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WAVELET ANALYSIS ON A BOEHMIAN SPACE

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We extend the wavelet transform to the space of periodic Boehmians and discuss some of its properties.

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1. Introduction. The concept of Boehmians was introduced by J. Mikusiński and P. Mikusiński [7], and the space of Boehmians with two notions of convergences was well established in [8]. Many integral transforms have been extended to the context of Boehmian spaces, for example, Fourier transform [9, 10, 11], Laplace transform [13, 17], Radon transform [14], and Hilbert transform [3, 5].

On the other hand, the theory of wavelet transform is recently developed, and it has various applications in signal processing, especially to analyze nonstationary signals by providing the time-frequency representation of the signal. For a fixed $g \in \mathcal{L}^2(\mathbb{R})$, called a mother wavelet, the wavelet transform $\Phi_g : \mathcal{L}^2(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}\times\mathbb{R}^+)$ is defined by

$$\Phi_g(f)(a,b) = \int_{-\infty}^{\infty} f(x)\overline{g_{a,b}(x)}dx \quad \text{for } a>0,\ b\in\mathbb{R}, \tag{1.1}$$

where $g_{a,b}(x) = (1/\sqrt{a})g((x-b)/a)$, $x \in \mathbb{R}$, are called wavelets. For more details, we refer the reader to [6]. In [4], we extended the wavelet transform to a Boehmian space which properly contains $\mathcal{L}^2(\mathbb{R})$ and studied its properties.

Holschneider [2] introduced the wavelet transform on the space $C^\infty(\mathbb{T})$ of smooth functions on the unit circle $\mathbb{T}$ of the complex plane and gave an extension to the space of periodic distributions. In Section 2, we fix some notations and discuss the theory of wavelet transform on $C^\infty(\mathbb{T})$. In Section 3, we briefly recall the periodic Boehmians, construct a new Boehmian space $\mathcal{B}(\mathcal{S}(\mathbb{Y}),(C^\infty(\mathbb{T}),*),\odot,\Delta)$, and verify some auxiliary results. In Section 4, we define wavelet transform on the space of periodic Boehmians and prove that it is consistent with the wavelet transform on $C^\infty(\mathbb{T})$. Further, we establish that the extended wavelet transform is linear and continuous with respect to δ-convergence as well as Δ-convergence.

2. Preliminaries. The space $C^\infty(\mathbb{T})$ consists of infinitely differentiable, periodic functions on $\mathbb{R}$ of period 2π, with the Fréchet space topology induced by the increasing sequence of seminorms

$$\|\phi\|_{C^\infty(\mathbb{T});n} = \sum_{p=0}^{n} \sup_{t\in[0,2\pi]} \left|\partial^p \phi(t)\right|. \tag{2.1}$$

We know that

$$C^\infty(\mathbb{T}) = C_+^\infty(\mathbb{T}) \oplus C_-^\infty(\mathbb{T}) \oplus K(\mathbb{T}), \tag{2.2}$$

where $C_+^\infty(\mathbb{T})$ and $C_-^\infty(\mathbb{T})$ are the subspaces consisting of functions with positive and negative Fourier coefficients, respectively, and $K(\mathbb{T})$ is the space of constant functions.

Let $\mathcal{S}(\mathbb{R})$ denote the space of rapidly decreasing functions on $\mathbb{R}$. (See [1].) Given $f \in \mathcal{S}(\mathbb{R})$, $b \in [0,2\pi]$, and $a > 0$, define $f_a, f_{b,a} \in C^\infty(\mathbb{T})$ by

$$\begin{aligned} f_a(x) &= \sum_{n\in\mathbb{Z}} \frac{1}{a} f\left(\frac{x+2n\pi}{a}\right), \quad x \in [0,2\pi], \\ f_{b,a}(x) &= f_a(x-b), \quad x \in [0,2\pi]. \end{aligned} \tag{2.3}$$

Let $\mathcal{S}(\mathbb{Y})$ denote the Fréchet space of all smooth functions $\eta(b,a)$ of rapid descent on $\mathbb{R}\times\mathbb{R}^+$ which are periodic functions in the variable b of period 2π, with the following directed family of seminorms:

$$\|\eta\|_{\mathcal{S}(\mathbb{Y});n,\alpha,\beta} = \sum_{\substack{0\le p\le n\\ 0\le l\le\alpha\\ 0\le k\le\beta}} \sup_{a>0} \sup_{b\in[0,2\pi]} \left|a^p \partial_a^l \partial_b^k \eta(b,a)\right|. \tag{2.4}$$

We choose a mother wavelet $g \in \mathcal{S}(\mathbb{R})$ with all moments $\int_{-\infty}^{\infty} x^n g(x)\,dx$ are equal to zero.

Definition 2.1. The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{Y})$ is defined by

$$T_g(\phi) = \int_0^{2\pi} \phi(x)\overline{g_{b,a}(x)}\,dx, \quad b \in \mathbb{R},\ a > 0. \tag{2.5}$$

Theorem 2.2. *The wavelet transform $T_g : C^\infty(\mathbb{T}) \to \mathcal{S}(\mathbb{Y})$ is continuous and linear.*

Definition 2.3. The map $R_g : \mathcal{S}(\mathbb{Y}) \to C^\infty(\mathbb{T})$ is defined by

$$(R_g\eta)(x) = \int_0^{2\pi}\int_0^{\infty} g_{b,a}(x)\eta(b,a)\frac{da\,db}{a}. \tag{2.6}$$

Theorem 2.4. *The map $R_g : \mathcal{S}(\mathbb{Y}) \to C^\infty(\mathbb{T})$ is continuous and linear.*

A partial inversion formula is given by the following theorem.

Theorem 2.5. *If $\hat{g}$ is the Fourier transform of g and $C_g^+ = \int_0^\infty |\hat{g}(a)|^2 (da/a)$, $C_g^- = \int_0^\infty |\hat{g}(-a)|^2 (da/a)$, then*

$$R_g \circ T_g \phi = C_g^+ \phi, \quad \forall \phi \in C_+^\infty(\mathbb{T}),$$
$$R_g \circ T_g \phi = C_g^- \phi, \quad \forall \phi \in C_-^\infty(\mathbb{T}). \tag{2.7}$$

3. Boehmian spaces. The triplet $(C^\infty(\mathbb{T}), *, \Delta)$, where $* : C^\infty(\mathbb{T}) \times C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ is defined by

$$(\phi * \psi)(x) = \int_0^{2\pi} \phi(x-t)\psi(t)\,dt, \quad x \in [0, 2\pi] \tag{3.1}$$

and Δ is the collection of all sequences (δ_k) from $C^\infty(\mathbb{T})$ satisfying

(1) $\int_0^{2\pi} \delta_k(t)\,dt = 1$ for all $k \in \mathbb{N}$,
(2) $\int_0^{2\pi} |\delta_k(t)|\,dt \le M$ for all $k \in \mathbb{N}$, for some $M > 0$,
(3) $s(\delta_k) \to 0$ as $n \to \infty$ where $s(\delta_k) = \sup\{t \in [0, 2\pi] : \delta_k(t) \neq 0\}$,

is the collection of all equivalence classes $[\phi_k/\delta_k]$ given by the equivalence relation $\sim$ defined by

$$((\phi_k), (\delta_k)) \sim ((\psi_k), (\epsilon_k)) \quad \text{if } \phi_k * \epsilon_j = \psi_j * \delta_k \ \forall k, j \in \mathbb{N} \tag{3.2}$$

on the collection $\mathcal{A}$ of pair of sequences $((\phi_k), (\delta_k))$, $\phi_n \in C^\infty(\mathbb{T})$, $(\delta_k) \in \Delta$ satisfying

$$\phi_k * \delta_j = \phi_j * \delta_k, \quad \forall k, j \in \mathbb{N}. \tag{3.3}$$

This triplet with addition and scalar multiplication, defined by

$$\left[\frac{\phi_k}{\delta_k}\right] + \left[\frac{\psi_k}{\epsilon_k}\right] = \left[\frac{\phi_k * \epsilon_k + \psi_k * \delta_k}{\delta_k * \epsilon_k}\right],$$
$$\alpha\left[\frac{\phi_k}{\delta_k}\right] = \left[\frac{\alpha\phi_k}{\delta_k}\right], \tag{3.4}$$

is called the periodic Boehmian space [15, 16], and we denote it by $\mathcal{B}_{\mathbb{T}}$.

Definition 3.1 (δ-convergence). A sequence (x_n) δ-converges to x in $\mathcal{B}_{\mathbb{T}}$, denoted by $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{T}}$ if there exists $(\delta_k) \in \Delta$ such that

$x_n * \delta_k$, $x * \delta_k \in C^\infty(\mathbb{T})$, and for each $k \in \mathbb{N}$,

$$x_n * \delta_k \longrightarrow x * \delta_k \quad \text{as } n \longrightarrow \infty \text{ in } C^\infty(\mathbb{T}). \tag{3.5}$$

The following theorem is proved in [8].

Theorem 3.2. *Let x_n, $x \in \mathcal{B}_\mathbb{T}$, $n \in \mathbb{N}$. $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_\mathbb{T}$ if and only if there exist $\phi_{n,k}, \phi_k \in C^\infty(\mathbb{T})$ such that $x_n = [\phi_{n,k}/\delta_k], [\phi_k/\delta_k]$ and, for each $k \in \mathbb{N}$,*

$$\phi_{n,k} \longrightarrow \phi_k \quad \textit{as } n \longrightarrow \infty \textit{ in } C^\infty(\mathbb{T}). \tag{3.6}$$

Definition 3.3 (Δ-convergence)**.** A sequence (x_n) Δ-converges to x in $\mathcal{B}_\mathbb{T}$, denoted by $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_\mathbb{T}$ if there exists a delta-sequence (δ_n) such that $(x_n - x) * \delta_n \in C^\infty(\mathbb{T})$ for each $n \in \mathbb{N}$ and

$$(x_n - x) * \delta_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \text{ in } C^\infty(\mathbb{T}). \tag{3.7}$$

Now, we construct a new Boehmian space as follows.

As in the context of Boehmian space defined in [12], we take the vector space Γ and the commutative semi-group as $\mathcal{S}(\mathbb{Y})$ and $(C^\infty(\mathbb{T}), *)$, respectively.

Definition 3.4. Given $\eta \in \mathcal{S}(\mathbb{Y})$ and $\phi \in C^\infty(\mathbb{T})$, define

$$(\eta \odot \phi)(b,a) = \int_0^{2\pi} \eta(b-t,a)\phi(t)\,dt. \tag{3.8}$$

Lemma 3.5. *If $\eta \in \mathcal{S}(\mathbb{Y})$ and $\phi \in C^\infty(\mathbb{T})$, then $\eta \odot \phi \in \mathcal{S}(\mathbb{Y})$.*

Proof. To prove that $(\eta \odot \phi)(b,a)$ is infinitely differentiable, we show that

$$\begin{aligned} \partial_a(\eta \odot \phi)(b,a) &= (\partial_a \eta \odot \phi)(b,a), \\ \partial_b(\eta \odot \phi)(b,a) &= (\partial_b \eta \odot \phi)(b,a). \end{aligned} \tag{3.9}$$

Fix $a_0 > 0$, $b_0 \in \mathbb{R}$ arbitrarily.

Consider $((\eta \odot \phi)(b_0,a) - (\eta \odot \phi)(b_0,a_0))/(a - a_0) = \int_0^{2\pi} (\eta(b_0 - t,a) - \eta(b_0 - t,a_0))/(a - a_0)\phi(t)\,dt$. Using the mean-value theorem (in the variable a), we get that the integrand is dominated by $\|\eta\|_{\mathcal{S}(\mathbb{Y});0,1,0}\|\phi\|_{C^\infty(\mathbb{T}),0}$. Therefore, we can apply Lebesgue dominated convergence theorem [18], and we get

$$\begin{aligned} \partial_a(\eta \odot \phi)(b_0,a_0) &= \lim_{a \to a_0} \int_0^{2\pi} \frac{\eta(b_0-t,a) - \eta(b_0-t,a_0)}{a-a_0}\phi(t)\,dt \\ &= \int_0^{2\pi} \lim_{a \to a_0} \frac{\eta(b_0-t,a) - \eta(b_0-t,a_0)}{a-a_0}\phi(t)\,dt \\ &= \int_0^{2\pi} \partial_a \eta(b_0-t,a_0)\phi(t)\,dt \\ &= (\partial_a \eta \odot \phi)(b_0,a_0). \end{aligned} \tag{3.10}$$

By a similar argument, we can prove that $\partial_b(\eta\odot\phi)(b,a)=(\partial_b\eta\odot\phi)(b,a)$. Finally by a routine manipulation, we get

$$\|\eta\odot\phi\|_{\mathcal{S}(\mathbb{Y});n,\alpha,\beta}\le\|\phi\|_{\mathcal{L}^1(\mathbb{T})}\|\eta\|_{\mathcal{S}(\mathbb{Y});n,\alpha,\beta}, \tag{3.11}$$

where $\|\phi\|_{\mathcal{L}^1(\mathbb{T})}=\int_0^{2\pi}|\phi(t)|dt$. Hence, $\eta\odot\phi\in\mathcal{S}(\mathbb{Y})$. □

Lemma 3.6. *If $\eta\in\mathcal{S}(\mathbb{Y})$ and $(\delta_n)\in\Delta$, then $\eta\odot\delta_n\to\phi$ as $n\to\infty$ in $\mathcal{S}(\mathbb{Y})$.*

Proof. Let $p,k,l\in\mathbb{N}_0$ be arbitrary. Using the mean-value theorem and a property of δ-sequence, we get

$$\begin{aligned}|a^p\partial_a^l\partial_b^k(\eta\odot\delta_n-\eta)(b,a)| &= |a^p((\partial_a^l\partial_b^k\eta)\odot\delta_n)(b,a)-a^p\partial_a^l\partial_b^k\eta(b,a)|\\ &\le\int_0^{2\pi}|a^p(\partial_a^l\partial_b^k\eta(b-t,a)-\partial_a^l\partial_b^k\eta(b,a))\delta_n(t)|dt\\ &\le\|\eta\|_{\mathcal{S}(\mathbb{Y});p,l,k+1}\int_0^{2\pi}|t||\delta_n(t)|dt\\ &\le Ms(\delta_n)\|\eta\|_{\mathcal{S}(\mathbb{Y});p,l,k+1},\end{aligned} \tag{3.12}$$

which tends to 0 as $n\to\infty$. This completes the proof of the lemma. □

Lemma 3.7. *If $\eta_n\to\eta$ as $n\to\infty$ in $\mathcal{S}(\mathbb{Y})$ and $\psi\in C^\infty(\mathbb{T})$, then $\eta_n\odot\psi\to\eta\odot\psi$ as $n\to\infty$.*

Proof. Let $p,k,l\in\mathbb{N}_0$ be arbitrary. Now,

$$\begin{aligned}&|a^p\partial_a^l\partial_b^k(\eta_n\odot\psi-\eta\odot\psi)(b,a)|\\ &\quad\le\int_0^{2\pi}a^p|\partial_a^l\partial_b^k(\eta_n-\eta)(b,a)||\psi(t)|dt\\ &\quad\le\|\psi\|_{\mathcal{L}^1(\mathbb{T})}||\eta_n-\eta||_{\mathcal{S}(\mathbb{Y});p,l,k}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.\end{aligned} \tag{3.13}$$

Hence, the lemma follows. □

Lemma 3.8. *If $\eta_n\to\eta$ as $n\to\infty$ in $\mathcal{S}(\mathbb{Y})$ and $\delta_n\in\Delta$, then $\eta_n\odot\delta_n\to\eta$ as $n\to\infty$.*

Proof. Since we have $\eta_n\odot\delta_n-\eta=\eta_n\odot\delta_n-\eta\odot\delta_n+\eta\odot\delta_n-\eta$ and Lemma 3.6, we merely prove that $\eta_n\odot\delta_n-\eta\odot\delta_n\to 0$ as $n\to\infty$.

If $p,k,l\in\mathbb{N}_0$, then, using a property of delta-sequence, we get

$$\begin{aligned}&|a^p\partial_a^l\partial_b^k(\eta_n-\eta)\odot\delta_n(b,a)|\\ &\quad\le||\eta_n-\eta||_{\mathcal{S}(\mathbb{Y});p,l,k}\int_0^{2\pi}|\delta_n(t)|dt\le M||\eta_n-\eta||_{\mathcal{S}(\mathbb{Y});p,l,k}.\end{aligned} \tag{3.14}$$

The above inequalities prove the lemma. □

Now using the above lemmas we can construct the Boehmian space $\mathscr{B}_{\mathbb{Y}} = (\mathscr{S}_{\mathbb{Y}}, (C^\infty, *), \odot, \Delta)$ in a canonical way.

4. Generalized wavelet transform

Definition 4.1. Define $\mathscr{T}_g : \mathscr{B}_{\mathbb{T}} \to \mathscr{B}_{\mathbb{Y}}$ by

$$\mathscr{T}_g\left(\left[\frac{\phi_n}{\delta_n}\right]\right) = \left[\frac{T_g\phi_n}{\delta_n}\right]. \tag{4.1}$$

Theorem 4.2. *The generalized wavelet transform $\mathscr{T}_g : \mathscr{B}_{\mathbb{T}} \to \mathscr{B}_{\mathbb{Y}}$ is well defined.*

First, we state and prove a lemma that will be useful.

Lemma 4.3. *If $\phi, \psi \in C^\infty(\mathbb{T})$, then $T_g(\phi * \psi) = T_g\phi \odot \psi$.*

Proof. Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ be arbitrary. Now

$$\begin{aligned} T_g(\phi * \psi)(b,a) &= \int_0^{2\pi} (\phi * \psi)(x)\overline{g_a(x-b)}dx \\ &= \int_0^{2\pi} \overline{g_a(x-b)}dx \int_0^{2\pi} \phi(x-t)\psi(t)dt. \end{aligned} \tag{4.2}$$

By an easy verification, we can apply Fubini's theorem and the last integral equals

$$\begin{aligned} &\int_0^{2\pi} \psi(t)dt \int_0^{2\pi} \phi(x-t)\overline{g_a(x-b)}dx \\ &\quad = \int_0^{2\pi} \psi(t)dt \int_0^{2\pi} \phi(x)\overline{g_a(x-(b-t))}dx \\ &\quad = (T_g\phi \odot \psi)(b,a). \end{aligned} \tag{4.3}$$

□

Proof of Theorem 4.2. First, we show that $((T_g\phi_n), (\delta_n))$ is a quotient. Since $[\phi_n/\delta_n] \in \mathscr{B}_{\mathbb{T}}$, we have

$$\phi_k * \delta_j = \phi_j * \delta_k, \quad \forall j,k \in \mathbb{N}. \tag{4.4}$$

Applying the classical wavelet transform T_g on both sides, we get

$$T_g\phi_k \odot \delta_j = T_g\phi_j \odot \phi_k, \quad \forall j,k \in \mathbb{N} \text{ (by Lemma 4.3)}. \tag{4.5}$$

Next, we show that the definition of $\mathscr{T}_g$ is independent of the choice of the representative.

Let $[\phi_k/\epsilon_k] = [\psi_k/\delta_k]$ in $\mathscr{B}_{\mathbb{T}}$. Then, we have

$$\phi_k * \epsilon_j = \psi_j * \delta_k, \quad \forall j,k \in \mathbb{N}. \tag{4.6}$$

Again, applying the wavelet transform and using Lemma 4.3, we get

$$\mathcal{T}_g \phi_k \odot \epsilon_j = \mathcal{T}_g \psi_j \odot \delta_k, \quad \forall j,k \in \mathbb{N}. \tag{4.7}$$

Hence, the theorem follows. □

Theorem 4.4 (consistency). *Let $\mathcal{I}_{\mathbb{T}} : C^\infty(\mathbb{T}) \to \mathscr{B}_{\mathbb{T}}$ and $\mathcal{I}_{\mathbb{Y}} : \mathcal{S}(\mathbb{Y}) \to \mathscr{B}_{\mathbb{Y}}$ be the canonical identification defined, respectively, by*

$$\phi \longmapsto \left[\frac{\phi * \delta_n}{\delta_n}\right], \quad \eta \longmapsto \left[\frac{\eta \odot \delta_n}{\delta_n}\right], \tag{4.8}$$

where $(\delta_n) \in \Delta$, then $\mathcal{T}_g \circ \mathcal{I}_{\mathbb{T}} = \mathcal{I}_{\mathbb{Y}} \circ T_g$.

Proof. Let $\phi \in C^\infty(\mathbb{T})$, then

$$\begin{aligned}\mathcal{T}_g(\mathcal{I}_{\mathbb{T}}(\phi)) &= \mathcal{T}_g\left(\left[\frac{\phi * \delta_n}{\delta_n}\right]\right) = \left[\frac{T_g(\phi * \delta_n)}{\delta_n}\right] \\ &= \left[\frac{T_g \phi \odot \delta_n}{\delta_n}\right] \quad (\text{by Lemma 4.3}) \\ &= \mathcal{I}_{\mathbb{Y}}(T_g(\phi)).\end{aligned} \tag{4.9}$$

□

Theorem 4.5. *The wavelet transform $\mathcal{T}_g : \mathscr{B}_{\mathbb{T}} \to \mathscr{B}_{\mathbb{Y}}$ is a linear map.*

Proof. If $[\phi_n/\delta_n], [\psi_n/\epsilon_n] \in \mathscr{B}_{\mathbb{T}}$, then

$$\begin{aligned}\mathcal{T}_g\left(\left[\frac{\phi_n}{\delta_n}\right] + \left[\frac{\psi_n}{\epsilon_n}\right]\right) &= \mathcal{T}_g\left(\left[\frac{\phi_n * \epsilon_n + \psi_n * \delta_n}{\delta_n * \epsilon_n}\right]\right) = \left[\frac{T_g(\phi_n * \epsilon_n + \psi_n * \delta_n)}{\delta_n * \epsilon_n}\right] \\ &= \left[\frac{T_g \phi_n \odot \epsilon_n + T_g \psi_n \odot \delta_n}{\delta_n * \epsilon_n}\right] = \left[\frac{T_g \phi_n}{\delta_n}\right] + \left[\frac{T_g \psi_n}{\epsilon_n}\right] \\ &= \mathcal{T}_g\left(\left[\frac{\phi_n}{\delta_n}\right]\right) + \mathcal{T}_g\left(\left[\frac{\psi_n}{\epsilon_n}\right]\right).\end{aligned} \tag{4.10}$$

If $\alpha \in \mathbb{C}$ and $[\phi_n/\delta_n] \in \mathscr{B}_{\mathbb{T}}$, then

$$\begin{aligned}\mathcal{T}_g\left(\alpha\left[\frac{\phi_n}{\delta_n}\right]\right) &= \mathcal{T}_g\left(\left[\frac{\alpha\phi_n}{\delta_n}\right]\right) = \left[\frac{T_g(\alpha\phi_n)}{\delta_n}\right] = \left[\frac{\alpha T_g \phi_n}{\delta_n}\right] \\ &= \alpha\left[\frac{T_g \phi_n}{\delta_n}\right] = \alpha\mathcal{T}_g\left(\left[\frac{\phi_n}{\delta_n}\right]\right).\end{aligned} \tag{4.11}$$

In the above proof, we have used the fact that T_g is linear wherever it is required. □

From the following two theorems, we say that the generalized wavelet transform is continuous with respect to δ-convergence as well as Δ-convergence.

Theorem 4.6. *If $x_n \xrightarrow{\delta} x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{T}}$, then $\mathcal{T}_g x_n \xrightarrow{\delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{Y}}$.*

Proof. If $x_n \xrightarrow{\delta} x$ as $n \to \infty$, then, by Theorem 3.2, there exist $\phi_{n,k}, \phi_k \in C^\infty(\mathbb{T})$ and $(\delta_k) \in \Delta$ such that $x_n = [\phi_{n,k}/\delta_k]$ and $x = [\phi_k/\delta_k]$ and, for each $k \in \mathbb{N}$, $\phi_{n,k} \to \phi_k$ as $n \to \infty$ in $C^\infty(\mathbb{T})$.

By the continuity of the classical wavelet transform, we have, for each $k \in \mathbb{N}$,

$$T_g \phi_{n,k} \longrightarrow T_g \phi_k \quad \text{as } n \longrightarrow \infty \text{ in } \mathcal{S}_{\mathbb{Y}}. \tag{4.12}$$

Since $\mathcal{T}_g(x_n) = [T_g \phi_{n,k}/\delta_k]$ and $\mathcal{T}_g(x) = [T_g \phi_k/\delta_k]$, we get $\mathcal{T}_g(x_n) \xrightarrow{\delta} \mathcal{T}_g(x)$ as $n \to \infty$. Hence, the theorem follows. □

Theorem 4.7. *If $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{T}}$, then $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{Y}}$.*

Proof. Let $x_n \xrightarrow{\Delta} x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{T}}$. Then, by definition, we can find $\phi_n \in C^\infty(\mathbb{T})$ and $(\delta_n) \in \Delta$ such that $(x_n - x) * \delta_n = [\phi_n * \delta_k/\delta_k]$ and

$$\phi_n \longrightarrow 0 \quad \text{as } n \longrightarrow 0 \text{ in } C^\infty(\mathbb{T}). \tag{4.13}$$

Applying the classical wavelet transform and using Lemma 4.3, we get

$$T_g \phi_n \longrightarrow 0 \quad \text{as } n \longrightarrow 0 \text{ in } \mathcal{S}(\mathbb{Y}). \tag{4.14}$$

Hence, we get $\mathcal{T}_g x_n \xrightarrow{\Delta} \mathcal{T}_g x$ as $n \to \infty$ in $\mathcal{B}_{\mathbb{Y}}$. □

Lemma 4.8. *If $\eta \in \mathcal{S}(\mathbb{Y})$ and $\phi \in C^\infty(\mathbb{T})$, then $R_g(\eta \odot \phi) = R_g \eta * \phi$.*

Proof. Using Fubini's theorem, we get

$$\begin{aligned}
R_g(\eta \odot \phi)(x) &= \int_0^{2\pi} \int_0^\infty g_a(x-b)(\eta \odot \phi)(b,a) \frac{dadb}{a} \\
&= \int_0^{2\pi} \int_0^\infty g_a(x-b) \frac{dadb}{a} \int_0^{2\pi} \eta(b-t,a)\phi(t)dt \\
&= \int_0^{2\pi} \phi(t)dt \int_0^{2\pi} \int_0^\infty g_a(x-b)\eta(b-t,a) \frac{dadb}{a} \\
&= \int_0^{2\pi} \phi(t)dt \int_0^{2\pi} \int_0^\infty g_a((x-t)-c)\eta(c,a) \frac{dadc}{a} \quad (b-t=c) \\
&= \int_0^{2\pi} R_g \eta(x-t)\phi(t)dt \\
&= (R_g \eta * \phi)(x).
\end{aligned} \tag{4.15}$$

□

Therefore, we can give the following definition.

Definition 4.9. Define $\mathcal{R}_g : \mathcal{B}_{\mathbb{Y}} \to \mathcal{B}_{\mathbb{T}}$ by

$$\mathcal{R}_g\left(\left[\frac{\eta_n}{\delta_n}\right]\right) = \left[\frac{R_g\eta_n}{\delta_n}\right]. \tag{4.16}$$

Theorem 4.10. *The map $\mathcal{R}_g : \mathcal{B}_{\mathbb{Y}} \to \mathcal{B}_{\mathbb{T}}$ is linear.*

Theorem 4.11. *The map $\mathcal{R}_g : \mathcal{B}_{\mathbb{Y}} \to \mathcal{B}_{\mathbb{T}}$ is continuous with respect to δ-convergence as well as Δ-convergence.*

Using Lemma 4.8 and Theorem 2.4, we get a proof similar to that of Theorems 4.6 and 4.7.

Theorem 4.12 (an inversion formula)**.** *If $x = [\phi_n/\delta_n] \in \mathcal{B}_{\mathbb{T}}$ such that $\phi_n \in C^{\infty}_{+(-)}(\mathbb{T})$ for all $n \in \mathbb{N}$, then*

$$\mathcal{R}_g \circ \mathcal{T}_g(x) = C_g^{+(-)}x. \tag{4.17}$$

Proof. Now,

$$\begin{aligned}\mathcal{R}_g \circ \mathcal{T}_g(x) &= \mathcal{R}_g\left(\left[\frac{T_g\phi_n}{\delta_n}\right]\right) = \left[\frac{(R_g\circ T_g)\phi_n}{\delta_n}\right] \\ &= \left[\frac{C_g^{+(-)}\phi_n}{\delta_n}\right] = C_g^{+(-)}\left[\frac{\phi_n}{\delta_n}\right] = C_g^{+(-)}x.\end{aligned} \tag{4.18}$$

□

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