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ON DEFINING THE PRODUCT $r^{-k} \cdot \nabla^l \delta$

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Let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0,\infty)$ having the properties: (i) $\rho(s) \ge 0$, (ii) $\rho(s) = 0$ for $s \ge 1$, and (iii) $\int_{R^m} \delta_n(x)dx = 1$ where $\delta_n(x) = c_m n^m \rho(n^2 r^2)$ and c_m is the constant satisfying (iii). We overcome difficulties arising from computing $\nabla^l \delta_n$ and express this regular sequence by two mutual recursions and use a Java swing program to evaluate corresponding coefficients. Hence, we are able to imply the distributional product $r^{-k} \cdot \nabla^l \delta$ for $k = 1,2,\dots$ and $l = 0,1,2,\dots$ with the help of Pizetti's formula and the normalization.

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1. Introduction. The sequential approach has been one of the main tools in dealing with products, powers, and convolutions of distributions introduced by L. Schwartz in 1951 as linear and continuous functionals on the testing function space whose elements have compact support. In 1972, Antosik, Mikusiński, and Sikorski gave a definition for a product of distributions using a delta sequence. However, δ^2 as a product of δ with itself was unable to be defined. Fisher [2] has actively used Jones' δ-sequence of one variable and the concept of neutrix limit of van der Corput [7] to deduce numerous products, powers, convolutions, and compositions of distributions on R since 1969. To extend such an approach from one-dimensional to m-dimensional, Li and Fisher [6] constructed several "useful" δ-sequences on R^m for noncommutative neutrix products such as $r^{-k} \cdot \triangle\delta$ as well as the more general $r^{-k} \cdot \triangle^l \delta$, where $\triangle$ denotes the Laplacian. Their methods of completing such products are totally based on the facts that $\triangle\delta_n$ is computable and a bridge distribution $(d^s/dr^s)\delta$ can act like $\triangle^l\delta$, except for a constant difference.

To make this paper as self-contained as possible, we start with a fixed infinitely differentiable function $\rho(x)$ on R with the following properties:

(i) $\rho(x) \ge 0$;
(ii) $\rho(x) = 0$ for $|x| \ge 1$;
(iii) $\rho(x) = \rho(-x)$;
(iv) $\int_{-1}^{1} \rho(x)dx = 1$.

The function $\delta_n(x)$ is defined by $\delta_n(x) = n\rho(nx)$ for $n = 1,2,\dots$ (due to Jones). It follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta function $\delta(x)$ in the distributional sense.

Now let $\mathcal{D}$ be the testing function space of infinitely differentiable functions of a single variable with compact support, and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = (f(t), \delta_n(x-t)) \tag{1.1}$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The following definition for the noncommutative neutrix product $f \circ g$ of two distributions f and g in $\mathcal{D}'$ was given by Fisher in [2].

Definition 1.1. Let f and g be distributions in $\mathcal{D}'$ and let $g_n = g * \delta_n$. The neutrix product $f \circ g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi) \tag{1.2}$$

for all functions ϕ in $\mathcal{D}$, where N is the neutrix (see [7]) having domain $N' = \{1, 2, \ldots\}$ and range N'', the real numbers, with negligible functions that are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0,\ r = 1, 2, \ldots) \tag{1.3}$$

and all functions of n which converge to zero in the normal sense as n tends to infinity.

The product of Definition 1.1 is not symmetric, and hence, $f \circ g \neq g \circ f$ in general.

Extending definitions of products from one-dimensional space R to an m-dimensional space R^m by using appropriate delta sequences has recently been an interesting topic in distribution theory. The following work on the noncommutative neutrix product of distributions on R^m can be found in [6].

Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ and let $\rho(s)$ be a fixed infinitely differentiable function defined on $R^+ = [0, \infty)$ having the following properties:

(i) $\rho(s) \geq 0$;
(ii) $\rho(s) = 0$ for $s \geq 1$;
(iii) $\int_{R^m} \delta_n(x)\,dx = 1$;

where $\delta_n(x) = c_m n^m \rho(n^2 r^2)$ and c_m is the constant satisfying (iii).

It follows that $\{\delta_n(x)\}$ is a regular δ-sequence of infinitely differentiable functions converging to $\delta(x)$ in $\mathcal{D}'_m$ (an m-dimensional space of distributions).

Definition 1.2. Let f and g be distributions in $\mathcal{D}'_m$ and let

$$g_n(x) = (g * \delta_n)(x) = (g(x-t), \delta_n(t)), \tag{1.4}$$

where $t = (t_1, t_2, \ldots, t_m)$. The neutrix product $f \cdot g$ of f and g exists and is equal to h if

$$N - \lim_{n\to\infty} (f g_n, \phi) = (h, \phi), \tag{1.5}$$

where $\phi \in \mathcal{D}_m$ (an m-dimensional Schwartz space) and the N-limit is defined as above.

With Definition 1.2, Li [4] shows that the noncommutative neutrix product $r^{-k}\cdot\nabla\delta$ exists. Furthermore,

$$\begin{gathered} r^{-2k}\cdot\nabla\delta = -\frac{1}{2^{k+1}(k+1)!(m+2)\cdots(m+2k)}\sum_{i=1}^{m}(x_i\triangle^{k+1}\delta), \\ r^{1-2k}\cdot\nabla\delta = 0, \end{gathered} \tag{1.6}$$

where k is a positive integer and ∇ is the gradient operator. By using Lemma 3.2,

$$\sum_{i=1}^{m}(x_i\triangle^{k+1}\delta) = -2(k+1)\nabla(\triangle^k\delta), \quad k\geq 0, \tag{1.7}$$

in [5], we have

$$r^{-2k}\cdot\nabla\delta = \frac{\nabla(\triangle^k\delta)}{2^k k!(m+2)\cdots(m+2k)}. \tag{1.8}$$

2. The distributions r^λ and $\mu(x)x_+^\lambda$. We consider the functional r^λ (see [3]) defined by

$$(r^\lambda,\phi) = \int_{R^m} r^\lambda\phi(x)\,dx, \tag{2.1}$$

where $\operatorname{Re}(\lambda) > -m$ and $\phi(x)\in\mathcal{D}_m$. Because the derivative

$$\frac{\partial}{\partial\lambda}(r^\lambda,\phi) = \int r^\lambda \ln r\,\phi(x)\,dx \tag{2.2}$$

exists, the functional r^λ is an analytic function of λ for $\operatorname{Re}(\lambda) > -m$.

For $\operatorname{Re}(\lambda)\leq -m$, we should use the following identity (2.4) to define its analytic continuation. For $\operatorname{Re}(\lambda) > 0$, we could deduce

$$\triangle(r^{\lambda+2}) = (\lambda+2)(\lambda+m)r^\lambda \tag{2.3}$$

simply by calculating the left-hand side, where $\triangle$ is the Laplacian operator. By iteration, we find for any integer k that

$$r^\lambda = \frac{\triangle^k r^{\lambda+2k}}{(\lambda+2)\cdots(\lambda+2k)(\lambda+m)\cdots(\lambda+m+2k-2)}. \tag{2.4}$$

On making substitution of spherical coordinates in (2.1), we come to

$$(r^\lambda,\phi) = \int_0^\infty r^\lambda\left\{\int_{r=1}\phi(r\omega)\,d\omega\right\}r^{m-1}\,dr, \tag{2.5}$$

where $d\omega$ is the hypersurface element on the unit sphere. The integral appearing in the above integrand can be written in the form

$$\int_{r=1}\phi(r\omega)\,d\omega = \Omega_m S_\phi(r), \tag{2.6}$$

where Ω_m is the hypersurface area of the unit sphere imbedded in the Euclidean space of m dimensions and S_ϕ is the mean value of ϕ on the sphere of radius r.

It was proven in [3] that $S_\phi(r)$ is infinitely differentiable for $r \geq 0$, bounded support, and

$$S_\phi(r) = \phi(0) + a_1 r^2 + a_2 r^4 + \cdots + a_k r^{2k} + o(r^{2k}) \tag{2.7}$$

for any positive integer k. From (2.5) and (2.6), we obtain

$$(r^\lambda, \phi) = \Omega_m \int_0^\infty r^{\lambda+m-1} S_\phi(r)\, dr, \tag{2.8}$$

which indicates the application of $\Omega_m x_+^\mu$ with $\mu = \lambda + m - 1$ to the testing function $S_\phi(r)$. Using the following Laurent series for x_+^λ about $\lambda = -k$:

$$x_+^\lambda = \frac{(-1)^{k-1}\delta^{(k-1)}(x)}{(k-1)!(\lambda+k)} + x_+^{-k} + (\lambda+k) x_+^{-k} \ln x + \cdots, \tag{2.9}$$

we can write out the Taylor series for $S_\phi(r)$, namely,

$$\begin{aligned} S_\phi(r) &= \phi(0) + \frac{1}{2!} S_\phi''(0) r^2 + \cdots + \frac{1}{(2k)!} S_\phi^{(2k)}(0) r^{2k} + \cdots \\ &= \sum_{k=0}^\infty \frac{\triangle^k \phi(0) r^{2k}}{2^k k! m(m+2)\cdots(m+2k-2)}, \end{aligned} \tag{2.10}$$

which is the well-known Pizetti's formula.

Let $\mu(x)$ be an infinitely differentiable function on R^+ having the following properties:

(i) $\mu(x) \geq 0$;

(ii) $\mu(0) \neq 0$;

(iii) $\mu(x) = 0$ for $x \geq 1$.

Let $\phi(x)$ be a testing function. Then the functional

$$(\mu(x) x_+^\lambda, \phi) = \int_0^1 \mu(x) x^\lambda \phi(x)\, dx \tag{2.11}$$

is regular for $\operatorname{Re}\lambda > -1$. It can be extended to the domain $\operatorname{Re}\lambda > -n-1$ ($\lambda \neq -1, -2, \ldots$) by analytic continuation following Gel'fand and Shilov (see [3]):

$$\begin{aligned} (\mu(x) x_+^\lambda, \phi) &= \int_0^1 \mu(x) x^\lambda \phi(x)\, dx \\ &= \sum_{k=1}^n \frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(\lambda+k)} \\ &\quad + \int_0^1 \mu(x) x^\lambda \left[\phi(x) - \phi(0) - x\phi'(0) - \cdots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right] dx \end{aligned} \tag{2.12}$$

on applying the mean value theorem with $0 < \theta_{k-1} < 1$ for $1 \leq k \leq n$. This means that the generalized function $\mu(x) x_+^\lambda$ is well defined for $\lambda \neq -1, -2, \ldots$.

We thus normalize the value of the functional $(\mu(x)x_+^{\lambda},\phi)$ at $-n$ by

$$\begin{aligned}(\mu(x)x_+^{-n},\phi) &= \sum_{k=1}^{n-1}\frac{\phi^{(k-1)}(0)\mu(\theta_{k-1})}{(k-1)!(-n+k)} \\ &+\int_0^1\mu(x)x^{-n}\cdot\left[\phi(x)-\phi(0)-x\phi'(0)-\cdots-\frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\right]dx.\end{aligned}\tag{2.13}$$

3. An approach to $\nabla^l\delta_n$. In order to derive $\nabla^l\delta_n$, we define

$$\begin{aligned}\tilde{D}^k &= 2^k n^{2k}\rho^{(k)}(n^2r^2),\quad k=0,1,2,\dots,\\ X &= \sum_{i=1}^m x_i,\end{aligned}\tag{3.1}$$

for a fixed function ρ.

Hence,

$$\begin{aligned}\nabla\tilde{D}^k &= \sum_{i=1}^m\frac{\partial}{\partial x_i}(2^k n^{2k}\rho^{(k)}(n^2r^2)) = 2^{k+1}n^{2(k+1)}\rho^{(k+1)}(n^2r^2)\sum_{i=1}^m x_i \\ &= 2^{k+1}n^{2(k+1)}\rho^{(k+1)}(n^2r^2)X = \tilde{D}^{(k+1)}X.\end{aligned}\tag{3.2}$$

It follows that

$$\nabla(\tilde{D}^kX^j) = \nabla\tilde{D}^kX^j+\tilde{D}^k\nabla X^j = \tilde{D}^{(k+1)}X^{j+1}+mj\tilde{D}^kX^{j-1}.\tag{3.3}$$

The following lemma will play an important role in computing the product $r^{-k}\cdot\nabla^l\delta$.

Lemma 3.1. *The expressions $\nabla^{2l}\rho(n^2r^2)$ and $\nabla^{2l+1}\rho(n^2r^2)$ must be in the following forms, respectively:*

$$\nabla^{2l}\rho(n^2r^2) = \sum_{j=0}^{l} b_j^l m^{l-j}\tilde{D}^{l+j}X^{2j},\tag{3.4}$$

$$\nabla^{2l+1}\rho(n^2r^2) = \sum_{j=1}^{l+1} c_j^{l+1} m^{l-j+1}\tilde{D}^{l+j}X^{2j-1},\tag{3.5}$$

where $\{b_j^l\}$ and $\{c_j^l\}$ are constant coefficients with conditions $b_j^j = c_{j+1}^{j+1} = 1$ for $j = 0,1,2,\dots$ and $l = 0,1,2,\dots$.

Proof. We use an inductive method to show Lemma 3.1. It is obviously true for (3.4) as $l=0$. Indeed, $\nabla^0\rho(n^2r^2) = \rho(n^2r^2) = \sum_{j=0}^0 b_j^0 m^{0-j}\tilde{D}^{0+j}X^{2j} = \tilde{D}^0 = \rho(n^2r^2)$. Setting $l=0$ in (3.5), the left-hand side is $\nabla\rho(n^2r^2) = \tilde{D}X$ whereas the right-hand side is equal to

$$\sum_{j=1}^{0+1} c_j^1 m^{0-j+1}\tilde{D}^{0+j}X^{2j-1} = \tilde{D}X.\tag{3.6}$$

Hence, Lemma 3.1 holds for $l=0$.

By an inductive hypothesis, we assume Lemma 3.1 has been established for $l = 0,1,2,\dots,k$. Hence we need only to show the case when $l = k+1$ by using our assumptions:

$$\begin{aligned}
\nabla^{2(k+1)}\rho(n^2r^2) &= \nabla(\nabla^{2k+1}\rho(n^2r^2)) \\
&= \sum_{j=1}^{k+1} c_j^{k+1} m^{k-j+1}\nabla(\tilde{D}^{k+j}X^{2j-1}) \\
&= \sum_{j=1}^{k+1} c_j^{k+1} m^{k-j+1}(\tilde{D}^{k+j+1}X^{2j} + m(2j-1)\tilde{D}^{k+j}X^{2j-2}) \\
&= \sum_{j=1}^{k+1} (c_j^{k+1} m^{k-j+1}\tilde{D}^{k+j+1}X^{2j} + c_j^{k+1} m(2j-1)m^{k-j+1}\tilde{D}^{k+j}X^{2j-2}) \\
&= c_{k+1}^{k+1} m^{k+1-(k+1)}\tilde{D}^{k+k+1+1}X^{2(k+1)} \\
&\quad + \sum_{j=2}^{k+1} (c_{j-1}^{k+1} + (2j-1)c_j^{k+1}) m^{k-j+2}\tilde{D}^{k+j}X^{2j-2} + c_1^{k+1} m^{k+1}\tilde{D}^{k+1}.
\end{aligned} \tag{3.7}$$

By setting $b_{k+1}^{k+1} = c_{k+1}^{k+1} = 1$, $b_{j-1}^{k+1} = c_{j-1}^{k+1} + (2j-1)c_j^{k+1}$, for $2 \le j \le k+1$, and $b_0^{k+1} = c_1^{k+1}$, we come to

$$\begin{aligned}
\nabla^{2(k+1)}\rho(n^2r^2) &= b_{k+1}^{k+1} m^{k+(k+1)-1}\tilde{D}^{k+k+1+1}X^{2(k+1)} \\
&\quad + \sum_{j=2}^{k+1} b_{j-1}^{k+1} m^{k-j+2}\tilde{D}^{k+j}X^{2j-2} + b_0^{k+1} m^{k+1}\tilde{D}^{k+1} \\
&= \sum_{j=0}^{k+1} b_j^{k+1} m^{k-j+1}\tilde{D}^{k+1+j}X^{2j},
\end{aligned} \tag{3.8}$$

which completes the first part of Lemma 3.1.

As for $\nabla^{2(k+1)+1}\rho(n^2r^2)$, we have the following by hypothesis:

$$\begin{aligned}
\nabla^{2(k+1)+1}\rho &= \nabla(\nabla^{2(k+1)}\rho(n^2r^2)) \\
&= \nabla\left(\sum_{j=0}^{k+1} b_j^{k+1} m^{k-j+1}\tilde{D}^{k+1+j}X^{2j}\right) \\
&= \sum_{j=0}^{k+1} b_j^{k+1} m^{k-j+1}\nabla(\tilde{D}^{k+1+j}X^{2j}) \\
&= b_0^{k+1} m^{k+1}\nabla\tilde{D}^{k+1} + \sum_{j=1}^{k+1} b_j^{k+1} m^{k-j+1}\nabla(\tilde{D}^{k+1+j}X^{2j})
\end{aligned}$$

$$
\begin{aligned}
&= b_0^{k+1}m^{k+1}\tilde{D}^{k+2}X \\
&\quad+\sum_{j=1}^{k+1}\left(b_j^{k+1}m^{k-j+1}\tilde{D}^{k+j+2}X^{2j+1}+2jb_j^{k+1}m^{k-j+2}\tilde{D}^{k+j+1}X^{2j-1}\right) \\
&= b_0^{k+1}m^{k+1}\tilde{D}^{k+2}X+2b_1^{k+1}m^{k+1}\tilde{D}^{k+2}X \\
&\quad+\sum_{j=2}^{k+1}\left(b_{j-1}^{k+1}+2jb_j^{k+1}\right)m^{k-j+2}\tilde{D}^{k+j+1}X^{2j-1} \\
&\quad+b_{k+1}^{k+1}m^{k+1-k-1}\tilde{D}^{k+k+1+2}X^{2(k+1)+1}.
\end{aligned}
\tag{3.9}
$$

By denoting $c_j^{k+2}=b_{j-1}^{k+1}+2jb_j^{k+1}$, $c_1^{k+2}=2b_1^{k+1}+b_0^{k+1}$, and $c_{k+2}^{k+2}=b_{k+1}^{k+1}=1$, we finally get

$$
\nabla^{2k+3}\rho=\sum_{j=1}^{k+2}c_j^{k+2}m^{k-j+2}\tilde{D}^{k+j+1}X^{2j-1}, \tag{3.10}
$$

which has the form in Lemma 3.1. □

Lemma 3.2. *The coefficient $\{b_j^l\}$ in Lemma 3.1 can be recursively obtained by the following relations, which are used to compute $\{c_j^{l+1}\}$ in (3.5):*

$$
\begin{aligned}
b_0^{l+1}&=2b_1^l+b_0^l,\quad b_1^0=0,\\
b_j^j&=1,\quad j=0,1,2,\dots,\\
b_{j-1}^{l+1}&=b_{j-2}^l+2(j-1)b_{j-1}^l+(2j-1)\left(b_{j-1}^l+2jb_j^l\right),\quad 2\le j\le l,\\
b_l^{l+1}&=4l+b_{l-1}^l+1.
\end{aligned}
\tag{3.11}
$$

Proof. We can deduce Lemma 3.2 from the following identities obtained in the verification of Lemma 3.1:

$$
\begin{aligned}
c_j^{l+1}&=b_{j-1}^l+2jb_j^l,\\
c_1^{l+1}&=2b_1^l+b_0^l,\\
c_j^j&=1,\quad j=1,2,\dots,\\
b_0^{l+1}&=c_1^{l+1},\\
b_{j-1}^{l+1}&=c_{j-1}^{l+1}+(2j-1)c_j^{l+1},\quad 2\le j\le l+1.
\end{aligned}
\tag{3.12}
$$

□

At the end of this paper, we have attached a nicely structured Java application program to compute $\{b_j^l\}$ and $\{c_j^l\}$ by using the Java swing package. One only needs to enter values of j and l in order to find the corresponding coefficients (this work is mainly due to my research assistant Jeff Liu).

4. The main results. In this section, we utilize the results of $\nabla^l\delta_n$ obtained in Section 3 to derive the distributional product $r^{-k}\cdot\nabla^l\delta$.

Theorem 4.1. *The noncommutative neutrix product $r^{-k}\cdot\nabla^{2l}\delta$ exists. Furthermore,*

$$r^{-2k}\cdot\nabla^{2l}\delta=\sum_{j=0}^{l}\frac{b_j^l m^{l-j}(-1)^{l+j}X^{2j}\triangle^{k+l+j}\delta}{(k+l+j)!2^{k+l+j}(m+2l+2j)\cdots(m+2k+2l+2j-2)},\qquad r^{-2k+1}\cdot\nabla^{2l}\delta=0, \tag{4.1}$$

where $k=1,2,\ldots$ and $l=0,1,2,\ldots$.

In particular, we let $l=0$ to get

$$\begin{aligned} r^{-2k}\cdot\delta &= b_0^0 m^0(-1)^0\frac{\triangle^k\delta}{k!2^k m(m+2)\cdots(m+2k-2)}\\ &=\frac{\triangle^k\delta}{k!2^k m(m+2)\cdots(m+2k-2)}, \end{aligned} \tag{4.2}$$

which is identically equal to the result obtained by Li and Fisher in [6].

Proof. From (3.4), we have

$$\begin{aligned} (r^{-k}\cdot\nabla^{2l}\delta_n,\phi) &= \int_{R^m} r^{-k}\nabla^{2l}\delta_n\phi\,dx\\ &= c_m n^m\int_{R^m} r^{-k}\left(\sum_{j=0}^{l} b_j^l m^{l-j}\tilde{D}^{l+j}X^{2j}\right)\phi\,dx\\ &= \sum_{j=0}^{l} c_m n^m\int_{R^m} r^{-k} b_j^l m^{l-j}\tilde{D}^{l+j}X^{2j}\phi\,dx\\ &= \sum_{j=0}^{l} b_j^l m^{l-j}c_m\int_{R^m} r^{-k}2^{l+j}n^{2(l+j)+m}\rho^{(l+j)}(n^2r^2)X^{2j}\phi\,dx\\ &= \sum_{j=0}^{l} b_j^l m^{l-j}2^{l+j}n^{2(l+j)+m}c_m\int_{R^m} r^{-k}\rho^{(l+j)}(n^2r^2)X^{2j}\phi\,dx\\ &= \sum_{j=0}^{l} b_j^l m^{l-j}2^{l+j}n^{2(l+j)+m}c_m\Omega_m\int_0^{1/n} r^{m-k-1}\rho^{(l+j)}(n^2r^2)S_{\psi_j}(r)\,dr, \end{aligned} \tag{4.3}$$

where $\psi_j=X^{2j}\phi$.

Using Taylor's formula, we obtain

$$\begin{aligned} S_{\psi_j}(r) &= \sum_{i=0}^{k+2(l+j)-1}\frac{S_{\psi_j}^{(i)}(0)}{i!}r^i+\frac{S_{\psi_j}^{(k+2l+2j)}(0)}{(k+2l+2j)!}r^{k+2l+2j}+\frac{S_{\psi_i}^{(k+2l+2j+1)}(\theta_j r)}{(k+2l+2j+1)!}r^{k+2l+2j+1}\\ &= I_1+I_2+I_3, \end{aligned} \tag{4.4}$$

where $0<\theta_j<1$.

Substituting I_1, I_2, and I_3 into (4.3) above, we have three corresponding parts, namely, T_1, T_2, and $T3$, respectively. Following the techniques used in [1, 4] and the normalization of (2.13), we can show that

$$N-\lim_{n\to\infty} T_1 = 0, \qquad N-\lim_{n\to\infty} T_3 = 0, \tag{4.5}$$

for $k = 1,2,\dots$ and $l = 0,1,2,\dots$.

On making the change $nr = t$, it follows from above that

$$\begin{aligned} T_2 &= 2^{l+j} n^{2(l+j)+m} c_m \Omega_m \int_0^{1/n} r^{m-k-1} \rho^{(l+j)}(n^2r^2) \frac{S_{\psi_j}^{(k+2l+2j)}(0)}{(k+2l+2j)!} r^{k+2l+2j} dr \\ &= \frac{S_{\psi_j}^{(k+2l+2j)}(0)}{(k+2l+2j)!} 2^{l+j} c_m \Omega_m \int_0^1 t^{m+2l+2j-1} \rho^{(l+j)}(t^2) dt. \end{aligned} \tag{4.6}$$

Integrating by parts, we get

$$\begin{aligned} \int_0^1 t^{m+2l+2j-1} \rho^{(l+j)}(t^2) dt &= \frac{1}{2} \int_0^1 t^{m+2l+2j-2} \rho^{(l+j)}(t^2) dt^2 = \cdots \\ &= \frac{(-1)^{l+j}}{2^{l+j}} (m+2l+2j-2)\cdots(m+2)m \cdot \int_0^1 \rho(t^2) t^{m-1} dt. \end{aligned} \tag{4.7}$$

Therefore,

$$T_2 = (-1)^{l+j} m(m+2)\cdots(m+2l+2j-2) \frac{S_{\psi_j}^{(k+2l+2j)}(0)}{(k+2l+2j)!}. \tag{4.8}$$

It follows from Pizetti's formula in (2.10) that

$$\frac{S_{\psi_j}^{(2k+2l+2j)}(0)}{(2k+2l+2j)!} = \frac{\triangle^{k+l+j} \psi_j(0)}{(k+l+j)! 2^{k+l+j} m(m+2)\cdots(m+2k+2l+2j-2)}, \tag{4.9}$$

which completes the proof of Theorem 4.1. □

Theorem 4.2. *The noncommutative neutrix product $r^{-k}\cdot\nabla^{2l+1}\delta$ exists. Furthermore,*

$$\begin{gathered} r^{-2k}\cdot\nabla^{2l+1}\delta = m \sum_{j=1}^{l+1} \frac{c_j^{l+1} m^{l-j} (-1)^{l+j} X^{2j-1} \triangle^{k+l+j}\delta}{(k+l+j)! 2^{k+l+j} (m+2l+2j)\cdots(m+2k+2l+2j-2)}, \\ r^{-2k+1}\cdot\nabla^{2l+1}\delta = 0, \end{gathered} \tag{4.10}$$

where $k = 1,2,\dots$ and $l = 0,1,2,\dots$.

In particular, we let $l = 0$ to get

$$\begin{aligned} r^{-2k}\cdot\nabla\delta &= c_1^1 m^0 (-1)^{0+1} \frac{X\triangle^{k+1}\delta}{(k+1)! 2^{k+1}(m+2)\cdots(m+2k)} \\ &= -\frac{X\triangle^{k+1}\delta}{(k+1)! 2^{k+1}(m+2)\cdots(m+2k)}, \end{aligned} \tag{4.11}$$

which is obviously identical with the result obtained by Li in [4].

Proof. The proof immediately follows the above. □

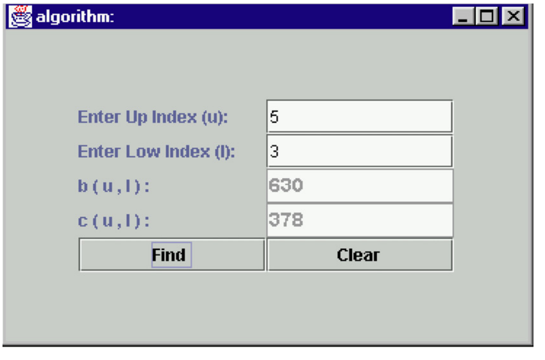
From our result, we can easily derive the product $\nabla^k r^{-n} \cdot \nabla^l \delta$ where $\nabla = \sum_{i=1}^{m} \partial/\partial x_i$. The author leaves this for interested readers.

Appendix

```
/***            APPENDIX            ***

 This program computes coefficients

*/

import javax.swing.*;
import java.awt.*;
import java.awt.event.*;

public class algorithm
{
     public Component createC()
     {   final JTextField up=new JTextField("");
         final JTextField low=new JTextField("");
         final JTextField valueB = new JTextField("");
         final JTextField valueC = new JTextField("");
         JButton button1 = new JButton("Find ");
         JButton button2 = new JButton("Clear");

         valueB.setEnabled(false);
         valueB.setFont(new Font("Dialog",Font.BOLD,14));
         valueC.setEnabled(false);
         valueC.setFont(new Font("Dialog",Font.BOLD,14));

         button1.addActionListener(new ActionListener()
         {   public void actionPerformed(ActionEvent e) throws NumberFormatException
             {   String ut=up.getText();
                 String lt=low.getText();
                 try
                 {  long x= Integer.parseInt(ut);
                    long y= Integer.parseInt(lt);
                    if(y<0 || x<y)
                    { if(y==1 && x==0)
                         valueB.setText("0");
                      else
                         valueB.setText("Undefined!");
                      valueC.setText("Undefined!");
                    }
                    else
                    { valueB.setText(""+findB(x,y));
                      if(y<1)
                         valueC.setText("Undefined!");
                      else
                         valueC.setText(""+findC(x,y));
                    }
                 }
                 catch(NumberFormatException ne)
```

```
            { valueB.setText("Invalid input!");
              valueC.setText("Invalid input!");
            }
        }
    });

    button2.addActionListener(new ActionListener()
    {   public void actionPerformed(ActionEvent e)
        {
             up.setText("");
             low.setText("");
             valueB.setText("");
             valueC.setText("");
        }
    });

    JPanel pane = new JPanel();
    pane.setBorder(BorderFactory.createEmptyBorder(50,50,50,50) );
    pane.setLayout(new GridLayout(5, 2));
    pane.add(new JLabel("Enter Up Index (u):      "));
    pane.add(up);
    pane.add(new JLabel("Enter Low Index (l):     "));
    pane.add(low);
    pane.add(new JLabel("b ( u , l ) :"));
    pane.add(valueB);
    pane.add(new JLabel("c ( u , l ) :"));
    pane.add(valueC);
    pane.add(button1);
    pane.add(button2);

    return pane;
  }

  public static long findB(long up,long low)
  {   if(up==low)
          return 1;
      else if(low==0&&up>0)
      {   if(up==1)
             return 1;
          else
             return 2*findB(up-1,1)+findB(up-1,0);
      }
      else if(low>0&&up>low)
      {   if(up-low==1)
             return 4*low+findB(low,low-1)+1;
          else
             return findB(up-1,low-1)+2*low*findB(up-1,low)+
                    (2*low+1)*(findB(up-1,low)+2*(low+1)*findB(up-1,low+1));
      }
      else
         return 0;
}

public static long findC(long up,long low)
{    if(low>0&&low==up)
```

```
            return 1;
         else if(low==1&&up>low)
            return findB(up,0);
         else
            return findB(up-1,low-1)+2*low*findB(up-1,low);
   }

   public static void main(String[] args)
   {
       JFrame frame=new JFrame("algorithm:");
       algorithm app = new algorithm();
       Component contents = app.createC();
       frame.getContentPane().add(contents, BorderLayout.CENTER);
       frame.addWindowListener(new WindowAdapter()
       {   public void windowClosing(WindowEvent e)
           {   System.exit(0);
           }
       });
       frame.pack();
       frame.setVisible(true);
   }
}
```



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