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THE CONJUGATION OPERATOR ON $A_q(G)$

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Let $q > 2$. We prove that the conjugation operator H does not extend to a bounded operator on the space of integrable functions defined on any compact abelian group with the Fourier transform in l_q.

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Let G be a compact abelian group with dual Γ. For $1 \le q < \infty$, the space A_q is defined as

$$A_q(G) = \{f : f \in L^1(G),\ \hat{f} \in l_q(\Gamma)\} \tag{1}$$

with the norm $\|f\|_{A_q} = \|f\|_{L^1} + \|\hat{f}\|_{l_q}$. Then $A_q(G)$ is a commutative semisimple Banach algebra with maximal ideal space Γ, in which the set of trigonometric polynomials is dense [4]. The A_p-spaces have been studied in [1, 6].

If G is, in addition, a connected group, then its dual can be ordered; there exists a semigroup $P \subset \Gamma$ such that $P \cap -P = \{0\}$, $P \cup -P = \Gamma$ (see [5]), and we say that $\gamma \in \Gamma$ is positive if $\gamma \in P$. If $f = \sum_{\gamma \in F} \hat{f}(\gamma)\gamma$ is a trigonometric polynomial, the conjugation operator H is defined as

$$Hf = \sum_{\gamma \in F} \operatorname{sgn}(\gamma)\hat{f}(\gamma)\gamma, \tag{2}$$

where $\operatorname{sgn}(\gamma) = +1$ if $\gamma \in P$, -1 if $\gamma \in -P$, and 0 if $\gamma = 0$.

If $1 \le q \le 2$, then $A_q(G) \subset L^2(G)$, and it is easy to see that H extends to a bounded operator on $A_q(G)$. It was mentioned in [5] that the corresponding result for $q > 2$ is not known. Note that H extends to a bounded operator on $A_q(G)$ if and only if H extends to a bounded operator from $A_q(G)$ to $L^1(G)$. In [5], the following theorem was proved.

Theorem 1. *Let G be a compact, connected abelian group and P any fixed order on Γ. If $q > 2$ and ϕ is a Young's function, then the conjugation operator H does not extend to a bounded operator from $A_q(G)$ to $L^\phi(G)$.*

We prove in Theorem 2 that H does not extend to a bounded operator on $A_q(G)$, $q > 2$, thus answering the problem mentioned in [5]. Also, Theorem 1 follows from our theorem (Theorem 2). Theorem 2 was proved for the circle group in [2] but for the completeness sake, we give it below.

Theorem 2. *Let G be a compact, connected abelian group and P any fixed order on Γ. If $q > 2$, then the conjugation operator H does not extend to a bounded operator on $A_q(G)$.*

Proof. We will show that H does not extend to a bounded operator from $A_q(G)$ to $L^1(G)$. The proof is divided into three steps.

Step 1. Let $G = \mathbf{T}$, the circle group. Suppose that H extends as a bounded operator from A_q to L^1. Choose $\mu_0 \in M(\mathbf{T})$, $\hat{\mu}_0 \in l_q$ such that μ_0 is not absolutely continuous. Define $T : L^1 \to L^1$ by

$$Tf = H(f * \mu_0), \tag{3}$$

where T is well defined as $f * \mu_0 \in A_q$ and H maps A_q into L^1 by our assumption on H. Hence, T is a multiplier from L^1 to L^1, and therefore is given by a measure $\mu \in M(\mathbf{T})$ (see [3]). Hence

$$\operatorname{sgn}(n)\hat{\mu}_0(n) = \hat{\mu}(n). \tag{4}$$

Observe that

$$\hat{\mu}_0 = \frac{\hat{\mu}_0 + \hat{\mu}}{2} + \frac{\hat{\mu}_0 - \hat{\mu}}{2}. \tag{5}$$

Now, $(\hat{\mu}_0 + \hat{\mu})/2$ and $(\hat{\mu}_0 - \hat{\mu})/2$ are absolutely continuous by F. and M. Riesz theorem. Hence, $\hat{\mu}_0$ is absolutely continuous, which contradicts the choice of μ_0. Hence, H is unbounded on A_q, $q > 2$.

Step 2. Let I be a closed subgroup of G such that H does not extend as a bounded operator on $A_q(G/I)$. Then H does not extend as a bounded operator on $A_q(G)$.

Proof. Let (f_n) be a sequence of trigonometric polynomials on G/I such that

$$\|Hf_n\|_{L^1(G/I)} \longrightarrow \infty, \quad \|f_n\|_{A_q(G/I)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{6}$$

Let $g_n = f_n \circ q$, where $q : G \to G/I$ is the quotient map. Then it is easily seen that $Hg_n = (Hf_n) \circ q$, $\|Hg_n\|_{L^1(G)} = \|Hf_n\|_{L^1(G/I)}$, and $\|f_n \circ q\|_{A_q(G)} = \|f_n\|_{A_q(G/I)}$. Hence

$$\|Hg_n\|_{L^1(G)} \longrightarrow \infty, \quad \|g_n\|_{A_q(G)} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{7}$$

and Step 2 follows. □

Step 3. Since G is connected, Γ contains an element of infinite order, say γ_0 (see [3]). Let S denote the subgroup generated by γ_0 and let $I = S^{\perp}$. Then G/H is isomorphic to the circle group $\mathbf{T}$. Now, the proof of the theorem follows from Steps 1 and 2. □

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