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THE BOOLEAN ALGEBRA OF GALOIS ALGEBRAS

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Let B be a Galois algebra with Galois group G, $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$, and $BJ_g = Be_g$ for a central idempotent e_g, B_a the Boolean algebra generated by $\{0, e_g \mid g \in G\}$, e a nonzero element in B_a, and $H_e = \{g \in G \mid ee_g = e\}$. Then, a monomial e is characterized, and the Galois extension Be, generated by e with Galois group H_e, is investigated.

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1. Introduction. The Boolean algebra of central idempotents in a commutative Galois algebra plays an important role for the commutative Galois theory (see [1, 3, 6]). Let B be a Galois algebra with Galois group G, C the center of B, and $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$ for each $g \in G$. In [2], it was shown that $BJ_g = Be_g$ for some idempotent e_g of C. Let B_a be the Boolean algebra generated by $\{0, e_g \mid g \in G\}$. Then in [5], by using B_a, the following structure theorem for B was given. There exist $\{e_i \in B_a \mid i = 1, 2, \ldots, m$ for some integer $m\}$ and some subgroups H_i of G such that $B = \oplus \sum_{i=1}^{m} Be_i \oplus Bf$ where $f = 1 - \sum_{i=1}^{m} e_i$, Be_i is a central Galois algebra with Galois group H_i for each $i = 1, 2, \ldots, m$, and $Bf = Cf$ which is a Galois algebra with Galois group induced by and isomorphic with G in case $1 \neq \sum_{i=1}^{m} e_i$. In [4], let K be a subgroup of G. Then, K is called a nonzero subgroup of G if $\prod_{k \in K} e_k \neq 0$ in B_a, and K is called a maximal nonzero subgroup of G if $K \subset K'$, where K' is a nonzero subgroup of G such that $\prod_{k \in K} e_k = \prod_{k \in K'} e_k$, then $K = K'$. We note that any nonzero subgroup is contained in a unique maximal nonzero subgroup of G. In [4], it was shown that there exists a one-to-one correspondence between the set of nonzero monomials in B_a and the set of maximal nonzero subgroups of G, and that, for a nonzero monomial e in B_a such that $H_e \neq \{1\}$, Be is a central Galois algebra with Galois group H_e if and only if e is a minimal nonzero monomial in B_a. The purpose of the present paper is to characterize a monomial e in B_a in terms of the maximal nonzero subgroups of G. Then, the Galois extension Be, generated by a nonzero idempotent e and by a monomial e with Galois group H_e, is investigated, respectively. Let $G(e) = \{g \in G \mid g(e) = e\}$ for each $e \neq 0$ in B_a. We will show that (1) H_e is a normal subgroup of $G(e)$, and (2) Be is a Galois extension of $(Be)^{H_e}$ with Galois group H_e and $(Be)^{H_e}$ is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)/H_e$. In particular, when e is a monomial, $G(e) = N(H_e)$ (the normalizer of H_e), and when e is an atom (a minimal nonzero element) of B_a, Be is a central Galois algebra over Ce with Galois group H_e and Ce is a commutative Galois algebra with Galois group $G(e)/H_e$. This generalizes and improves the result of the components of B in [5, Theorem 3.8] for a Galois algebra.

2. Definitions and notations. Let B be a ring with 1, C the center of B, G an automorphism group of B of order n for some integer n, and B^G the set of elements in B, fixed under each element in G. B is called a Galois extension of B^G with Galois group G if there exist elements $\{a_i, b_i$ in B, $i = 1,2,\dots,m\}$ for some integer m such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g}$ for each $g \in G$. B is called a Galois algebra over R if B is a Galois extension of R which is contained in C, and B is called a central Galois extension if B is a Galois extension of C. In this paper, we assume that B is a Galois algebra with Galois group G. Let $J_g = \{b \in B \mid bx = g(x)b$ for all $x \in B\}$. In [2], it was shown that $BJ_g = Be_g$ for some central idempotent e_g of B. We denote $(B_a; \dotplus, \cdot)$, the Boolean algebra generated by $\{0, e_g \mid g \in G\}$, where $e \cdot e' = ee'$ and $e \dotplus e' = e + e' - ee'$ for any e and e' in B_a. An order relation $\le$ is defined as usual, that is, $e \le e'$ in B_a if $e \cdot e' = e$. Throughout, $e + e'$, for $e, e' \in B_a$, means the sum in the Boolean algebra $(B_a; \dotplus, \cdot)$, $H_e = \{g \in G \mid e \le e_g\}$ for an $e \ne 0$ in B_a, and a monomial e in B_a is $\prod_{g \in S} e_g \ne 0$ for some $S \subset G$.

3. The Boolean algebra. In this section, we will characterize a monomial e in B_a in terms of the maximal nonzero subgroups of G. We begin with several lemmas.

Lemma 3.1. *Let* $\{e_i, f \mid i = 1,2,\dots,m\}$ *be given in* [5, Theorem 3.8]. *Then,*

(1) $\{e_i, f \mid i = 1,2,\dots,m\}$ *is the set of all minimal elements of* B_a *in case* $f \ne 0$,

(2) *for each* $e \ne 0$ *in* B_a, *there exists a unique subset* Z_e *of the set* $\{1,2,\dots,m\}$ *such that* $e = \sum_{i \in Z_e} e_i$ *or* $e = \sum_{i \in Z_e} e_i + f$.

Proof. (1) By the proof of [5, Theorem 3.8], either $e_i = \prod_{g \in H_i} e_g$, where H_i is a maximum subset (subgroup) of G such that $\prod_{g \in H_i} e_g \ne 0$, or $e_i = (1 - \sum_{j=1}^t e_j)\prod_{g \in H_i} e_g$ for some $t < i$, where H_i is a maximum subset (subgroup) of G such that $(1 - \sum_{j=1}^t e_j)\prod_{g \in H_i} e_g \ne 0$; so, either e_i is a minimal element of B_a or e_i is a minimal element of $(1 - \sum_{j=1}^t e_j)B_a$. Noting that any minimal element in $(1 - \sum_{j=1}^t e_j)B_a$ is also a minimal element in B_a, we conclude that each e_i is a minimal element in B_a. Next, we show that f is also a minimal element of B_a in case $f \ne 0$. In fact, by the proof of [5, Theorem 3.8], $e_g f = 0$ for any $g \ne 1$ in G; so, for any $e \in B_a$, $ef = 0$ or $ef = f$. This implies that f is a minimal element of B_a in case $f \ne 0$. Moreover, $\sum_{i=1}^m e_i + f = 1$; so, $\{e_i, f \mid i = 1,2,\dots,m\}$ is the set of all minimal elements of B_a in case $f \ne 0$.

(2) Since $1 = \sum_{i=1}^m e_i + f$, a sum of all minimal elements of B_a, the statement is immediate. □

Lemma 3.2. *Let e be a nonzero element in B_a. Then,*
(1) *there exists a monomial e' of B_a such that $e \le e'$ and $H_e = H_{e'}$,*
(2) *H_e is a maximal nonzero subgroup of G.*

Proof. (1) For any nonzero element e in B_a, let $e' = \prod_{g\in H_e} e_g$. We claim that $e \le e'$ and $H_e = H_{e'}$. In fact, for any $h \in H_e$, $e \le e_h$; so, $e \le \prod_{h\in H_e} e_h = e'$. Moreover, for any $h \in H_e$, $e_h \ge \prod_{g\in H_e} e_g = e'$; so, $h \in H_{e'}$. Hence, $H_e \subset H_{e'}$. On the other hand, for any $h \in H_{e'}$, $e_h \ge e' = \prod_{g\in H_e} e_g \ge e$; so, $h \in H_e$. Thus, $H_{e'} \subset H_e$. Therefore, $H_e = H_{e'}$.

(2) By [4, Theorem 3.2], $H_{e'}$ is a maximal nonzero subgroup of G for e' is a monomial. Hence, H_e $(= H_{e'})$ is a maximal nonzero subgroup of G. □

Next is an expression of H_e for a nonzero $e \in B_a$.

Theorem 3.3. *For any $e \ne 0$ in B_a, $H_e = \cap_{i\in Z_e} H_{e_i}$ or H_1, where $e = \sum_{i\in Z_e} e_i$ or $e = \sum_{i\in Z_e} e_i + f$ as given in Lemma 3.1(2).*

Proof. We first show that for $e = e' + e''$ for some $e', e'' \ne 0$ in B_a, $H_e = H_{e'} \cap H_{e''}$. In fact, since $e \ge e'$ and $e \ge e''$, we have $H_e \subset H_{e'} \cap H_{e''}$. Conversely, for any $g \in H_{e'} \cap H_{e''}$, $e_g \ge e'$ and $e_g \ge e''$; so, $e_g \ge e' + e'' = e$. Hence, $g \in H_e$; so, $H_e = H_{e'} \cap H_{e''}$. Therefore, by induction, if $e = \sum_{i\in Z_e} e_i$, then $H_e = \cap_{i\in Z_e} H_{e_i}$. Now, by Lemma 3.1, for any $e \ne 0$ in B_a, $e = \sum_{i\in Z_e} e_i$ or $e = \sum_{i\in Z_e} e_i + f$. Similarly, if $e = \sum_{i\in Z_e} e_i + f$, then $H_e = H_{(\sum_{i\in Z_e} e_i)+f} = (\cap_{i\in Z_e} H_{e_i}) \cap H_f$. But, for $g \in G$ such that $e_g \ne 1$, $e_g f = 0$; so, $H_f = H_1$. Therefore, $H_e = (\cap_{i\in Z_e} H_{e_i}) \cap H_1 = H_1$ for $H_1 \subset H_{e_i}$ for each i. □

We observe that there exist some $e \ne 0$ such that $H_e = \cap_{i\in Z_e} H_{e_i}$ and $H_e \subset H_{e_j}$ for some $j \notin Z_e$, and that not all $e \ne 0$ are monomials. Next, we identify which element $e \ne 0$ in B_a is a monomial. Two characterizations are given. We begin with a definition.

Definition 3.4. An $e \ne 0$ in B_a is called a maximal G-element if $H_e \ne H_1$ and, for any $e' \in B_a$ such that $e \le e'$ and $H_e = H_{e'}$, $e = e'$.

Lemma 3.5. (1) *If $e \ne 0$ such that $ef = 0$, then $e = \sum_{i\in Z_e} e_i$.*
(2) *If e is a monomial, $e = \prod_{g\in S} e_g$ for some $S \subset G$, then $e = 1$ or $e = \sum_{i\in Z_e} e_i$.*

Proof. (1) By Lemma 3.1, $e = \sum_{i\in Z_e} e_i$ or $e = \sum_{i\in Z_e} e_i + f$. If $e \ne \sum_{i\in Z_e} e_i$, then $e = \sum_{i\in Z_e} e_i + f$ and $f \ne 0$. But then, $f = (\sum_{i\in Z_e} e_i + f) f = ef = 0$. This is a contradiction. Hence, $e = \sum_{i\in Z_e} e_i$.

(2) In case $e = 1$, we are done. In case $e \ne 1$. Since $e_g f = 0$ for each $g \in G$ such that $e_g \ne 1$, $ef = \prod_{g\in S} e_g f = 0$. Thus, by (1), $e = \sum_{i\in Z_e} e_i$. □

Theorem 3.6. *Keeping the notations of Lemma 3.1 for any $e \ne 0, 1$ in B_a, the following statements are equivalent:*
(1) *$e = \prod_{g\in S} e_g$ for some $S \subset G$, a monomial in B_a;*
(2) *e is a maximal G-element in B_a;*

(3) $e = \sum_{i\in Z_e} e_i$ *where* $\{e_i \mid i \in Z_e\}$ *are all atoms such that* $H_e \subset H_{e_i}$ *and* $H_e \neq H_1$.

Proof. (1)⇒(2). Since e is a monomial and $e \neq 1$, $e = \prod_{g\in H_e} e_g$ where $e_g \neq 1$ for some $g \in H_e$. Thus, $H_e \neq H_1$. Next, for any e' such that $e \leq e'$ and $H_e = H_{e'}$,

$$e \leq e' \leq \prod_{g\in H_{e'}} e_g = \prod_{g\in H_e} e_g = e. \tag{3.1}$$

Hence, $e = e'$. This implies that e is a maximal G-element in B_a.

(2)⇒(1). Let e be a maximal G-element and $e' = \prod_{g\in H_e} e_g$. Then, by Lemma 3.2, $e \leq e'$ and $H_e = H_{e'}$. But e is a maximal G-element; so, $e = e'$ which is a monomial.

(1)⇒(3). By Lemma 3.5, $e = \sum_{i\in Z_e} e_i$. Now, let e_j be an atom such that $H_e \subset H_{e_j}$. Then, $e_j \leq \prod_{g\in H_{e_j}} e_g \leq \prod_{g\in H_e} e_g$. But, by hypothesis, e is a monomial; so, $e = \prod_{g\in H_e} e_g$. Hence, $e_j \leq e$. This implies that e_j is a term in e. Thus, $e = \sum_{i\in Z_e} e_i$ where $\{e_i \mid i \in Z_e\}$ are all atoms such that $H_e \subset H_{e_i}$. Moreover, since $e = \prod_{g\in S} e_g \neq 1$, there exists $g \in G$ such that $e \leq e_g \neq 1$. Thus, $g \in H_e$ and $g \notin H_1$. Therefore, $H_e \neq H_1$.

(3)⇒(1). Let $e' = \prod_{g\in H_e} e_g$. Then, by Lemma 3.2, $e \leq e'$ and $H_e = H_{e'}$. Since $H_e \neq H_1$, $H_{e'} \neq H_1$. Also, since e' is a monomial, $e' = \sum_{j\in Z_{e'}} e_j$ by Lemma 3.5(2). Now, suppose that $e \neq e'$. Then, there is a $j \in Z_{e'}$ but $j \notin Z_e$, that is, e_j is a term of $e' = \sum_{j\in Z_{e'}} e_j$ but not a term of $e = \sum_{i\in Z_e} e_i$. But then, $H_e = H_{e'} = \cap_{j\in Z_{e'}} H_{e_j} \subset H_{e_j}$ such that $j \notin Z_e$. This contradicts the hypothesis that $e = \sum_{i\in Z_e} e_i$ where $\{e_i \mid i \in Z_e\}$ are all atoms such that $H_e \subset H_{e_i}$. Thus, $e = e'$ which is a monomial in B_a. □

4. Galois extensions. In [5], it was shown that Be is a central Galois algebra with Galois group H_e for any atom $e \neq f$ of B_a. Also, for any $e \neq 0$ in B_a, Be is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)|_{Be} \cong G(e)$ where $G(e) = \{g \in G \mid g(e) = e\}$ (see [5, Lemma 3.7]). In this section, we are going to show that, for any $e \neq 0$ in B_a (not necessary an atom), (1) H_e is a normal subgroup of $G(e)$, and (2) Be is a Galois extension of $(Be)^{H_e}$ with Galois group H_e and $(Be)^{H_e}$ is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)/H_e$. This generalizes and improves the result for Be when e is an atom of B_a as given in [5, Theorem 3.8]. In particular, for a monomial e, $G(e) = N(H_e)$, the normalizer of H_e in G.

Lemma 4.1. *Let* $e \neq 0$ *in* B_a. *Then,* H_e *is a normal subgroup of* $G(e)$ *where* $G(e) = \{g \in G \mid g(e) = e\}$.

Proof. We first claim that $H_e \subset G(e)$. In fact, by Lemma 3.1, for any $e \neq 0$ in B_a, there exists a unique subset Z_e of the set $\{1,2,\dots,m\}$ such that $e = \sum_{i\in Z_e} e_i$ or $e = \sum_{i\in Z_e} e_i + f$ where e_i are given in Lemma 3.1. Moreover, for each i,

$e_i = \prod_{h\in H_{e_i}} e_h$ or $e_i = (1-\sum_{j=1}^{t} e_j)\prod_{g\in H_{e_i}} e_g$ for some $t < i$. Noting that g permutes the set $\{e_i \mid i = 1,2,\ldots,t\}$ for each $g \in G$ by the proof of [5, Theorem 3.8], we have, for each $g \in G$,

$$g(e_i) = g\left(\prod_{h\in H_{e_i}} e_h\right) = \prod_{h\in H_{e_i}} e_{ghg^{-1}} \geq \prod_{h\in H_{e_i}} e_g e_h e_{g^{-1}} = e_g e_i e_{g^{-1}} \tag{4.1}$$

or

$$\begin{aligned} g(e_i) &= g\left(\left(1-\sum_{j=1}^{t} e_j\right)\prod_{h\in H_{e_i}} e_h\right) = \left(1-\sum_{j=1}^{t} e_j\right)\prod_{h\in H_{e_i}} e_{ghg^{-1}} \\ &\geq \left(1-\sum_{j=1}^{t} e_j\right)\prod_{h\in H_{e_i}} e_g e_h e_{g^{-1}} \\ &= e_g\left(\left(1-\sum_{j=1}^{t} e_j\right)\prod_{h\in H_{e_i}} e_h\right)e_{g^{-1}} = e_g e_i e_{g^{-1}}. \end{aligned} \tag{4.2}$$

Now, in case $e = \sum_{i\in Z_e} e_i$, for any $h \in H_e$,

$$e = e_h e e_{h^{-1}} = \sum_{i\in Z_e} e_h e_i e_{h^{-1}} \leq \sum_{i\in Z_e} h(e_i) = h(e). \tag{4.3}$$

Thus, $h(e) = e$ using Lemma 3.1(2). Noting that g permutes the set $\{e_i \mid i = 1,2,\ldots,m\}$ for each $g \in G$, we have $g(f) = f$ for each $g \in G$. Thus, we have $h(e) = e$ for each $h \in H_e$ in case $e = \sum_{i\in Z_e} e_i + f$. This proves that $H_e \subset G(e)$. Next, we show that H_e is a normal subgroup of $G(e)$. Since for each $g \in G$, $g(e_i)$ is also an atom, $g(e) = e$ (i.e., $g \in G(e)$) implies that g permutes the set $\{e_i \mid i \in Z_e\}$. Therefore, for each $i \in Z_e$, $g(e_i) = e_j$ and $gH_{e_i}g^{-1} = H_{e_j}$ for some $j \in Z_e$. But, by Theorem 3.3, $H_e = \cap_{i\in Z_e} H_{e_i}$ (or $H_e = H_1$ which is normal); so, for any $g \in G(e)$, $gH_e g^{-1} = g(\cap_{i\in Z_e} H_{e_i})g^{-1} = \cap_{i\in Z_e} gH_{e_i}g^{-1} = \cap_{j\in Z_e} H_{e_j} = H_e$. Therefore, H_e is a normal subgroup of $G(e)$. □

Theorem 4.2. *Let e be a nonzero element in B_a. Then,*

(1) *Be is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)$,*

(2) *Be is a Galois extension of $(Be)^{H_e}$ with Galois group H_e and $(Be)^{H_e}$ is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)/H_e$.*

Proof. (1) Since B is a Galois algebra with Galois group G, B is a Galois extension with Galois group $G(e)$. But $g(e) = e$ for each $g \in G(e)$; so, by [5, Lemma 3.7], Be is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)$.

(2) Clearly, Be is a Galois extension of $(Be)^{H_e}$ with Galois group H_e by part (1). Next, we claim that $|H_e|$, the order of H_e, is a unit in Be. In fact, by [5, Theorem 3.8], for each atom e_i of B_a, Be_i is a central Galois algebra over Ce_i with Galois group H_{e_i}; so, $|H_{e_i}|$, the order of H_{e_i}, is a unit in Be_i (see [2, Corollary 3]). Hence, $|H_e|$ $(= |\cap H_{e_i}|)$ is a unit in Be if $e = \sum_{i\in Z_e} e_i$. If $e = \sum_{i\in Z_e} e_i + f$ and $f \neq 0$, then $H_e = H_1 = \{g \in G \mid e_g = 1\} = \{g \in G \mid g(c) = c$ for each $c \in C\}$. Hence, by

[2, Proposition 5], $|H_e|$ is a unit in B. Thus, $(Be)^{H_e}$ is a Galois extension of $(Be)^{G(e)}$ with Galois group $G(e)/H_e$ for H_e is a normal subgroup of $G(e)$ by Lemma 4.1. □

Lemma 4.1 shows that, for any nonzero element e in B_a, $G(e)$ is contained in (not necessarily equal to) the normalizer $N(H_e)$ of H_e in G. Next, we want to show that $G(e) = N(H_e)$ when e is a monomial. Consequently, for any nonzero element e in B_a, Be is embedded in a Galois extension Be' of $(Be')^{H_e}$ with the same Galois group H_e, and $(Be')^{H_e}$ is a Galois extension of $(Be')^{G(e')}$ with Galois group $G(e')/H_e$ such that $G(e') = N(H_e)$ for some monomial e' in B_a.

Lemma 4.3. *Let e be a nonzero element in B_a. Then, there exists a monomial e' in B_a such that $e \le e'$, $H_e = H_{e'}$, and $N(H_e) = G(e')$ where $G(e') = \{g \in G \mid g(e') = e'\}$ and $N(H_e)$ is the normalizer of H_e in G.*

Proof. By Lemma 3.2, there exists a monomial e' in B_a such that $e \le e'$ and $H_e = H_{e'}$; so, it suffices to show that $N(H_e) = G(e')$. For any $g \in N(H_e)$, $g \in N(H_{e'})$; so, by Theorem 3.3, $H_{e'} = gH_{e'}g^{-1} = g(\cap_{i\in Z_{e'}} H_{e_i})g^{-1} = \cap_{i\in Z_{e'}} gH_{e_i}g^{-1} = \cap_{i\in Z_{e'}} H_{g(e_i)} = H_{\sum_{i\in Z_{e'}} g(e_i)} = H_{g(e')}$. Noting that e' is a monomial, we have $g(e') = e'$ by Lemma 3.2, that is, $g \in G(e')$. This implies that $N(H_e) \subset G(e')$. Conversely, $G(e') \subset N(H_{e'})$ by Lemma 4.1. But $H_e = H_{e'}$; so, $G(e') \subset N(H_{e'}) = N(H_e)$. Therefore, $N(H_e) = G(e')$. □

Theorem 4.4. *Let e be a nonzero element in B_a. Then, there exists a monomial e' in B_a such that Be is embedded in Be', Be' is a Galois extension of $(Be')^{H_e}$ with Galois group H_e, and $(Be')^{H_e}$ is a Galois extension of $(Be')^{N(H_e)}$ with Galois group $N(H_e)/H_e$.*

Proof. By Lemma 4.3, there exists a monomial e' in B_a such that $e \le e'$, H_e is a normal subgroup of $G(e')$, and $N(H_e) = G(e')$. Hence, $Be \subset Be'$. But Be' is a Galois extension of $(Be')^{H_{e'}}$ with Galois group $H_{e'}$ and $(Be')^{H_{e'}}$ is a Galois extension of $(Be')^{G(e')}$ with Galois group $G(e')/H_{e'}$ by Theorem 4.2; so, Theorem 4.4 holds. □

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