (see, e.g., [17, Theorem 1.7.18] or [4, Chapter II, Theorem 5.4]).

An important consequence of the Slice theorem is that if X is a G-space with the orbits all of the same type, then the orbit map $X \to X/G$ is a locally trivial fibration [4, Chapter II, Theorem 5.8].

In what follows, $\cong_G$ will mean "*is G-homeomorphic*."

We write $\widetilde{X} = X/G$ for the orbit space of X.

The following definition is due to Jaworowski [12] even for G-spaces of finitely many orbit types.

Definition 2.3. We say that a G-space X with a single orbit type (H) is of *finite structure* if the orbit map $p : X \to \widetilde{X}$ has a finite trivializing cover, that is to say, there exists a finite open cover $\{U_1, \dots, U_n\}$ of $\widetilde{X}$ such that each $p^{-1}(U_i)$ is G-equivalent to $(G/H) \times U_i$, that is, there exists a G-homeomorphism $f_i : p^{-1}(U_i) \to (G/H) \times U_i$ such that $\pi(f_i(x)) = p(x)$ for every $x \in p^{-1}(U_i)$.

Here, we remark that the claim "$p : X \to \widetilde{X}$ has a finite trivializing cover" is equivalent to "X can be covered by finitely many H-tubes." Namely, in this form, we will use the definition in what follows.

It is evident from Definition 2.3 that any invariant subspace of a G-space of finite structure is again a G-space of finite structure.

3. G-compactifications of a single orbit type.

Recall that the cone $\operatorname{con}(X)$ over a compact metric space X is the quotient set $[0,1] \times X / \{0\} \times X$ equipped with the quotient topology. This topology is metrizable too (see [10, Chapter VI, Lemma 1.1]). The image of the point $(t,x) \in [0,1] \times X$ under the canonical projection $p : [0,1] \times X \to \operatorname{con}(X)$ will be denoted by tx, and we will simply write θ (think of zero) instead of $0x$; this is the vertex of the cone. It is convenient to call the number t in tx the norm of tx and denote it by $\|tx\|$.

If $X_1,\dots,X_k$ are compact metric spaces, the join $X_1 * \cdots * X_k$ is defined to be the subset of the product $\mathrm{con}(X_1)\times\cdots\times\mathrm{con}(X_k)$ consisting of all those points $(t_1x_1,\dots,t_kx_k)$ for which $\sum_{i=1}^{n} t_i = 1$. Below, we will consider the case when $X_1 = \cdots = X_k = G/H$, where H is a closed subgroup of G. In this case, G acts coordinatewise on the k-fold join $G/H * \cdots * G/H$ by left translations; so, $G/H * \cdots * G/H$ is a G-space, which we will denote shortly by $(G/H)^{*k}$.

In what follows, by a Euclidean G-space, we mean a real Euclidean space E on which G acts by means of orthogonal transformations.

It is convenient to introduce the following notion that is closely related to the notion of the finite structure introduced by Jaworowski (see Section 2).

Definition 3.1. We say that a G-space X is of Euclidean type if there exists an isovariant map $f : X \to E$ into a Euclidean G-space E.

In [12], Jaworowski proved that each normal G-space of finite structure is of Euclidean type. Here, we need the following more precise version of Jaworowski's result.

Lemma 3.2. *Any normal G-space X of a single orbit type (H) and of finite structure admits an isovariant map into a finite-dimensional, compact, metrizable G-space D of type (H).*

Proof. It is known that, under the conditions of the lemma, the orbit map $p : X \to \widetilde{X}$ is a locally trivial fibration (see [4, Chapter II, Theorem 5.8]).

Let $\{U_1,U_2,\dots,U_k\}$ be a finite open cover of the orbit space $\widetilde{X}$ such that, for every $1 \le n \le k$, $p^{-1}(U_n)$ is equivariantly homeomorphic to the product $G/H \times U_n$, where the group G acts on the left on G/H and acts trivially on U_n.

Further, for each $n \ge 1$, the first projection of the product $p^{-1}(U_n) = G/H \times U_n$ gives an isovariant map $\varphi_n : p^{-1}(U_n) \to G/H$.

Since the orbit space $\widetilde{X}$ is normal, there exists a closed shrinking $\{F_1,\dots,F_k\}$ for $\{U_1,U_2,\dots,U_k\}$ in $\widetilde{X}$, that is, $F_n \subset U_n$ for all $1 \le n \le k$ and $\bigcup_{n=1}^{k} F = X$ [9, Theorem 1.7.8].

Let $\psi_n : \widetilde{X} \to [0,1]$ be a continuous function such that $F_n \subset \psi_n^{-1}(1)$ and $\widetilde{X} \setminus U_n \subset \psi_n^{-1}(0)$. Now, we define a map $f_n : X \to \mathrm{con}(G/H)$ by the formula

$$f_n(x) = \begin{cases} \theta & \text{if } x \notin p^{-1}(U_n), \\ \psi_n(p(x))\varphi_n(x) & \text{if } x \in p^{-1}(U_n). \end{cases} \tag{3.1}$$

It is clear that f_n is an equivariant map, and that its restriction to $p^{-1}(F_n)$ coincides with φ_n and is, therefore, isovariant. We consider the diagonal product

$$f = \triangle_{n=1}^{k} f_n : X \longrightarrow \left(\mathrm{con}(G/H)\right)^k. \tag{3.2}$$

Then, f is an equivariant map. Since $\sum_{n=1}^{k}\|f_n(x)\| = 1$, for all $x \in X$, we conclude that $f(x)$ belongs, in fact, to the k-fold join $(G/H)^{*k}$.

If $x \in X$ and $x \in p^{-1}(F_n)$, then

$$G_{f(x)} = \bigcap_{k=1}^{\infty} G_{f_k(x)} \subset G_{f_n(x)} = G_x. \tag{3.3}$$

On the other hand, $G_x \subset G_{f(x)}$ since f is equivariant. Consequently, $G_x = G_{f(x)}$, that is, f is an isovariant map.

Now, we define D to be the closure of $f(X)$ in $(G/H)^{*k}$. Then, D is finite-dimensional, compact and metrizable. It remains to see that D has the orbit type (H). Let $d \in D$ be an arbitrary point. Since each orbit type in $(G/H)^{*k}$ is $\preceq (H)$, we see that $(G_d) \leq (H)$. On the other hand, since $f(X)$ is dense in D and $f(X)$ is of type (H), it follows from the Slice theorem [4, Chapter II, Corollary 5.5] that $(H) \preceq (G_d)$. Thus, $(G_d) = (H)$, and hence $f : X \to D$ is the desired map. □

Theorem 3.3. *For a G-space X of a single orbit type (H), the following are equivalent:*

(1) *X has a G-compactification of type (H),*
(2) *X has an isovariant map in a finite-dimensional, compact metrizable G-space D of type (H),*
(3) *X is of Euclidean type,*
(4) *X has an isovariant map in a compact G-space of type (H),*
(5) *X has a G-compactification of type (H) and of the same weight wX.*

Proof. (1)⇒(2). Let b_GX be a G-compactification of X of type (H). By Lemma 3.2, there is an isovariant map $\varphi : b_GX \to D$ in some finite-dimensional, compact metrizable G-space D of type (H). The restriction $\varphi|_X$ is the desired map.

(2)⇒(3). Let $\varphi : X \to D$ be an isovariant map in a finite-dimensional, compact metrizable G-space D of type (H). Since there exists an equivariant embedding $i : D \to E$ in a Euclidean G-space E (see [4, Chapter II, Theorem 10.1]), the composition $f = \varphi i$ maps X isovariantly into E.

(3)⇒(4). Let $\psi : X \to E$ be an isovariant map in a Euclidean G-space E. Then, by Lemmas 3.2 and 3.6, there exists an isovariant map $j : \psi(X) \to D$ in a finite-dimensional, compact metrizable G-space D of type (H). Then, the composition $j\psi : X \to D$ is the required map.

(4)⇒(1). Let $f : X \to Y$ be an isovariant map in a compact G-space Y of type (H).

Let $p : X \to X/G$ be the orbit map. By Lemma 2.1, the diagonal product $i = \varphi\Delta p : X \to Y \times (X/G)$ is an equivariant embedding.

Let B be any compactification of the orbit space X/G. Then, X can be regarded as an invariant subset of the compact G-space $Y \times B$, where G acts on B trivially. Now, the closure $\overline{X}$ of X in $Y \times B$ is a G-compactification of X. Since

Y is of type (H), we see that $Y \times B$ is also of type (H). Hence, $b_G X = \overline{X}$ is a G-compactification of X of type (H).

(2)⇒(5) can be proved like the implication (4)⇒(1) using D instead of Y. In that case, if we choose the compactification B of X/G such that $w(B) = w(X/G)$ [8, Theorem 3.5.2], then the G-compactification $b_G X$ will have the weight $w(b_G X) = w(X/G)$ because $w(D) = \aleph_0$. It remains only to observe that $w(X) = w(X/G)$.

(5)⇒(1) is evident. □

Recall that a paracompact space X is said to be *finitistic* if every open cover of X has a refinement ω of a finite order, that is, there is a natural number n such that any point $x \in X$ can belong at most to n elements of ω (see [19]).

Evidently, each compact space, as well as each paracompact finite-dimensional space, is finitistic.

A wide class of G-spaces that admit G-compactifications of a single orbit type is provided by the following theorem.

Theorem 3.4. *Every finitistic G-space X of type (H) has a G-compactification $b_G X$ of the same type (H) and of the same weight wX.*

For the proof, we need the following result, which was established first by Milnor for finite-dimensional spaces (cited in [17, Theorem 1.8.2]).

Lemma 3.5. *Let X be a finitistic space and let $\{U_\alpha\}$ be an open covering of X. Then, there exist a natural number n and an open covering $\{V_{i\beta}\}_{\beta \in B_i}$, $i = 0,\dots,n$, of X refining $\{U_\alpha\}$ such that $V_{i\beta} \cap V_{i\beta'} = \varnothing$ whenever $\beta \neq \beta'$ and $0 \le i \le n$.*

Proof. As X is finitistic, there are a natural number n and a refinement $\{W_\mu\}$ of $\{U_\alpha\}$ such that the order of the cover $\{W_\mu\}$ is at most n. Let $\{\varphi_\mu\}$ be a locally finite partition of unity with $\varphi_\mu^{-1}((0,1]) \subset W_\mu$. For every $0 \le i \le n$, let B_i be the set of all subsets β of the indexing set of the cover $\{W_\mu\}$ with cardinality $|\beta| = i+1$. Given $\beta = (\mu_0,\dots,\mu_i) \in B_i$, we set

$$V_{i\beta} = \{x \in X \mid \varphi_{\mu_j}(x) > 0 \text{ and } \varphi_\mu(x) < \varphi_{\mu_j}(x)\ \forall 0 \le j \le i,\ \mu \notin \beta\}. \tag{3.4}$$

As in a neighborhood of any point x, only a finite number of φ_μ is not identically zero, and it follows that each $V_{i\beta}$ is open.

Let us check that $V_{i\beta} \cap V_{i\beta'} = \varnothing$ if $\beta \neq \beta'$. Indeed, since $|\beta| = i+1 = |\beta'|$ and $\beta \neq \beta'$, we infer that there are $\mu \in \beta \setminus \beta'$ and $\mu' \in \beta' \setminus \beta$. Now, if $x \in V_{i\beta} \cap V_{i\beta'}$, it then follows that $\varphi_{\mu(x)} < \varphi_{\mu'}(x) < \varphi_\mu(x)$, a contradiction.

Check that $\{V_{i\beta}\}$ is a covering for X. If $x \in X$ and $\mu_0,\dots,\mu_m$ are all the indices with $\varphi_{\mu_k}(x) > 0$ so arranged that

$$\varphi_{\mu_0}(x) = \varphi_{\mu_1}(x) = \cdots = \varphi_{\mu_i}(x) > \varphi_{\mu_{i+1}}(x) \ge \cdots \ge \varphi_{\mu_m}(x), \tag{3.5}$$

then, evidently, $x \in V_{m\beta}$, where $\beta = \{\mu_0, \dots, \mu_m\}$. Since

$$x \in \bigcap_{j=0}^{m} \operatorname{supp} \varphi_{\mu_j} \subset \bigcap_{j=0}^{m} W_{\mu_j} \tag{3.6}$$

and $\{W_\mu\}$ has order $\le n$, it follows that $m \le n$. Consequently, $i \le n$, and, clearly, $x \in V_{i\{\mu_0,\dots,\mu_i\}}$. Thus, $\{V_{i\beta}\}_{\beta\in B_i}$, $i = 0, \dots, n$, is an open cover of X, and since $V_{i\beta} \subset W_\mu$ for every $\mu \in i$, we see that $\{V_{i\beta}\}$ refines the cover $\{W_\mu\}$, and hence, the original cover $\{U_\alpha\}$. Thus, $\{V_{i\beta}\}$ is the desired cover. □

Lemma 3.6. *Every finitistic G-space X having only one orbit type is of finite structure.*

Proof. Let (H) be the only orbit type of X. Let $\{S_\alpha\}$ be a family of H-slices in X such that $X = \bigcup G(S_\alpha)$. Then, $G(S_\alpha) \cong_G (G/H) \times p(S_\alpha)$ and the sets $p(G(S_\alpha)) = p(S_\alpha)$ constitute an open cover of the orbit space X/G. Now, by [6], X/G is also finitistic, so, by the preceding lemma, we can find a natural number n and an open cover $\{\widetilde{U}_{i\beta}\}_{\beta\in B_i}$, $i = 0, \dots, n$ of X/G which refines $\{p(S_\alpha)\}$ and is such that $\widetilde{U}_{i\beta} \cap \widetilde{U}_{i\beta'} = \varnothing$ if $\beta \ne \beta'$. Then, the set $U_{i\beta} = p^{-1}(\widetilde{U}_{i\beta})$ is an H-slice in $\widetilde{U}_{i\beta}$, that is, $G(U_{i\beta}) \cong_G (G/H) \times \widetilde{U}_{i\beta}$ [17, Proposition 1.7.2], and $\widetilde{U}_{i\beta} = p(U_{i\beta})$. It then follows that the union $U_i = \bigcup_{\beta\in B_i} U_{i\beta}$ is an H-slice over $\widetilde{U}_i = \bigcup_{\beta\in B_i} \widetilde{U}_{i\beta}$ (see [17, Proposition 1.7.3]). Thus, $G(U_i) \cong_G (G/H) \times \widetilde{U}_i$, and hence $\{\widetilde{U}_i\}_{i=1}^{n}$ is a finite trivializing cover for X/G. □

Proof of Theorem 3.4. It follows from Lemmas 3.6 and 3.2 that X is of Euclidean type. Now, the claim follows from Theorem 3.3. □

Proposition 3.7. *If a G-space X of type (H) admits a G-compactification $b_G X$ of the same type (H), then its maximal G-compactification $\beta_G X$ is also of the same type (H).*

Proof. Indeed, there exists a G-map $f : \beta_G X \to b_G X$. Hence, $(G_t) \preceq (G_{f(t)}) = (H)$ for every $t \in \beta_G X$. On the other hand, since X is dense in $\beta_G X$ and X is of type (H), it follows from the Slice theorem that $(H) \preceq (G_t)$ for every $t \in \beta_G X$ (see [4, Chapter II, Corollary 5.5]). Thus, $(G_t) = (H)$ for every $t \in \beta_G X$. □

The following is an example of a free $\mathbb{Z}_2$-action on the Hilbert cube with a removed point, which does not have a free $\mathbb{Z}_2$-compactification.

Example 3.8 (see [12]). Let $X = [-1,1]^\infty \setminus \{0\}$, where $0 = (0,0,\dots) \in [-1,1]^\infty$, and $G = \mathbb{Z}_2$, the cyclic group of order two. So, X is the Hilbert cube with a removed point. Consider the free action of $\mathbb{Z}_2$ on X defined by the standard involution $\{x_i\} \to \{-x_i\}$. It turns out that the free $\mathbb{Z}_2$-space X does not have a free $\mathbb{Z}_2$-compactification.

Assume the contrary. Then, by Theorem 3.3, there exists an isovariant map $f : X \to E$ in a Euclidean $\mathbb{Z}_2$-space. Since X is a free $\mathbb{Z}_2$-space, $f^{-1}(0) = \varnothing$, where 0 denotes the origin of E. Clearly, the radial retraction $r : E \setminus \{0\} \to S$ onto the unit sphere of E is an isovariant map. Hence, the composition $\varphi = rf : X \to S$ is isovariant too.

Let S^k be a sphere of arbitrary dimension $k > 0$, considered as a G-space with the antipodal action of $\mathbb{Z}_2$.

Claim 1. Each sphere S^k can $\mathbb{Z}_2$-equivariantly be embedded into the $\mathbb{Z}_2$-space X.

Indeed, it suffices to show that the $\mathbb{Z}_2$-maps from S^k to $[-1,1]$ separate points of S^k. Let $a, b \in S^k$, $a \neq b$. If $b = -a$, then we first choose a continuous map $f : S^k \to [-1,1]$ with $f(a) = 1$ and $f(b) = -1$ and then define $f'(x) = (f(x) - f(-x))/2$, $x \in S^k$. Clearly, f' is a $\mathbb{Z}_2$-map with $f'(a) = 1$ and $f'(b) = -1$. If $b \neq -a$, then we first choose a continuous map $f : S^k \to [-1,1]$ with $f(a) = f(-b) = 1$ and $f(b) = f(-a) = -1$ and then define $f'(x) = (f(x) - f(-x))/2$, $x \in S^k$. Clearly, f' is a $\mathbb{Z}_2$-map with $f'(a) = 1$ and $f'(b) = -1$.

Now, by Claim 1 there exists a G-embedding $i : S^k \hookrightarrow X$. The composition $q = \varphi i : S^k \to S$ is then an equivariant (i.e., an antipodal) map. But, according to the classical Borsuk-Ulam theorem (see, e.g., [18, Chapter 5, Section 8, Corollary 8]), there is no such a map for $k > \dim S$.

This example also has the following interesting property in spirit of Douwen's paper [20].

Corollary 3.9. *Let $f : X \to X$ be the standard involution on the Hilbert cube with a removed point (Example 3.8). Then, the Stone-Čech compactification $\beta f : \beta X \to \beta X$ has a fixed point.*

Proof. Indeed, otherwise βX is a free $\mathbb{Z}_2$-compactification of X, which contradicts the claim of Example 3.8. □

4. Universal finite-dimensional compact free G-spaces. In this section, we prove the following theorem.

Theorem 4.1. *For every infinite cardinal number τ and for every nonnegative integer $n \geq \dim G$, there exists a compact free G-space $\mathcal{F}_\tau^n$ with $w(\mathcal{F}_\tau^n) = \tau$, $\dim(\mathcal{F}_\tau^n) = n$ which is universal in the following sense: $\mathcal{F}_\tau^n$ contains a G-homeomorphic copy of any free G-space X of Euclidean type with $wX \leq \tau$ and $\dim \beta_G X \leq n$. In particular, $\mathcal{F}_\tau^n$ contains a G-homeomorphic copy of each paracompact free G-space X with $wX \leq \tau$ and $\dim X \leq n$.*

We notice that a similar result for the nonfree case was established earlier in [13].

Before proceeding with the proof, we will establish the following lemma.

Lemma 4.2. *Let X be a paracompact free G-space. Then, the following two properties are fulfilled:*

(1) $\dim X = \dim(X/G) + \dim G$;
(2) $\dim \beta_G X = \dim X$.

Proof. (1) Let $p : X \to X/G$ be the orbit map. It is well known [4, Chapter II, Theorem 5.8] that p is a locally trivial fibration with fibers homeomorphic to G. Let $\{U_\alpha\}$ be an open trivializing cover of the orbit space X/G, that is, $p^{-1}(U_\alpha) \cong_G G \times U_\alpha$. By compactness of G, the map p is closed, and by a theorem of E. Michael [8, Theorem 5.1.13], the orbit space X/G is paracompact, too. Then, there exists a locally finite closed cover $\{F_\alpha\}$ of X/G such that $F_\alpha \subset U_\alpha$ for each index α. It follows that $p^{-1}(F_\alpha) \cong_G G \times F_\alpha$ and the family $\{p^{-1}(F_\alpha)\}$ constitute a locally finite closed cover of X. Then, according to the Sum theorem [9, Theorem 3.1.10], $\dim X = \max_\alpha\{\dim p^{-1}(F_\alpha)\}$. But $\dim p^{-1}(F_\alpha) = \dim(G \times F_\alpha)$. Being a closed subset of a paracompact space, F_α is itself paracompact. On the other hand, G is a polyhedron. Hence, Morita's theorem [16] is applicable here and, accordingly this logarithmic low holds true: $\dim(G \times F_\alpha) = \dim G + \dim F_\alpha$. Thus, we have $\dim X = \dim G + \max_\alpha\{\dim F_\alpha\}$. Applying once more the sum theorem, we get $\dim(X/G) = \max_\alpha\{\dim F_\alpha\}$. Consequently, $\dim X = \dim(X/G) + \dim G$.

(2) We will use the formula $\beta(X/G) = (\beta_G X)/G$ (see [3]). Consider two cases.

(a) Let $\dim X < \infty$. Then, X has finite structure (Lemma 3.6) and then $\beta_G X$ is a free G-space (Proposition 3.7). Applying twice the equality established in the previous step, we get

$$\begin{aligned}
\dim \beta_G X &= \dim(\beta_G X)/G + \dim G \\
&= \dim \beta(X/G) + \dim G \\
&= \dim(X/G) + \dim G \\
&= \dim X.
\end{aligned} \tag{4.1}$$

(b) Let $\dim X = \infty$. By Claim 1, we have $\dim X = \dim G + \dim(\beta_G X)/G$, which implies that $\dim(\beta_G X)/G = \infty$. But the orbit map does not rise dimension [6]; in particular,

$$\dim \beta_G X = \dim(\beta_G X)/G = \infty = \dim X. \tag{4.2}$$

□

The following lemma in the nonfree case was proved by Megrelishvili [13] even for noncompact acting groups.

Lemma 4.3. *Let $f : X \to Y$ be a G-map of a compact free G-space X into a compact G-space Y. Then, there exist a compact free G-space Z and G-maps $\varphi : X \to Z$, $\psi : Z \to Y$ such that $f = \psi\varphi$ and* $\dim Z \le \dim X$, $wZ \le wY$.

Proof. We will first prove the claim in case when Y is a free G-space, too. Consider the induced map $f' : X/G \to Y/G$. By Mardešić's factorization theorem [9, Theorem 3.3.2], there exist a compact space Z' and continuous maps $\varphi' : X/G \to Z'$, $\psi' : Z' \to Y/G$ such that $f' = \psi'\varphi'$ and $\dim Z' \le \dim(X/G)$, $wZ' \le w(Y/G)$.

Denote by p the orbit map $Y \to Y/G$. It is well known [11, Chapter IV, Proposition 4.1] that we have the following (pullback) commutative diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\psi} & Y \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ Z' & \xrightarrow{\psi'} & Y/G \end{array} \tag{4.3}$$

where Z is a compact G-space with $Z/G{=}Z'$, $\pi : Z \to Z'$—the orbit map and ψ—an equivariant map that induces the map ψ'. In fact, Z is the G-invariant subset of $Z' \times Y$ defined as follows: $Z = \{(z', y) \mid \psi'(z') = p(y)\}$, where G acts on $Z' \times Y$ by $g(z', y) = (gz', y)$ for $g \in G$ and $(z', y) \in Z' \times Y$. Thus, Z is a compact free G-space and $\psi : Z \to Y$ is the restriction of the second projection $Z' \times Y \to Y$.

Now, we define $\varphi : X \to Z$ by $\varphi(x) = (\varphi' q(x), f(x))$, where $q : X \to X/G$ is the orbit map. It is easy to check that $f = \psi\varphi$.

On the other hand, $wZ = wZ' \le w(Y/G) = wY$.

Let us check that $\dim Z \le \dim X$. As Z is a paracompact free G-space, we can apply Lemma 4.2, according to which $\dim Z = \dim Z' + \dim G \le \dim(X/G) + \dim G = \dim X$.

Now we pass to the general case. By Lemma 3.2, there is an isovariant map $h : X \to D$ to a compact free G-space D. Consider the product $T = h(X) \times Y$ and the map $r : X \to T$ defined by $r(x) = (h(x), f(x))$, $x \in X$. Since X is free and h is isovariant, we infer that T is a free G-space. It is clear that r is equivariant and $wT = wY$. Now, we apply the preceding case, according to which there exist a compact G-space Z and G-maps $\varphi : X \to Z$, $\psi_1 : Z \to T$ such that $\dim Z \le \dim X$, $wZ \le wT$ and $r = \psi_1\varphi$. Observe that $wT = wY$ because $wh(X) = \aleph_0$; so, $wZ \le wY$. Put $\psi = \pi_2\psi_1$, where $\pi_2 : T \to Y$ is the second projection. Then, $\psi : Z \to Y$ is a G-map such that $f = \psi\varphi$. It remains to observe that Z is a free G-space; this is immediate from the equivariance of ψ_1 and from the freeness of T. □

Proof of Theorem 4.1. Let B_τ be a universal Tychonoff G-cube of weight τ (see [3]), that is, B_τ is a G-space homeomorphic to the Tychonoff cube $[0,1]^\tau$ and contains a G-homeomorphic copy of every G-space of weight $\le \tau$. Let $\{Y_t\}_{t\in T}$ be the family of all invariant free subsets $Y_t \subset B_\tau$ of Euclidean type such that $\dim \beta_G Y_t \le n$. This family is nonempty because the group G belongs to it. For each $t \in T$, we denote by i_t the identical embedding of Y_t into B_τ. Consider the discrete sum $Y = \bigoplus_{t\in T} \beta_G Y_t$, which naturally becomes a G-space. By Proposition 3.7, each $\beta_G Y_t$ is a free G-space. Consequently, Y is a paracompact free G-space. As $\dim \beta_G Y_t \le n$ for all $t \in T$, then, by the Sum theorem [9, Theorem 3.1.10], we have $\dim Y \le n$. Consequently, by Lemma 4.2, $\dim \beta_G Y = \dim Y \le n$.

Next, each map $i_t : Y_t \to B_\tau$ can be extended to a G-map $i'_t : \beta_G Y_t \to B_\tau$ (see [17, Section 5]); so, a map $i : Y \to B_\tau$ arises defined by $i(y) = i'_t(y)$ for $y \in \beta_G Y_t$. Applying once more [17, Section 5], we extend i to a G-map $j : \beta_G Y \to B_\tau$. As Y has finite structure according to Proposition 3.7, $\beta_G Y$ is a compact free G-space. By virtue of Lemma 4.3, there exist a compact free G-space $\mathcal{F}^n_\tau$ and G-maps $\varphi : \beta_G Y \to \mathcal{F}^n_\tau$, $\psi : \mathcal{F}^n_\tau \to B_\tau$ such that $i = \psi\varphi$ and $\dim \mathcal{F}^n_\tau \le n$, $w\mathcal{F}^n_\tau \le wB_\tau = \tau$. We claim that $\mathcal{F}^n_\tau$ is the desired G-space.

Indeed, let X be an arbitrary free G-space such that $\dim X \le n$ and $wX \le \tau$. Since X is equivariantly embeddable in B_τ, there exists a $t \in T$ such that Y_t is G-homeomorphic to X. As the restriction of i on Y_t is a homeomorphism, the restriction $\varphi|_{Y_t}$ is also a homeomorphism. Besides, $\varphi|_{Y_t}$ is equivariant. Thus, X is equivariantly embeddable in $\mathcal{F}^n_\tau$.

If X is paracompact, then, by Lemma 4.2, $\dim \beta_G X = \dim X \le n$, and hence X can be embedded equivariantly in $\mathcal{F}^n_\tau$.

To complete the proof, it remains to see that $\dim \mathcal{F}^n_\tau = n$ and $w\mathcal{F}^n_\tau = \tau$. As $\mathcal{F}^n_\tau$ contains an equivariant homeomorphic copy of the n-dimensional, compact free G-space $G \times I^k$ with $k = n - \dim G$, we infer that $\dim \mathcal{F}^n_\tau = n$. On the other hand, the discrete sum Z of τ many copies of G is a metrizable free G-space of weight $wZ = \tau$, and hence $\mathcal{F}^n_\tau$ contains an equivariant homeomorphic copy of Z. This yields that $w\mathcal{F}^n_\tau = \tau$. □

From Theorem 4.1, the following corollary follows immediately.

Corollary 4.4. *Any paracompact free G-space X has a free G-compactification $b_G X$ of weight $w(b_G X) \le wX$ and of dimension $\dim b_G X \le \dim X$.*

Corollary 4.5. *Let G be a finite group. Then, for any integer $n \ge 0$, there is a free action of G on the Menger compactum μ^n such that every separable, metrizable, free G-space X with $\dim X \le n$ admits an equivariant embedding into μ^n.*

Proof. By the preceding corollary, X has a compact, metrizable, free G-compactification $b_G X$ of $\dim b_G X \le \dim X$. It remains to apply Dranishnikov's result [7, Corollary and Theorem 3] to the effect that there is a unique free action of G on the Menger compactum μ^n such that μ^n contains an equivariant homeomorphic copy of each compact, metrizable, free G-space of dimension less than or equal to n. □

5. The case of G-spaces of a single orbit type. In this section, we generalize Theorem 4.1 to the case of G-spaces of Euclidean type that may not be free, but have a single orbit type.

Let H be a closed subgroup of G and X be a G-space of type (H). Let $N(H)$ be the normalizer of H in G and $W(H) = N(H)/H$, the Weyl group. Below, for any $n \in N(H)$, we denote by $\tilde{n}$ the lateral class nH. The group $W(H)$ acts freely on X^H, the H-fixed point set of X. At the same time, $W(H)$ acts on G/H by the formula

$$\tilde{n} * gH = gn^{-1}H, \quad \tilde{n} \in W(H),\ gH \in G/H. \tag{5.1}$$

The twisted product $(G/H)\times_{W(H)} X^H$ is just the $W(H)$-orbit space of the product $G/H \times X^H$ endowed with the diagonal action of $W(H)$. It is well known (see [4, Chapter II, Corollary 5.11]) that X is G-homeomorphic to the G-space $(G/H)\times_{W(H)} X^H$, equipped with the action of G given by the formula

$$g' * [gH,x] = [g'gH,x], \quad g' \in G,\ [gH,x] \in (G/H)\times_{W(H)} X^H. \tag{5.2}$$

Lemma 5.1. *If H is a closed subgroup of G and Y is a free $W(H)$-space, then the twisted product $T = (G/H)\times_{W(H)} Y$ has only one orbit type (H). Besides, $wT = wY$ and $\dim T = \dim Y + \dim(G/N(H))$.*

Proof. Indeed, let $[gH,x]$ be a point of $(G/H)\times_{W(H)} Y$ fixed under an element $g' \in G$. Then, $[g'gH,x] = [gH,x]$ or, equivalently, $(g'gH,x) = (\tilde{n} * gH, \tilde{n}x)$, for some $n \in N(H)$. Then, $g'gH = gn^{-1}H$ and $x = \tilde{n}x$. Since $W(H)$ acts freely on Y, the equality $x = \tilde{n}x$ implies that $n \in H$. The equality $g'gH = gn^{-1}H$ yields that $g' = gn^{-1}hg^{-1}$ for some $h \in H$, and hence, $g' \in gHg^{-1}$. Consequently, the stabilizer of $[gH,x]$ is just the group gHg^{-1}, and hence, the G-space $(G/H)\times_{W(H)} Y$ has only one orbit type (H).

Since $w(G/H) \le \aleph_0$, we see that

$$w\big((G/H)\times_{W(H)} Y\big) \le wY. \tag{5.3}$$

On the other hand, Y is a subset of T, so $wY \le wT$.

For the second equality, by Lemma 4.2 and by the above quoted Morita's theorem [16], we have

$$\begin{aligned}
\dim T &= \dim\big((G/H)\times_{W(H)} Y\big) \\
&= \dim\big((G/H)\times Y\big) - \dim W(H) \\
&= \dim(G/H) + \dim Y - \big(\dim N(H) - \dim H\big) \\
&= \dim Y + \dim G - \dim H - \dim N(H) + \dim H \\
&= \dim Y + \dim G - \dim N(H) \\
&= \dim Y + \dim\big(G/N(H)\big).
\end{aligned} \tag{5.4}$$

□

Theorem 5.2. *For every closed subgroup $H \subset G$, every infinite cardinal number τ and for every nonnegative integer $n \ge \dim G$, there exists a compact G-space $\mathcal{F}^n_\tau(H)$ of type (H) with $w(\mathcal{F}^n_\tau(H)) = \tau$, $\dim(\mathcal{F}^n_\tau(H)) = n$ which is universal in the following sense: $\mathcal{F}^n_\tau(H)$ contains a G-homeomorphic copy of any G-space X of Euclidean type and of the single orbit type (H) such that $wX \le \tau$ and $\dim \beta_G X \le n$. In particular, $\mathcal{F}^n_\tau(H)$ contains a G-homeomorphic copy of each paracompact G-space X of type (H) with $wX \le \tau$ and $\dim X \le n$.*

Proof. Let $k = n - \dim(N(H)/H)$. Then, we have

$$\begin{aligned} k &= n - \dim\left(G/N(H)\right) \\ &= n - \dim G + \dim N(H) \\ &\geq \dim N(H) - \dim H \\ &= \dim\left(N(H)/H\right) \\ &= \dim W(H). \end{aligned} \tag{5.5}$$

Hence, by Theorem 4.1, there exists a universal compact free $W(H)$-space $\mathcal{F}_\tau^k$ of dimension k and weight τ.

Set $\mathcal{F}_\tau^n(H) = (G/H) \times_{W(H)} \mathcal{F}_\tau^k$. By Lemma 5.1, $\mathcal{F}_\tau^n(H)$ is a compact G-space of the single orbit type (H). We claim that it is the required one.

Indeed, by Lemma 5.1,

$$\begin{aligned} w\left((G/H) \times_{W(H)} \mathcal{F}_\tau^k\right) &= w\mathcal{F}_\tau^k = \tau, \\ \dim \mathcal{F}_\tau^n(H) &= \dim\left((G/H) \times_{W(H)} \mathcal{F}_\tau^k\right) \\ &= \dim \mathcal{F}_\tau^k + \dim\left(G/N(H)\right) \\ &= k + \dim\left(G/N(H)\right) = n. \end{aligned} \tag{5.6}$$

Now, if X is a G-space with the single orbit type (H) such that $wX \leq \tau$ and $\dim X \leq n$, then, since $X = (G/H) \times_{W(H)} X^H$, it follows from Lemma 5.1 that $w(X^H) \leq \tau$ and $\dim X^H \leq k$.

By Theorem 4.1, there is a $W(H)$-equivariant embedding $f : X^H \hookrightarrow \mathcal{F}_\tau^k$. Then, the map $F : (G/H) \times_{W(H)} X^H \to (G/H) \times_{W(H)} \mathcal{F}_\tau^k$, generated by f, is a G-equivariant embedding. We recall that F is defined as follows: $F([gh, x]) = [gH, f(x)]$ for every $[gh, x] \in (G/H) \times_{W(H)} X^H$ (see [17, Theorem 1.7.10]).

It remains only to recall that

$$X = (G/H) \times_{W(H)} X^H, \qquad \mathcal{F}_\tau^n(H) = (G/H) \times_{W(H)} \mathcal{F}_\tau^k. \tag{5.7}$$

This completes the proof. □

From Theorem 5.2, the following corollary follows immediately.

Corollary 5.3. *Any paracompact G-space X of a single orbit type (H) has a G-compactification $b_G X$ of the same orbit type (H) such that $w(b_G X) \leq wX$ and $\dim b_G X \leq \dim X$.*

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