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QUASIREDUCIBLE OPERATORS

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We introduce the concept of *quasireducible* operators. Basic properties and illustrative examples are considered in some detail in order to situate the class of quasireducible operators in its due place. In particular, it is shown that *every quasinormal operator is quasireducible*. The following result links this class with the invariant subspace problem: *essentially normal quasireducible operators have a nontrivial invariant subspace*, which implies that *quasireducible hyponormal operators have a nontrivial invariant subspace.* The paper ends with some open questions on the characterization of the class of all quasireducible operators.

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1. Introduction. Let $\mathcal{H}$ be a complex Hilbert space of dimension greater than one. By an operator on $\mathcal{H}$, we mean a bounded linear transformation of $\mathcal{H}$ into itself. A subspace $\mathcal{M}$ of $\mathcal{H}$ is a closed linear manifold of $\mathcal{H}$. It is nontrivial if $\mathcal{M}$ is nontrivial if $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. If T is an operator on $\mathcal{H}$ and $T(\mathcal{M}) \subseteq \mathcal{M}$, then $\mathcal{M}$ is invariant for T (or $\mathcal{M}$ is T-invariant), and hyperinvariant for T if it is invariant for every operator that commutes with T. If $\mathcal{M}$ is a nontrivial invariant subspace for T, then its orthogonal complement $\mathcal{M}^\perp = \mathcal{H} \ominus \mathcal{M}$ is a nontrivial invariant subspace for the adjoint T^* of T. If $\mathcal{M}$ is invariant for both T and T^* (equivalently, if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are T-invariant), then $\mathcal{M}$ reduces T (or $\mathcal{M}$ is a reducing subspace for T). An operator is reducible if it has a nontrivial reducing subspace or, equivalently, if it is the (orthogonal) direct sum of two operators on nonzero subspaces. Recall that a scalar operator is a (complex) multiple of the identity, and also that a projection is an idempotent operator whose range and kernel are orthogonal to each other. A projection P is nontrivial if $O \neq P \neq I$, where O and I denote the null operator and the identity, respectively. We begin with a well-known result on the characterization of reducible operators (see, e.g., [4, page 159]), which helps in the definition of quasireducible operators.

Proposition 1.1. *Let T be an operator. The following assertions are pairwise equivalent:*

(a) *T is reducible;*
(b) *T commutes with a nontrivial projection;*
(c) *T commutes with a nonscalar normal operator;*
(d) *there exists a nonscalar operator that commutes with T and with T^*.*

Thus, an operator T is reducible if and only if there exists a nonscalar operator L such that $LT = TL$ and $T^*L - LT^* = O$, that is, if and only if there exists a nonscalar operator L in $\{T\}' \cap \{T^*\}'$, where $\{T\}'$ denotes the commutant of T.

Definition 1.2. An operator T is *quasireducible* if there exists a nonscalar L such that

$$LT = TL, \qquad \operatorname{rank}\left(\left(T^*L - LT^*\right)T - T\left(T^*L - LT^*\right)\right) \leq 1. \tag{1.1}$$

In other words, T is quasireducible if there exists a nonscalar L in $\{T\}'$ such that either $T^*L - LT^*$ also lies in $\{T\}'$ or the commutator $[(T^*L - LT^*), T]$ is a rank-one operator.

Clearly, *every reducible operator is quasireducible*. Here is an alternative characterization of quasireducibility. Let D_T denote the self-commutator of T:

$$D_T = [T^*, T] = T^*T - TT^*. \tag{1.2}$$

If $LT = TL$, then $D_TL - LD_T = (T^*L - LT^*)T - T(T^*L - LT^*)$ so that *T is quasireducible if and only if there exists a nonscalar L such that*

$$LT = TL, \qquad \operatorname{rank}\left(D_TL - LD_T\right) \leq 1. \tag{1.3}$$

Elementary facts about quasireducibility, which will be needed in the sequel, are stated in Propositions 1.3 and 1.4. Proposition 1.4 says that quasireducibility (as reducibility) is preserved under unitary equivalence. Their proofs are straightforward, hence omitted.

Proposition 1.3. *If T is quasireducible, then*
(a) *λT is quasireducible for every $\lambda \in \mathbb{C}$;*
(b) *$\lambda I + T$ is quasireducible for every $\lambda \in \mathbb{C}$;*
(c) *T^* is quasireducible.*

Proposition 1.4. *Every operator unitarily equivalent to a quasireducible operator is quasireducible.*

2. Finite-dimensional examples.

First, we will exhibit a nonquasireducible operator that is similar to a reducible one. Thus, (as reducibility) quasireducibility also is not preserved under similarity.

Example 2.1. Set $\mathcal{H} = \mathbb{C}^3$ and identify the operators on $\mathbb{C}^3$ with their matrices with respect to the canonical basis for $\mathbb{C}^3$. Let

$$T = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{so that } D_T = \begin{pmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 2 & -1 & 0 \end{pmatrix}. \tag{2.1}$$

Any nonscalar L that commutes with T is of the form

$$L = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha-\gamma & 0 \\ 0 & -\beta & \alpha-\gamma \end{pmatrix}, \tag{2.2}$$

where β and γ cannot be both null. Thus

$$D_T L - L D_T = \begin{pmatrix} \beta-2\gamma & 2\gamma-6\beta & \beta-4\gamma \\ -\gamma & 0 & -\gamma \\ 2\gamma-\beta & 4\beta & 2\gamma-\beta \end{pmatrix}, \tag{2.3}$$

and hence $\operatorname{rank}(D_T L - L D_T) \geq 2$ for every nonscalar L that commutes with T. Outcome: T is not reducible. Now, put

$$\widetilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad W = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.4}$$

so that W is invertible and $WT = \widetilde{T}W$. Therefore, the reducible $\widetilde{T} = 1 \oplus \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ is similar to T, which is not even quasireducible.

An operator T is nilpotent if $T^n = O$ for some positive integer n. The least integer n such that $T^n = O$ is the nilpotence index of T.

Proposition 2.2. *Let T be an operator acting on an arbitrary Hilbert space $\mathcal{H}$. If T is a nilpotent operator of index $n+1$ for some $n \geq 1$, then either T^n is reducible or T is quasireducible with nilpotence index 2 on a two-dimensional space.*

Proof. Take a nonzero operator T on $\mathcal{H}$ and let $\mathcal{N}(T)$ denote the null space (kernel) of T. Since $\mathcal{N}(T)$ is T-invariant, we may write

$$T = \begin{pmatrix} O & X \\ O & Y \end{pmatrix} \quad \text{so that } T^n = \begin{pmatrix} O & XY^{n-1} \\ O & Y^n \end{pmatrix} \quad \text{for every } n \geq 1, \tag{2.5}$$

with respect to the decomposition $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^{\perp}$, where $X : \mathcal{N}(T)^{\perp} \to \mathcal{N}(T)$ and $Y : \mathcal{N}(T)^{\perp} \to \mathcal{N}(T)^{\perp}$ are bounded and linear. If $T^{n+1} = TT^n = O$ and $T^n \neq O$, then the invariant subspace $\mathcal{N}(T)$ is nontrivial (0 is an eigenvalue of T), so that both $\mathcal{N}(T)$ and $\mathcal{N}(T)^{\perp}$ are nonzero, and $Y^n = O$, $Y^{n-1} \neq O$, and $X \neq O$. Hence

$$T^n = \begin{pmatrix} O & Z \\ O & O \end{pmatrix} \quad \text{with } Z = XY^{n-1} : \mathcal{N}(T)^{\perp} \longrightarrow \mathcal{N}(T). \tag{2.6}$$

Therefore, with respect to the same decomposition $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^{\perp}$, set

$$Q = \begin{pmatrix} ZZ^* & O \\ O & Z^*Z \end{pmatrix} \quad \text{so that } QT^n = T^nQ = \begin{pmatrix} O & ZZ^*Z \\ O & O \end{pmatrix}, \tag{2.7}$$

where $Z^* : \mathcal{N}(T) \to \mathcal{N}(T)^\perp$ is the adjoint of Z. If the nonnegative Q is nonscalar, then T^n is reducible. Suppose that Q is scalar. In this case, $Z = \lambda^{1/2}U$ for some positive scalar λ and some unitary transformation U so that $\mathcal{N}(T)$ and $\mathcal{N}(T)^\perp$ are unitarily equivalent, and hence $\dim\mathcal{N}(T) = \dim\mathcal{N}(T)^\perp$. Now, take an arbitrary operator $A : \mathcal{N}(T) \to \mathcal{N}(T)$ and set, still on $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$,

$$N = \begin{pmatrix} A & O \\ O & \lambda^{-1}Z^*AZ \end{pmatrix} \quad \text{so that } NT^n = T^nN = \begin{pmatrix} O & AZ \\ O & O \end{pmatrix}. \tag{2.8}$$

If $\dim\mathcal{N}(T) \geq 2$, then let A be a nonscalar normal operator so that N is a nonscalar normal operator as well, and therefore T^n is reducible. If $\dim\mathcal{N}(T) = 1$, then $\dim\mathcal{H} = 2$. In this case, we may assume without loss of generality that $T^n = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathbb{C}^2$. This implies that $n = 1$, and hence any nonscalar L that commutes with T is of the form $L = \left(\begin{smallmatrix} \alpha & \beta \\ 0 & \alpha \end{smallmatrix}\right)$ with $\beta \neq 0$, which is never normal. Thus, T is irreducible. However, T is quasireducible because $\operatorname{rank}(D_TL - LD_T) = 1$. □

Particular case ($n = 1$): every nilpotent operator of index 2 is quasireducible. In fact, a nilpotent operator of index 2 acting on a Hilbert space of dimension greater than two is reducible; on a two-dimensional space, it is irreducible but quasireducible.

Remark 2.3. It is worth noticing that nilpotent operators of higher index are not necessarily quasireducible. Sample:

$$T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.9}$$

on $\mathbb{C}^3$ is a nilpotent of index 3 that is not quasireducible. In fact, any L that commutes with T is of the form

$$L = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{pmatrix} \quad \text{so that } D_TL - LD_T = \begin{pmatrix} \beta & -2\beta-\gamma & -2\beta-4\gamma \\ 0 & -2\beta & -2\beta-\gamma \\ 0 & 0 & \beta \end{pmatrix}, \tag{2.10}$$

and hence $\operatorname{rank}(D_TL - LD_T) \geq 2$ whenever L is nonscalar.

3. Infinite-dimensional examples. Recall that an operator T is quasinormal if it commutes with T^*T, subnormal if it has a normal extension (i.e., if it is the restriction of a normal operator to an invariant subspace), and hyponormal if D_T is nonnegative. These classes are related by proper inclusion (normal $\subset$ quasinormal $\subset$ subnormal $\subset$ hyponormal) if $\mathcal{H}$ is an infinite-dimensional space (otherwise, they all coincide with the class of normal operators). The techniques applied in this section are all standard from single-operator theory and can be found in many sources (see, e.g., [3, 4, 6, 9, 10, 15]).

Proposition 3.1. *Every quasinormal operator is quasireducible.*

Proof. We will split the proof into four parts.

(a) A normal operator is trivially reducible, and hence trivially quasireducible.

(b) A pure isometry (i.e., a completely nonunitary isometry) is precisely a unilateral shift (by the von Neumann-Wold decomposition). If its multiplicity is greater than one, then it is the direct sum of two unilateral shifts, thus reducible. If it is of multiplicity one, then it is not reducible but quasireducible. Indeed, if S_+ is a unilateral shift of multiplicity one, then $(S_+^*S_+ - S_+S_+^*)S_+ - S_+(S_+^*S_+ - S_+S_+^*) = S_+(S_+S_+^* - I)$ is a rank-one operator.

(c) The von Neumann-Wold decomposition says that every isometry is the direct sum of a unilateral shift and a unitary operator (i.e., a normal isometry), where any of the direct summands may be missing. Thus, parts (a) and (b) ensure that every isometry is quasireducible and so is every multiple of an isometry (Proposition 1.3(a)).

(d) An operator T is a multiple of an isometry if and only if the nonnegative operator T^*T is scalar (reason: an isometry is precisely an operator V such that $V^*V = I$). If T is quasinormal but not a multiple of an isometry, then T^*T is a nonscalar normal operator in $\{T\}'$ by the very definition of quasinormality. Thus, T is reducible, and hence quasireducible. □

A unilateral weighted shift with weight sequence $\{\omega_k\}_{k\geq 1}$ is unitarily equivalent to the unilateral weighted shift with weight sequence $\{|\omega_k|\}_{k\geq 1}$. Therefore, according to Proposition 1.4, there is no loss of generality in assuming weighted shifts with nonnegative weights as far as quasireducibility is concerned. Thus, from now on, all weight sequences will be assumed nonnegative. We will say that a diagonal operator has *multiplicity one* if the diagonal sequence is made up of distinct elements.

Proposition 3.2. *Every injective unilateral weighted shift whose self-commutator has multiplicity one is not quasireducible.*

Proof. Let $W_+ = \text{shift}(\{\omega_k\}_{k\geq 1})$ be a unilateral weighted shift on ℓ_+^2 with weight (nonnegative) sequence $\{\omega_k\}_{k\geq 1}$ so that $W_+^*W_+ = \text{diag}(\omega_1^2, \omega_2^2, \omega_3^2, \dots)$ and $W_+W_+^* = \text{diag}(0, \omega_1^2, \omega_2^2, \omega_3^2, \dots)$, and hence

$$D_{W_+} = W_+^*W_+ - W_+W_+^* = \text{diag}\left(\{\delta_k\}_{k\geq 1}\right), \tag{3.1}$$

a diagonal operator on ℓ_+^2 whose diagonal entries are $\delta_1 = \omega_1^2$ and $\delta_{k+1} = \omega_{k+1}^2 - \omega_k^2$ for every $k \geq 1$. Recall that W_+ is injective if and only if $W_+^*W_+$ is injective, which means that $\omega_k \neq 0$ for every $k \geq 1$. Let A be an arbitrary operator on ℓ_+^2 and identify it with the (infinite) matrix $[\alpha_{j,k}]_{j,k\geq 1}$ that represents it with respect to the canonical basis for ℓ_+^2.

Claim 1. If $D = \text{diag}(\{\lambda_k\}_{k\geq 1})$ is a diagonal of multiplicity one, then A commutes with D if and only if A is a diagonal.

Proof. This is readily verified once $DA-AD=[\alpha_{j,k}(\lambda_j-\lambda_k)]_{j,k\geq 1}$. □

Claim 2. If A commutes with an injective unilateral weighted shift W_+, then A is lower triangular (i.e., all entries above the main diagonal are zero) with a constant main diagonal. Moreover, the entries of each lower diagonal (i.e., of each diagonal below the main diagonal) are either all zero or all nonzero.

Proof. If $W_+A=AW_+$ and $\omega_k\neq 0$ for every $k\geq 1$, then (see [15, page 53])

$$\alpha_{j+i+1,j+1}=\frac{\omega_{j+i}}{\omega_j}\alpha_{j+i,j} \tag{3.2}$$

for every $j\geq 1$ and $i\geq 0$. Moreover, it is readily verified that A is lower triangular. For $i=0$, we get $\alpha_{j+1,j+1}=\alpha_{j,j}$, which means that A has a constant main diagonal. For any $i\geq 1$, this ensures that either the sequence $\{\alpha_{j+i,j}\}_{j\geq 1}$ is null or $\alpha_{j+i,j}\neq 0$ for every $j\geq 1$. That is, the ith lower diagonal of A is either zero or entirely made of nonzero entries. □

Claim 3. If a lower triangular A with lower diagonals either zero or entirely nonzero does not commute with a diagonal D of multiplicity one, then $DA-AD$ is not finite rank.

Proof. Let $e_k=(0,\dots,0,1,0,\dots)$, with the only nonzero entry equal to 1 at the kth position, be an arbitrary element of the canonical basis for ℓ^2_+. Recall that $DA-AD=[\alpha_{j,k}(\lambda_j-\lambda_k)]_{j,k\geq 1}$ for any diagonal $D=\operatorname{diag}(\{\lambda_k\}_{k\geq 1})$. If A is lower triangular (i.e., $\alpha_{j,k}=0$ whenever $j<k$), then $(DA-AD)e_k$ coincides with the kth column of the lower triangular $DA-AD$. Since A does not commute with D, it follows by Claim 1 that A is not a diagonal: there exists $\alpha_{j,k}\neq 0$ for some pair (j,k) with $j>k\geq 2$. Consequently, the lower diagonal to which it belongs is entirely nonzero. Since D is a diagonal of multiplicity one (i.e., $\lambda_j\neq\lambda_k$ whenever $j\neq k$), it follows that the lower triangular $DA-AD$ has at least one lower diagonal made up of nonzero entries. Hence, $\bigvee\{(DA-AD)e_k\}_{k\geq 1}$ is an infinite-dimensional subspace of $\operatorname{range}(DA-AD)$ so that $DA-AD$ is not finite rank. □

Outcome. Let W_+ be an injective unilateral weighted shift whose self-commutator $D_{W_+}=W_+^*W_+-W_+W_+^*$ has multiplicity one. If A commutes with W_+ and with D_{W_+}, then A is scalar (Claims 1 and 2). If A commutes with W_+ but does not commute with D_{W_+}, then $D_{W_+}A-AD_{W_+}$ is not finite rank (Claims 2 and 3). □

4. Invariant subspaces. In this section, we investigate a relationship between the concept of quasireducibility and two major invariant subspace results, namely, Lomonosov's for compact operators and Berger-Shaw's for hyponormal operators. As usual, if an operator T has a compact self-commutator D_T, then T is called essentially normal.

Theorem 4.1. *Every essentially normal quasireducible operator has a nontrivial invariant subspace.*

Proof. The Lomonosov theorem [11] (see also [14, page 158] or [12, page 42]) says that if a nonscalar operator commutes with a nonzero compact operator, then it has a nontrivial hyperinvariant subspace. A nice generalization of it was considered in [5, 7] (see also [8]): if an operator L is such that $\operatorname{rank}(KL-LK)=1$ for some compact operator K, then L has a nontrivial hyperinvariant subspace. If there exists a nonscalar L such that $LT=TL$ and $\operatorname{rank}(KL-LK)\le 1$ for some nonzero compact operator K, then the above two results ensure that T has a nontrivial invariant subspace. This proves the theorem whenever the self-commutator $D_T=[T^*,T]$ is nonzero and compact. If $D_T=O$, then T is normal and the result holds trivially. □

Corollary 4.2. *Every quasireducible hyponormal operator has a nontrivial invariant subspace.*

Proof. The Berger-Shaw theorem [1, 2] (see also [4, page 152]) ensures that if a hyponormal operator T has no nontrivial invariant subspace, then its self-commutator D_T is a trace-class operator, and hence compact, that is, T is essentially normal. Therefore, if a hyponormal operator has no nontrivial invariant subspace, then it is not quasireducible by the above theorem. □

5. Open questions. Apparently, there is a gap between the classes of reducible and quasireducible operators. Consider the class $\mathscr{C}$ of all operators T for which there exists a nonscalar L such that

$$LT=TL, \qquad D_TL=LD_T. \tag{5.1}$$

Clearly, $\mathscr{C}$ includes the class of all reducible operators and is included in the class of all quasireducible operators:

$$\text{Reducible} \subseteq \mathscr{C} \subset \text{Quasireducible}. \tag{5.2}$$

Note that the second inclusion is, in fact, proper (i.e., there exist quasireducible operators not in $\mathscr{C}$). For instance, take any quasireducible operator T for which $\operatorname{rank}(D_TL-LD_T)\ge 1$ for every nonscalar L in $\{T\}'$ (samples are unilateral shift of multiplicity one or, simply, $T=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ on $\mathbb{C}^2$).

Question 5.1. Does the class $\mathscr{C}$ coincide with the class of all reducible operators?

In other words, *is it true that if T is irreducible, then* $\operatorname{rank}(D_TL-LD_T)\ge 1$ *for every nonscalar L in* $\{T\}'$? There are many ways to reformulate the above question. We first consider the following proposition.

Proposition 5.2. *An operator T is reducible if and only if there exits a nonscalar L such that*

(a) $LT = TL$, $D_T L = LD_T$, *and*
(b) $(T^*D_L - D_L T^*)T = T(T^*D_L - D_L T^*)$.

Proof. If T is reducible, then there exists a nonscalar L in $\{T\}' \cap \{T^*\}'$. Thus, assertions (a) and (b) hold trivially. Conversely, take a nonscalar L and set $C = T^*L - LT^*$. Recall that assertion (a) is equivalent to $LT = TL$ and $CT = TC$. Hence, if (a) holds,

$$\begin{aligned} D_C = C^*C - CC^* &= (L^*T - TL^*)C - C(L^*T - TL^*) \\ &= (L^*C - CL^*)T - T(L^*C - CL^*). \end{aligned} \tag{5.3}$$

However, as $L^*T^* = T^*L^*$,

$$L^*C - CL^* = L^*(T^*L - LT^*) - (T^*L - LT^*)L^* = T^*D_L - D_L T^*. \tag{5.4}$$

Therefore, if assertion (b) also holds, $D_C = O$, that is, C is normal. If C is nonscalar, then T is reducible (since $CT = TC$). If C is scalar, then $C = O$ (reason: C is a commutator, and nonzero commutators are nonscalar—see, e.g., [6, page 128]), and hence the nonscalar L lies in $\{T\}' \cap \{T^*\}'$, that is, T is reducible. □

Thus, Question 5.1 can be rewritten as: *can we drop assertion* (b) *from the statement of* Proposition 5.2? Equivalently, *does there exist a nonscalar normal in* $\{T\}'$ *whenever there exists a nonscalar* L *that satisfies assertion* (a)? Another way to look upon the same question: fix an operator T and consider the unital algebra $\mathcal{A}_T$ of all operators that commute with T and with D_T:

$$\mathcal{A}_T = \{L : LT = TL,\ D_T L = LD_T\} = \{T\}' \cap \{D_T\}'. \tag{5.5}$$

Let $\mathbb{C}$ denote the trivial unital algebra of all scalar operators so that the inclusions $\mathbb{C} \subseteq \mathcal{A}_T \subseteq \{T\}'$ hold trivially. It is readily verified that $\mathcal{A}_T = \{T\}'$ if and only if T is normal. Indeed, if $D_T = O$, then $\mathcal{A}_T = \{T\}'$ and, conversely, if $\{T\}' \subseteq \mathcal{A}_T$, then T lies in $\mathcal{A}_T$ so that T commutes with D_T, which means that T is normal (see, e.g., [13, page 5]). On the opposite end, if $\mathbb{C} = \mathcal{A}_T$, then T is irreducible. In fact, if T is reducible, then any nonscalar L in $\{T\}' \cap \{T^*\}'$ lies in $\mathcal{A}_T$. The converse holds if and only if Question 5.1 has an affirmative answer: *is it true that* $\mathbb{C} = \mathcal{A}_T$ *for every irreducible* T?

Both product and (ordinary) sum of quasireducible operators are not necessarily quasireducible. For instance, set

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{so that } T^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.6}$$

The operator T^2 is a nilpotent of index 2 on $\mathbb{C}^3$ (thus reducible) and T is quasireducible (since $\operatorname{rank}(T^2D_T - D_T T^2) = 1$) so that $I + T$ is quasireducible by Proposition 1.3(b). However, $T(I+T) = T+T^2$, which is both a product and a sum of quasireducible operators, is not quasireducible (cf. Remark 2.3). Is the square of a quasireducible operator quasireducible?

Question 5.3. Is T^n quasireducible for every integer $n \geq 1$ whenever T is quasireducible?

Observe that there exist operators for which all (positive) powers are not quasireducible. Sample: $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ on $\mathbb{C}^2$ is idempotent and not quasireducible (actually, $D_T L - L D_T$ is full rank for every nonscalar L in $\{T\}'$), and hence every polynomial of T is not quasireducible by Proposition 1.3(a), (b). This prompts our final question.

Question 5.4. If every polynomial of T is quasireducible, must T be reducible?

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