IJMMS 2003:2, 87–96
PII. S0161171203201150
http://ijmms.hindawi.com

BOUNDEDNESS FOR MULTILINEAR MARCINKIEWICZ OPERATORS ON CERTAIN HARDY SPACES

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Received 14 January 2002

The boundedness for the multilinear Marcinkiewicz operators on certain Hardy and Herz-Hardy spaces are obtained.

2000 Mathematics Subject Classification: 42B25, 42B20.

1. Introduction and definitions. Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be homogeneous of degree zero and satisfy the following two conditions:

(i) $\Omega(x)$ is continuous on S^{n-1} and satisfies the Lip_γ condition on S^{n-1} $(0 \leq \gamma \leq 1)$, that is,

$$|\Omega(x') - \Omega(y')| \leq M|x' - y'|^\gamma, \quad x', y' \in S^{n-1}; \tag{1.1}$$

(ii) $\int_{S^{n-1}} S(x')dx' = 0$.

Let m be a positive integer and A be a function on $\mathbb{R}^n$. The multilinear Marcinkiewicz integral operator is defined by

$$\mu_\Omega^A(f)(x) = \left[\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right]^{1/2}, \tag{1.2}$$

where

$$\begin{aligned} F_t^A(f)(x) &= \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)dy, \\ R_{m+1}(A;x,y) &= A(x) - \sum_{|\alpha|\leq m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\beta. \end{aligned} \tag{1.3}$$

We denote that $F_t(f)(x) = f_{|x-y|\leq t}(\Omega(x-y)/|x-y|^{n-1})f(y)dy$. We also denote that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2}, \tag{1.4}$$

which is the Marcinkiewicz integral operator (see [5, 6, 12]).

Note that when $m=0$, μ_Ω^A is just the commutator of Marcinkiewicz operator (see [5, 12]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 3, 4, 5]). The main purpose of this paper is to consider the continuity of the multilinear Marcinkiewicz operators on certain Hardy and Herz-Hardy spaces. We first introduce some definitions (see [7, 8, 9, 10, 11]).

Definition 1.1. Let A be a function on $\mathbb{R}^n$, m a positive integer, and $0<p\le 1$. A bounded measurable function a on $\mathbb{R}^n$ is said to be a (p, D^mA)-atom if

(i) $\operatorname{supp} a \subset B = B(x_0, r)$,
(ii) $\|a\|_{L^\infty} \le |B|^{-1/p}$,
(iii) $\int a(y)\,dy = \int a(y)D^\alpha A(y)\,dy = 0$, $|\alpha| = m$.

A temperate distribution f is said to belong to $H^p_{D^mA}(\mathbb{R}^n)$, if, in the Schwartz distributional sense, it can be written as

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x), \tag{1.5}$$

where a_j's are (p, D^mA)-atoms, $\lambda_j \in C$, and $\sum_{j=0}^{\infty}|\lambda_j|^p < \infty$. Moreover, $\|f\|_{H^p_{D^mA}(\mathbb{R}^n)} \sim (\sum_{j=0}^{\infty}|\lambda_j|^p)^{1/p}$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $C_k = B_k \setminus B_{k-1}$, $k \in \mathbb{Z}$, and $m_k(\lambda, f) = |\{x \in C_k : |f(x)| > \lambda\}|$; for $k \in \mathbb{N}$, let $\tilde{m}_k(\lambda, f) = m_k(\lambda, f)$ and $\tilde{m}_0(\lambda, f) = |\{x \in B_0 : |f(x)| > \lambda\}|$.

Definition 1.2. Let $0 < p$, $q < \infty$, and $\alpha \in \mathbb{R}$.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p} = \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \tag{1.6}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p \right]^{1/p}. \tag{1.7}$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} < \infty \right\}, \tag{1.8}$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} \|f\chi_k\|_{L^q}^p + \|f\chi_{B_0}\|_{L^q}^p \right]^{1/p}, \tag{1.9}$$

where

$$\|f\|_{WK_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left[\sum_{k=0}^{\infty} 2^{k\alpha p} \tilde{m}_k(\lambda, f)^{p/q} \right]^{1/p}. \tag{1.10}$$

Definition 1.3. Let m be a positive integer and A a function on $\mathbb{R}^n$, $\alpha \in \mathbb{R}$, and $1 < q \le \infty$. A function $a(x)$ on $\mathbb{R}^n$ is called a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type), if

(1) $\operatorname{supp} a \subset B(0, r)$ for some $r > 0$ (or for some $r \ge 1$),

(2) $\|a\|_{L^q} \le |B(0,r)|^{-\alpha/n}$,

(3) $\int a(x)dx = \int a(x) D^\beta A(x) dx = 0$, $|\beta| = m$.

A temperate distribution f is said to belong to $H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)$ (or $HK^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)$) if it can be written as $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(\mathbb{R}^n)$ sense, where a_j is a central $(\alpha, q, D^m A)$-atom (or a central $(\alpha, q, D^m A)$-atom of restrict type) supported on $B(0, 2^j)$ and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$ (or $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$). Moreover, $\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}}$ (or $\|f\|_{HK^{\alpha,p}_{q,D^m A}}$) $\sim (\sum_j |\lambda_j|^p)^{1/p}$.

2. Theorems and proofs.

We begin with some preliminary lemmas.

Lemma 2.1 (see [2]). *Let A be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for $|\alpha| = m$ and some $q > n$. Then,*

$$|R_m(A : x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q}, \tag{2.1}$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2.2. *Let $1 < p < \infty$ and $D^\alpha A \in L^r(\mathbb{R}^n)$, $|\alpha| = m$, $1 < r \le \infty$, and $1/q = 1/p + 1/r$. Then, μ_Ω^A is bound from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is,*

$$\|\mu_\Omega^A(f)\|_{L^q} \le C \sum_{|\alpha|=m} \|D^\alpha A\|_{L^r} \|f\|_{L^p}. \tag{2.2}$$

Proof. By Minkowski inequality and the condition of Ω, we have

$$\begin{aligned} \mu_\Omega^A(f)(x) &\le \int_{\mathbb{R}^n} \frac{|f(y)|\,|R_{m+1}(A; x, y)|}{|x-y|^m} \left(\int_{|x-y|}^{\infty} \frac{dt}{t^3} \right)^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|R_{m+1}(A; x, y)|}{|x-y|^{m+n}} |f(y)| dy. \end{aligned} \tag{2.3}$$

Thus, the lemma follows from [3, 4]. □

Theorem 2.3. *Let* $1 \geq p > n/(n+\gamma)$*, and let* $D^{\beta}A \in \mathrm{BMO}(\mathbb{R}^n)$ *for* $|\beta| = m$*. Then,* μ_{Ω}^{A} *is bounded from* $H^{p}_{D^m A}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$*.*

Proof. It suffices to show that there exists a constant $c > 0$ such that, for every $(p, D^m A)$-atom a,

$$||\mu_{\Omega}^{A}(a)||_{L^p} \leq C. \tag{2.4}$$

Let a be a $(p, D^m A)$-atom supported on a ball $B = B(x_0, r)$. We write

$$\begin{aligned}\int_{\mathbb{R}^n} [\mu_{\Omega}^{A}(a)(x)]^p dx &= \int_{|x-x_0|\leq 2r} [\mu_{\Omega}^{A}(a)(x)]^p dx \\ &\quad + \int_{|x-x_0|>2r} [\mu_{\Omega}^{A}(a)(x)]^p dx \\ &\equiv I + II.\end{aligned} \tag{2.5}$$

For I, taking $q > 1$ and by Hölder's inequality and the L^q-boundedness of μ_{Ω}^{A} (see Lemma 2.2), we see that

$$\begin{aligned}I &\leq C||\mu_{\Omega}^{A}(a)||_{L^q}^{p} \cdot |B(x_0, 2r)|^{1-p/q} \\ &\leq C\|a\|_{L^q}^{p} |B|^{1-p/q} \\ &\leq C.\end{aligned} \tag{2.6}$$

To obtain the estimate of II, we need to estimate $\mu_{\Omega}^{A}(a)(x)$ for $x \in (2B)^c$. Let $\tilde{B} = 5\sqrt{n}B$, and let $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m}(1/\alpha!)(D^{\alpha}A)_{\tilde{B}} \cdot x^{\alpha}$. Then, $R_m(A; x, y) = R_m(\tilde{A}; x, y)$. By the vanishing moment of a, we write

$$\begin{aligned}F_t^{A}(a)(x) &= \int_{|x-y|\leq t} \left[\frac{\Omega(x-y)}{|x-y|^{m+n-1}} - \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}}\right] R_m(\tilde{A}; x, y) a(y) dy \\ &\quad + \int_{|x-y|\leq t} \frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, x_0)] a(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{|x-y|\leq t} \frac{\Omega(x-y)(x-y)^{\alpha}}{|x-y|^{m+n-1}} \Big(D^{\alpha}A(y) - (D^{\alpha}A)_B\Big) a(y) dy,\end{aligned} \tag{2.7}$$

thus,

$$\begin{aligned}\mu_\Omega^A(a)(x) \le &\left[\int_0^\infty\left(\int_{|x-y|\le t}\left|\frac{\Omega(x-y)}{|x-y|^{m+n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}}\right|\right.\right.\\ &\qquad\left.\left.\times|R_m(\tilde{A};x,y)||a(y)|dy\right)^2\frac{dt}{t^3}\right]^{1/2}\\ &+\left[\int_0^\infty\left(\int_{|x-y|\le t}\frac{|\Omega(x-x_0)|}{|x-x_0|^{m+n-1}}\right.\right.\\ &\qquad\left.\left.\times|R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)||a(y)|dy\right)^2\frac{dt}{t^3}\right]^{1/2}\\ &+\left[\int_0^\infty\left|\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{|x-y|\le t}\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}}\right.\right.\\ &\qquad\left.\left.\times\left(D^\alpha A(y)-(D^\alpha A)_B\right)a(y)dy\right|^2\frac{dt}{t^3}\right]^{1/2}\\ &\equiv II_1+II_2+II_3.\end{aligned} \tag{2.8}$$

By Lemma 2.1, for $y \in B$ and $x \in 2^{k+1}B \setminus 2^kB$, we know

$$|R_m(\tilde{A};x,y)| \le C|x-y|^m \sum_{|\alpha|=m}|D^\alpha A(x)-(D^\alpha A)_{2^kB}|. \tag{2.9}$$

By the condition of Ω and Minkowski's inequality, and noting that $|x-y| \sim |x-x_0|$ for $y \in B$ and $x \in \mathbb{R}^n \setminus B$, we obtain

$$\left|\frac{\Omega(x-y)}{|x-y|^{m+n-1}}-\frac{\Omega(x-x_0)}{|x-x_0|^{m+n-1}}\right| \le C\left(\frac{r}{|x-x_0|^{m+n}}+\frac{r^\gamma}{|x-x_0|^{m+n+\gamma-1}}\right). \tag{2.10}$$

Thus,

$$\begin{aligned}II_1 &\le C\int_B|R_m(\tilde{A};x,y)||a(y)|\left(\int_{|x-y|}^\infty\frac{dt}{t^3}\right)^{1/2}\\ &\qquad\times\left(\frac{r}{|x-x_0|^{m+n}}+\frac{r^\gamma}{|x-x_0|^{m+n+\gamma-1}}\right)dy\\ &\le C\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}\right)|B|^{1-1/p}\sum_{|\alpha|=m}\left|D^\alpha A(x)-(D^\alpha A)_{2^kB}\right|.\end{aligned} \tag{2.11}$$

On the other hand, by the following formula (see [2]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};y,x_0)(x-x_0)^\beta \tag{2.12}$$

and Lemma 2.1, we get

$$\begin{aligned}&|R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)|\\&\qquad\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x_0-y|^{m-|\beta|}|x-x_0|^{|\beta|}\|D^\alpha A\|_{\mathrm{BMO}},\end{aligned} \tag{2.13}$$

so that

$$\begin{aligned}II_2&\le C\int_B|x-x_0|^{-(n+m)}\sum_{|\beta|<m}|R_{m-|\beta|}(D^\beta\tilde{A};y,x_0)||x-x_0|^{|\beta|}|a(y)|dy\\&\le C\int_B|x-x_0|^{-(n+m)}\sum_{|\beta|<m}|x_0-y|^{m-|\beta|}|x-x_0|^{|\beta|}\\&\qquad\times\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|a(y)|dy\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}\int_B\frac{|x_0-y|}{|x-x_0|^{n+1}}|a(y)|dy\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|x-x_0|^{-n-1}|B|^{1/n-1/p+1}.\end{aligned} \tag{2.14}$$

For II_3, and by the vanishing moment of a, we write,

$$\begin{aligned}&\int\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^m}\big(D^\alpha A(y)-(D^\alpha A)_B\big)a(y)dy\\&=\int\left[\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^m}-\frac{\Omega(x-x_0)(x-x_0)^\alpha}{|x-x_0|^{m+n-1}}\right]\big[D^\alpha A(y)-(D^\alpha A)_B\big]a(y)dy.\end{aligned} \tag{2.15}$$

Similar to the estimate of II_1, we obtain

$$\begin{aligned}II_3&\le C\sum_{|\alpha|=m}\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}\right)\\&\qquad\times\int_B|x_0-y|\left|D^\alpha A(y)-(D^\alpha A)_B\right||a(y)|dy\\&\le C\sum_{|\alpha|=m}\|D^\alpha A\|_{\mathrm{BMO}}|B|^{1-1/p}\left(\frac{r}{|x-x_0|^{n+1}}+\frac{r^\gamma}{|x-x_0|^{n+\gamma}}\right).\end{aligned} \tag{2.16}$$

Recalling that $p > n/(n+\gamma)$, therefore,

$$\begin{aligned} II &\le \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^kB} [\mu_\Omega^A(a)(x)]^p dx \\ &\le C \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^kB} \left(\frac{r}{|x-x_0|^{n+1}} + \frac{r^\gamma}{|x-x_0|^{n+\gamma}} \right)^p |B|^{p-1} \\ &\quad \times \left(\sum_{|\alpha|=m} \left| D^\alpha A(x) - (D^\alpha A)_{2^{k+1}B} \right| \right)^p dx \\ &\quad + C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \right)^p \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^kB} |x-x_0|^{-p(n+1)} |B|^{p(1+1/n-1/p)} dx \\ &\le C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \right)^p \sum_{k=1}^{\infty} (2^{k(n-p-pn)} + 2^{k(n-pn-p\gamma)}) \\ &\le C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\mathrm{BMO}} \right)^p, \end{aligned} \tag{2.17}$$

which, together with the estimate for I, yields the desired result. This finishes the proof of Theorem 2.3. □

Theorem 2.4. *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < n(1-1/q)+\gamma$, and $D^\beta A \in \mathrm{BMO}(\mathbb{R}^n)$ for $|\beta| = m$. Then, μ_Ω^A is bounded from $H\dot{K}_{q,D^mA}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

Proof. Let $f \in H\dot{K}_{q,D^mA}^{\alpha,p}(\mathbb{R}^n)$ and $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$ be the atomic decomposition for f as in Definition 1.3. We write

$$\begin{aligned} \|\mu_\Omega^A(f)\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} &\le C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-3} |\lambda_j| \|\mu_\Omega^A(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\ &\quad + C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|\mu_\Omega^A(a_j)\chi_k\|_{L^q} \right)^p \right]^{1/p} \\ &\equiv I + II. \end{aligned} \tag{2.18}$$

For II, and by the boundedness of μ_Ω^A on $L^q(\mathbb{R}^n)$ (see Lemma 2.2), we have

$$II \le C \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-2}^{\infty} |\lambda_j| \|a_j\|_{L^q} \right)^p \right]^{1/p}$$

$$
\begin{aligned}
&\le C\left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j| 2^{-j\alpha}\right)^p\right]^{1/p} \\
&\le C\begin{cases}\left[\displaystyle\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0<p\le 1 \\ \left[\displaystyle\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-2}^{\infty} |\lambda_j|^p 2^{-j\alpha p/2}\right)\left(\sum_{j=k-2}^{\infty} 2^{-j\alpha p'/2}\right)^{p/p'}\right]^{1/p}, & p>1\end{cases} \\
&\le C\begin{cases}\left[\displaystyle\sum_{j=-\infty}^{\infty} |\lambda_j|^p\left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p}\right)|\lambda_j|^p 2^{-j\alpha p}\right]^{1/p}, & 0<p\le 1 \\ \left[\displaystyle\sum_{j=-\infty}^{\infty} |\lambda_j|^p\left(\sum_{k=-\infty}^{j+2} 2^{(k-j)\alpha p/2}\right)\right]^{1/p}, & p>1\end{cases} \\
&\le C\left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \le C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)}.
\end{aligned}
\tag{2.19}
$$

For I, and similar to the proof of Theorem 2.3, we have, for $x \in C_k$, $j \le k-3$,

$$
\begin{aligned}
\mu_\Omega^A(a_j)(x) \le\; & C\left(|x|^{-n-m-1}|B_j|^{1/n} + |x|^{-n-m-\gamma}|B_j|^{\gamma/n}\right) \\
& \times\left(\int_{B_j} |a_j(y)|\,|R_m(\tilde{A};x,y)|\,dy\right) \\
& + C\sum_{|\beta|=m} ||D^\beta A||_{\text{BMO}} |x|^{-n-1}|B_j|^{1/n}\int_{B_j}|a(y)|\,dy \\
& + C\left(|x|^{-n-1}|B_j|^{1+1/n} + |x|^{-n-\gamma}|B_j|^{1+\gamma/n}\right) \\
& \quad\times\sum_{|\beta|=m}\int_{B_j}\left|D^\beta A(y) - (D^\beta A)_{B_j}\right| |a(y)|\,dy \\
\le\; & C\left(2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)} + 2^{-k(n+\gamma)}2^{j(\gamma+n(1-1/q)-\alpha)}\right) \\
& \times\left(\sum_{|\beta|=m}\left|D^\beta A(x) - (D^\beta A)_{B_k}\right|\right) \\
& + C\sum_{|\alpha|=m} ||D^\alpha A||_{\text{BMO}}(k-j) \\
& \qquad\times\left(2^{-k(n+1)}2^{j(1+n(1-1/q)-\alpha)} + 2^{-k(n+\gamma)}2^{j(\gamma+n(1-1/q)-\alpha)}\right)
\end{aligned}
\tag{2.20}
$$

thus,

$$\begin{aligned}
I \le{}& C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Bigg(\sum_{j=-\infty}^{k-3} |\lambda_j| \big(2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} + 2^{-k(n+\gamma)+j(\gamma+n(1-1/q)-\alpha)}\big) \\
&\qquad \times \sum_{|\alpha|=m}\left(\int_{B_k}\left|D^\alpha A(x)-(D^\alpha A)_{B_k}\right|^q dx\right)^{1/q}\Bigg)^p\Bigg]^{1/p} \\
&+ C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Bigg(\sum_{j=-\infty}^{k-3} |\lambda_j| (k-j)\big(2^{-k(n+1)+j(1+n(1-1/q)-\alpha)} \\
&\qquad + 2^{-k(n+\gamma)+j(\gamma+n(1-1/q)-\alpha)}\big) 2^{kn/q} \sum_{|\alpha|=m} ||D^\alpha A||_{\mathrm{BMO}}\Bigg)^p\Bigg]^{1/p} \\
\equiv{}& I_1 + I_2.
\end{aligned} \tag{2.21}$$

To estimate I_1 and I_2, we consider two cases.

Case 1 $(0 < p \le 1)$**.** We have

$$\begin{aligned}
I_1 \le{}& C\Bigg[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} \sum_{j=-\infty}^{k-3} |\lambda_j|^p \big(2^{[-k(n+1)+j(1+n(1-1/q)-\alpha)]p} \\
&\qquad + 2^{[-k(n+\gamma)+j(\gamma+n(1-1/q)-\alpha)]p}\big) 2^{knp/q}\left(\sum_{|\beta|=m} ||D^\beta A||_{\mathrm{BMO}}\right)^p\Bigg]^{1/p} \\
={}& C\sum_{|\beta|=m} ||D^\beta A||_{\mathrm{BMO}} \Bigg[\sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+3}^{\infty} \big(2^{(j-k)(1+n(1-1/q)-\alpha)p} \\
&\qquad + 2^{(j-k)(\gamma+n(1-1/q)-\alpha)p}\big)\Bigg]^{1/p} \\
\le{}& C\sum_{|\beta|=m} ||D^\beta A||_{\mathrm{BMO}} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\right)^{1/p} \\
\le{}& C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)}.
\end{aligned} \tag{2.22}$$

Similarly,

$$I_2 \le C\|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)}. \tag{2.23}$$

Case 2 $(p > 1)$**.** By Hölder's inequality, we deduce that

$$I_1 \le C \sum_{|\beta|=m} ||D^\beta A||_{\text{BMO}} \left[\sum_{j=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-3} |\lambda_j|^p 2^{(j-k)p(\gamma+n(1-1/q)-\alpha)/2} \right) \times \left(\sum_{j=-\infty}^{k-3} 2^{(j-k)p'(\gamma+n(1-1/q)-\alpha)/2} \right)^{p/p'} \right]^{1/p} \le C \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \le C \|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)}, \tag{2.24}$$

$$I_2 \le C \|f\|_{H\dot{K}^{\alpha,p}_{q,D^m A}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2. □

Remark 2.5. Theorem 2.4 also holds for nonhomogeneous Herz-type space.

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