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REAL QUARTIC SURFACES CONTAINING 16 SKEW LINES

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It is well known that there is an open three-dimensional subvariety M^s of the Grassmannian of lines in $\mathbb{P}_3$ which parametrizes smooth irreducible complex surfaces of degree 4 which are Heisenberg invariant, and each quartic contains 32 lines but only 16 skew lines, being determined by its configuration of lines, are called a *double 16*. We consider here the problem of visualizing in a computer the real Heisenberg invariant quartic surface and the real double 16. We construct a family of points $l \in M^s$ parametrized by a two-dimensional semialgebraic variety such that under a change of coordinates of l into its Plüecker, coordinates transform into the real coordinates for a line L in $\mathbb{P}_3$, which is then used to construct a program in Maple 7. The program allows us to draw the quartic surface and the set of transversal lines to L. Additionally, we include a table of a group of examples. For each test example we specify a parameter, the viewing angle of the image, compilation time, and other visual properties of the real surface and its real double 16. We include at the end of the paper an example showing the surface containing the double 16.

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1. Introduction. Let H^t be the well-known *Heisenberg group* of level 2 (for the precise definition, we refer to Section 3). We consider H^t-invariant quartic surfaces $X_f = \{f(z_0,z_1,z_2,z_3) = 0\}$, where f is a homogeneous polynomial of degree 4 in the variables z_0, z_1, z_2, z_3, which is H^t-invariant over the complex numbers field. One of the problems posed in [1] is that if such a quartic surface contains a complex line, then determine the configuration of lines or find a characterization of X_f in terms of the configuration of lines contained in it. It is a classical fact of line geometry that the lines in $\mathbb{P}_3$ are parametrized by the Grassmannian variety denoted here by Gr. In order to formulate the problem more precisely, we introduce the complex vector space W of polynomials of degree 4 in the variables $z_0,\dots,z_3$ which are H^t-invariant. The condition that X_f contains a line is stated as $l \subset X_f$ if and only if $f|_l = 0$. It is not difficult to prove (see, e.g., [1, Section 4]) that $\dim_{\mathbb{C}}(W) = 5$ (in [1], one can give an explicit set of generators for W), and for a "generic" l, the last condition implies that there exist a 5×5 matrix $M(l)$ and $v \in \mathbb{CP}_4$ such that $M(l)\cdot v = 0$, which implies that $\det(M(l)) = 0$. By choosing the so-called K-coordinates associated to a line, l transforms into a point $(x_0 : \cdots : x_5)$ such that $x_0^2 + \cdots + x_5^2 = 0$ which is the well-known equation for Gr in the K-coordinates (c.f. Section 3). The equation involving the matrix $M(l)$ is written in the K-coordinates as $\Sigma_{i=0}^{5}\Pi_{j\neq i}x_j^2 = 0$. The previous arguments are only heuristical and will be made precise in Section 3. Both equations above are the equations which define the threefold $M \subset$ Gr. In this paper, we consider the converse problem. Namely, we are interested in finding points in a subvariety of $M^s \subset M$ (for which a precise definition will be given in Section 2) such that each point in M^s arises as a *real* line in $\mathbb{P}_3$ (i.e., defined by real coordinates), the quartic X_f determined by l is *real* (i.e., the points are defined in real three-dimensional projective space), and the coefficients of f are defined over the real numbers field. Moreover, if according to the theory of [1, Section 4], a point in M^s generates a complex line which determines a smooth, irreducible complex H^t-invariant quartic surface containing it and its H^t orbit, then an easy extension of the theory to the *real* case shows that for the special case treated here, the H^t-invariant quartic surface X_f is real and its defining polynomial f is real. The next problem is to find a special hyperplane $H \subset \mathbb{RP}_3$ to be able to graph $X_f \cap H$ and visualize it along with its configuration of lines called here a double 16 on a computer. It is of interest to note that the H^t-invariant quartic surfaces containing a complex line form a three-dimensional complex parameter space within the 34-dimensional space of quartic surfaces in $\mathbb{CP}_3$. Such surfaces of degree 4 defined over the complex field arise as projective models of linear systems of abelian surfaces of polarization type $(1,3)$ (c.f. [1, 5]).

We give an outline of the paper. In Section 2, we state sufficient conditions for an arbitrary point $l \in M$ to be in M^s (Proposition 2.7). We then construct a subset of M^s parametrized by a two-dimensional semialgebraic set (Remark 2.9). In Section 3, after giving a few basic facts of line geometry and the definition of the Heisenberg group, we show that the H^t-invariant quartic surfaces defined by the points of Proposition 2.7 are *real* quartic surfaces and contain the double 16 with *real* lines, and that the quartics are determined by the configuration.

In Section 4, we give a detailed description of how the Maple program works and is used to visualize these surfaces along with their line configurations (to the extent to which they can be shown) on the computer. In Section 5, we describe the results given by the program for a group of examples in a table which describes, for the surface drawn with its ten transversals, the following parameters: d (which defines the quartic surface; this is the value λ of (2.8)), the angle (u,v) of the image surface, compilation time for each surface, and the visual description of the double 16. We printed one test example of a surface for which one can see a line and most of the double 16, at one specific angle, and it is given at the end of the paper. The program can be used as a guide to produce Heisenberg invariant Kummer quartic surfaces, which contain other types of curves, not necessarily rational ones, and can be useful for other researchers working on similar problems.

2. An elementary proposition and a locus of real solutions. Another more suggestive way of writing the equation for M as defined in [1, Section 4] is

$$\Sigma_{i=0}^{5} x_i^2 = \Sigma_{i=0}^{5} \frac{1}{x_i^2} = 0. \tag{2.1}$$

In the sequel, the following set of equations, derived from the above definition, will be more useful to us:

$$x_0^2+x_1^2+x_2^2=-x_3^2-x_4^2-x_5^2, \tag{2.2}$$

$$\frac{1}{x_0^2}+\frac{1}{x_2^2}+\frac{1}{x_4^2}=-\frac{1}{x_3^2}-\frac{1}{x_5^2}-\frac{1}{x_1^2}, \tag{2.3}$$

which is of course defined away from $\{x_i=0\}$ for $i=0,\dots,5$. Otherwise, (2.3) becomes $\Sigma_{i=0}^5\Pi_{j\neq i}x_j^2=0$. According to [1, Section 4], there exists an open subvariety of M, a threefold denoted in [1] as M^s, defined as follows. For this, we let $\eta=\sqrt{-1}$,

$$x_1=\eta y_1, \qquad x_3=\eta y_3, \qquad x_5=\eta y_5. \tag{2.4}$$

We also introduce the following change of variables:

$$q_i=\begin{cases} x_i^2 & \text{for } i \text{ even},\\ y_i^2 & \text{for } i \text{ odd}.\end{cases} \tag{2.5}$$

We fix

$$\begin{gathered} Q_{jk}=\{x_j=x_k=0=\Sigma_{i\neq j,i\neq k}x_i^2=0\},\\ P=(\epsilon_1:\epsilon_2:\epsilon_3:\eta\epsilon_4:\epsilon_5\eta:\epsilon_6\eta), \quad \epsilon_i\in\{\pm1\},\ i=1,\dots,6,\\ \mathcal{P}_1=\{q_0-q_1=q_2-q_3=q_4-q_5=0\}. \end{gathered} \tag{2.6}$$

By defining

$$\mathfrak{Q}=\bigcup_{j\neq k}Q_{jk}, \qquad \Sigma=\{\sigma(P)\mid\sigma\in\mathbb{S}_6\}, \qquad \mathcal{P}=\bigcup_{\sigma\in\mathbb{S}_6}\sigma\mathcal{P}_1, \tag{2.7}$$

then $M^s=M-\mathfrak{Q}-\Sigma-\mathcal{P}$. M^s has the property that for each $l\in M^s$, there exists an equation of degree 4 invariant under the Heisenberg group H^t, irreducible, and uniquely determined by its configuration of lines, and the surface defined by it is smooth (see [1, Proposition 4.4]).

We can find the solutions to (2.2), (2.3) as a particular case of the following one-parameter set of equations for $q_0>0$, $q_1>0$, $q_2>0$, $q_3>0$, $q_4>0$, $q_5>0$, $\lambda>0$:

$$\begin{gathered} \lambda=\frac{1}{q_2}+\frac{1}{q_0}+\frac{1}{q_4}=\frac{1}{q_3}+\frac{1}{q_1}+\frac{1}{q_5},\\ 1=q_0+q_2+q_4=q_1+q_3+q_5. \end{gathered} \tag{2.8}$$

To solve the last set of equations is to solve, for real variables x, y, z and a real parameter λ, the system of equations

$$\begin{gathered} xy+xz+yz=\lambda xyz,\\ x+y+z=1,\\ x>0, \quad y>0, \quad z>0, \end{gathered} \tag{2.9}$$

and solve them for the case $(x,y,z)=(q_0,q_2,q_4)$ and (q_1,q_3,q_5). We obtain the following elementary proposition.

Proposition 2.1. *The system of equations (2.9) has a solution if and only if $\lambda > 9$. For such λ, set $\rho = (\lambda-9)(\lambda-1)/4\lambda^2$ and $\sigma = (\lambda-3)/2\lambda$; then*

$$\frac{1}{\lambda} < \sigma - \sqrt{\rho} < \sigma + \sqrt{\rho} < 1 \tag{2.10}$$

and the system of equations has a real solution if and only if $z \in (\sigma - \sqrt{\rho}, \sigma + \sqrt{\rho})$ or $z < 1/\lambda$ when the solutions for x and y are given by positive

$$\begin{aligned} x &= \frac{1}{2}(1-z)\left(1 \pm \sqrt{-\frac{\Omega}{N}}\right), \\ y &= \frac{1}{2}(1-z)\left(1 \mp \sqrt{-\frac{\Omega}{N}}\right), \end{aligned} \tag{2.11}$$

where $\Omega = \lambda z^2 - (\lambda-3)z + 1$, $N = (\lambda z - 1)(1-z)$.

Proof. Fixing $\lambda > 0$ and $1/\lambda < z < 1$, we must find the intersection points of the line $x+y = 1-z$ and the hyperbola $\alpha xy + x + y = 0$ (where $\alpha = (1-\lambda z)/z$). Solving these yields the expressions for x and y as stated above. These solutions are real exactly when $1 \pm \sqrt{-\Omega/N} > 0$. But $\sqrt{-\Omega/N}$ is positive if and only if $\Omega < 0$, $N > 0$ or $\Omega > 0$, $N < 0$, which are equivalent to $\Omega < 0$, $z > 1/\lambda$ or $\Omega > 0$, $z < 1/\lambda$. The condition $\Omega > 0$ (resp., $\Omega < 0$) is equivalent to saying that $z > \sigma + \sqrt{\rho}$ or $z < \sigma - \sqrt{\rho}$ (resp., $z \in (\sigma - \sqrt{\rho}, \sigma + \sqrt{\rho})$), hence the claim. Clearly, $\rho > 0$ if and only if $\lambda > 9$ or $\lambda < 1$, but (2.10) rules out the second possibility. Equation (2.10) can easily be verified if $\lambda > 0$. This can be applied with $(x,y,z) = (q_0,q_2,q_4)$ or (q_1,q_3,q_5). $\square$

Proposition 2.2. *The case $\lambda = 9$ in Proposition 2.1 is exceptional. Fix the affine plane $H = \{g = q_0 + q_2 + q_4 - 1 = 0\}$ in the $\mathbb{R}^3$ defined by the coordinates q_0, q_2, q_4, and define, for each $\lambda \in \mathbb{R}_{\geq 0}$,*

$$f_\lambda = q_2q_4 + q_0q_4 + q_0q_2 - \lambda q_0q_2q_4 \tag{2.12}$$

which is a surface in the $\mathbb{A}_3$ defined by the coordinates q_0, q_2, q_4. Let $C_\lambda = H \cap \{f_\lambda = 0\}$. Then the linear system of curves $\{C_\lambda = \{f_\lambda = g = 0\}\}_{\lambda \in \mathbb{R}_{\geq 0}}$ is always smooth except for $\lambda = 1, 9$ with singularities

$$\operatorname{Sing}(C_9) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}, \qquad \operatorname{Sing}(C_1) = \{Q = (-1,1,1)\}. \tag{2.13}$$

Proof. To simplify the computations, let $\partial_i f_\lambda = \partial f_\lambda/\partial q_i$. Recall that $f_\lambda = q_2q_4 + q_0q_2 + q_0q_4 - \lambda q_0q_2q_4$ and $-g = 1 - q_0 - q_2 - q_4$. Note that $\partial_i f_\lambda = q_j + q_k - \lambda q_j q_k$ for $i,j,k \in \{0,2,4\}$ with $i \neq j \neq k$ and $\partial_i g = 1$. The Jacobian matrix is

$$\begin{pmatrix} \partial_2 f_\lambda & \partial_0 f_\lambda & \partial_4 f_\lambda \\ 1 & 1 & 1 \end{pmatrix}. \tag{2.14}$$

It is of rank less than 1 if and only if

$$0 = (\partial_i - \partial_j) f_\lambda = 0 \quad \text{for } i,j \in \{0,2,4\}. \tag{2.15}$$

It follows that

$$(q_0-q_2)(1-\lambda q_4)=(q_2-q_4)(1-\lambda q_0)=(q_4-q_0)(1-\lambda q_2)=0. \tag{2.16}$$

It is enough to prove the following three cases (the others are derived from these).

(I) $1-\lambda q_4=q_0-q_4=q_2-q_4=0$. Thus $q_2=q_4=q_0$. $q_4=1/\lambda$. Substituting in $g=0$, one obtains $g=1-3q_0$. Therefore, $q_0=1/3$. Substituting this value of q_0 in $0=f_\lambda$, one obtains $0=2q_2^2$, a contradiction.

(II) $0=q_2-q_0=q_0-q_4=q_2-q_4$. It follows that $q_0=q_4=q_2$. Hence $g=1-3q_0=0$. Substituting $q_0=1/3$ in f_λ, one obtains $f_\lambda=3/9-\lambda/27=0$, therefore $\lambda=9$.

(III) $1-\lambda q_4=1-\lambda q_0=q_4-q_0=0$. Therefore, $q_0=q_4=1/\lambda$. From $0=g=1-2/\lambda-q_2$, it follows that $q_2=1-2/\lambda$ and $0=f_\lambda=q_4(1-1/\lambda)$, hence $\lambda=1$. □

Fix once again the $\mathbb{R}^3$ defined by the coordinates x, y, z. Then an easy computation shows that the equation of $\{C_\lambda=\{0=f_\lambda(x,y,z)=1-(x+y+z)\}\}_{\lambda\in\mathbb{R}_{>0}}$ can be written for $\lambda=1,9$ as

$$\begin{aligned} f_9&=9(x^2y+xy^2)+x+y-10xy-(x^2+y^2),\\ f_1&=x^2y+xy^2+x+y-2xy-(x^2+y^2). \end{aligned} \tag{2.17}$$

For the next lemma, recall that $P=(1/3,1/3,1/3)$ is a singular point for C_9 and $Q=\mathrm{Sing}(C_1)$.

Lemma 2.3. *The equation for the tangent cone of C_9 (resp., of C_1) passing through P (resp., Q of C_1) is $(x-1/3)^2+(x-1/3)(y-1/3)+(y-1/3)^2$ (resp., $4(y-1)(x+y)$). In particular, P (resp., Q) is a nonordinary double point of C_9 at P (resp., Q of C_1).*

Proof. The second partial derivatives at P are given as $\partial_x^2 f_9=-2+18y$, $\partial_{xy}^2 f_9=-10+18x+18y$, $\partial_y^2 f_9=-2+18x$. Summarizing, $\partial_x^2 f_9(P)=4$, $\partial_y^2 f_9(P)=4$, $\partial_{xy}^2 f_9(P)=2$. The equation for the tangent cone at P is then

$$4\left(x-\frac{1}{3}\right)^2+4\left(x-\frac{1}{3}\right)\left(y-\frac{1}{3}\right)+4\left(y-\frac{1}{3}\right)^2. \tag{2.18}$$

The calculation for C_1 can be done analogously. □

Remark 2.4. A direct computation shows that f_9 is irreducible over $\mathbb{R}$. Under the linear change of coordinates $u=x-1/3$, $v=y-1/3$, the equation for $f_9=0$ is transformed into $f_9=9uv(u+v)+2uv+2(u^2+v^2)$. Under this linear change of coordinates, the cubic curve C_9 is transformed into a real cubic with isolated singularity at the origin which is to be expected from the classification of irreducible cubic curves over the real numbers field.

Useful notation. Let $e=\sqrt{(\lambda-9)(\lambda-1)}$, $R=(\lambda-3-e)/2\lambda$, $S=(\lambda-3+e)/2\lambda$. By Proposition 2.1, $R<S$, for $x,u\in(R,S)$, we will adapt the convention of writing these numbers as $x=(R(n-1)+S)/n$, $u=(R(m-1)+S)/m$ for some $n,m>1$.

We need the following lemma.

Lemma 2.5. (1) *If $n,m \in \mathbb{R}$ are chosen so that $1 < n < M = me/(3m+e(m-1))$, $2 < m$, then $M < 2 < m < e/3$ and for such n, m, the numbers x,u satisfy $1-u < x$. In particular, $e > 6$, which is equivalent to $\lambda > 5(1+\sqrt{2})$.*

(2) *If (u,v,w), (x,y,z) are solutions to (2.9) and are chosen so that $x > 1-u$ and $x \notin \{(\lambda+3\pm e)/4\lambda\}$, then $x \neq v$, $x \neq w$, $x \neq y$, $x \neq z$. Indeed, $x \in \{(\lambda+3\pm e)/4\lambda\}$ only if $n = (3(\lambda-1)+e)/2\lambda$ or $n = (\lambda-1+e)/2$. In particular, if $1 < n < 2$ only, the value for $\lambda = (3n-2)/(n-1)(2-n)$ is possible.*

Proof of part (1). $1-u < x$ if and only if $(m-(R(m-1)+S))/m < (R(n-1)+S)/n$, hence $n < (S-R)/(1-(R(2m-1)+S)/m)$. Substitute the values for S, R. The value for the numerator $S-R = e/\lambda$ and the denominator is equal to $(m(3+e)-e)/m\lambda$, hence, by substituting in the original expression, we obtain $n < M$. The last inequality follows from $1 < M$ if and only if $m < e/3$. To prove the inequality, $m > M$ if and only if $m > 2e/(e+3)$ and note that the last number is always less than 2. Thus, if $m > 2$, it is sufficient. We finally note that $M < 2$ if and only if $m > 2e/(e+6)$, but if $m > 2$, then $m > 2e/(e+3) > 2e/(e+6)$. □

Proof of part (2). Note that

$$\begin{aligned} \Omega &= t(\lambda t+3)-(\lambda t-1), \\ \frac{\Omega}{\lambda t-1} &= \frac{t(\lambda t+3)}{\lambda t-1}-1 > t-1. \end{aligned} \tag{2.19}$$

Fix u as one solution to (2.11); namely, $v = (1/2)(1-u)+Z/2$, where $Z = \sqrt{-(1-u)\Omega/(\lambda u-1)}$. Assume to the contrary that $x = v$, therefore $2x-(1-u) = Z$. By (2.19), we obtain $-Z^2 = (1-u)(u(\lambda u+3)/(\lambda u-1)-1) > (1-u)(u-1)$, hence $(1-u)^2 > Z^2$. For $u > 1/\lambda$, the quantity on the right-hand side is positive, hence $(1-u)^2 - Z^2 = (1-u+Z)(1-u-Z) > 0$, which is equal to $2x(2(1-u)-2x) > 0$. Since $x > 0$, $x < 1-u$, which is a contradiction.

Assuming $x = w = 1-(u+v)$, substituting in v, we obtain $2x-(1-u) = -Z \le 0$. If $x > 1-u$, then $0 \ge 2x-(1-u) > 2(1-u)-(1-u) = 1-u > 0$, again a contradiction.

To see that none of u,v,w (resp., x, y, z) is equal to the other using the equation

$$v^2(\lambda u-1)+v(1-u)(1-\lambda u)+u(1-u) = 0, \tag{2.20}$$

we consider the following cases.

(I) If $v = u$ such that $u(\lambda u-1)+(1-u)(1-\lambda u)+(1-u) = 0$, hence $2\lambda u^2 - u(\lambda+3)+2 = 0$. It follows that $u = (\lambda+3\pm e)/4\lambda$.

(II) If $u = w = 1-u-v$, $v = 1-2u$, thus $v^2 = 1-4u(1-u)$ and, substituting in (2.20), we obtain again the same quadratic equation in u as in case (I).

A completely analogous reasoning applies to the variables x, y, z. The verification of (2.20) is an easy verification using (2.9).

To conclude the proof, if we write $x = ((\sigma-\sqrt{\rho})(n-1)+\sigma+\sqrt{\rho})/n$, then n is to be eliminated from

$$(\sigma-\sqrt{\rho})(n-1)+\sigma+\sqrt{\rho} = n\left(\frac{\sigma\pm\sqrt{\rho}}{2}+\frac{3}{2\lambda}\right), \tag{2.21}$$

where the right-hand side is $\{(\lambda+3\pm e)/4\lambda\}$. Hence, $n(\sigma-2\sqrt{\rho}\pm\sqrt{\rho}-3/\lambda) = -4\sqrt{\rho}$. We are to solve two cases corresponding to the sign in the last expression.

(I) $n(\sigma-3\sqrt{\rho}-3/\lambda) = -4\sqrt{\rho}$; then substituting the definitions for σ, ρ in the last expression gives $n = -4e/(\lambda-9-3e)$ or, by multiplying the denominator by $\lambda-9+3e$, we obtain $n = e(\lambda-9+3e)/2\lambda(\lambda-9)$ and, after simplification, we obtain the claimed value. To obtain the value for n, one solves the quadratic equation in terms of λ by the value found for n and obtains $\lambda = -(n-1)(n+2)/(2-n) < 0$.

(II) $n(\sigma-\sqrt{\rho}-3/\lambda) = -4\sqrt{\rho}$; then as in case (I), we obtain $n = -4e/(\lambda-9-e)$ by multiplying again by $\lambda-9+e$, and, simplifying the expression, we obtain the claimed value. The value for n is obtained in the same way as in case (I). □

An application of Lemma 2.5 is the following.

Corollary 2.6. *$7 < e/3$ if and only if $e > 21$ if and only if $\lambda > 5+\sqrt{407}$ and the last number is greater than 25. For example, if $\lambda = 30.0000$, then $e = 24.6779$. Consider values for $m < e/3 = 8.2259$, say, for example, $m = 7$; then $M = 7e/(21+6e) = 1.0217$, so choosing $n = 1.01$ will suffice.*

Proposition 2.7. *Let $l = (\sqrt{q_0}:\eta\sqrt{q_1}:\sqrt{q_2}:\eta\sqrt{q_3}:\sqrt{q_4}:\eta\sqrt{q_5})$ be such that $q_i \in \mathbb{R}_{>0}$ for all $i = 0,\dots,5$ with $q_0 > 1-q_1$ such that $n = 1.01$, $m = 7$ for $\lambda \geq 30.00$, $\lambda \neq 104.04$, and the triples (q_0,q_2,q_4), (q_1,q_3,q_5) are solutions to (2.9); then $l \in M^s$.*

Proof. The point l is not in

(1) $\mathcal{Q}$ trivially since $\Pi_{i=0}^5 q_i \neq 0$;
(2) Σ since $l \in \Sigma$, then $\sqrt{q_0} = \sqrt{q_1}$, hence $q_0 = q_1$, contrary to the assumption;
(3) $\mathcal{P}$; indeed, we need to check that $q_0 \notin \{q_2,q_3,q_4,q_5\}$ since already $q_0 \neq q_1$. But this is the statement of Lemma 2.5; we only need to verify that n, λ are not the exceptional case of the statement of Lemma 2.5(2). For that, note that only $\lambda = 104.04$ is possible. The chosen value for $m = 7$ is in accordance with Corollary 2.6.

By hypothesis, $l \in M$ and (1), (2), (3) above show that $l \in M^s$. □

Remark 2.8. It is interesting to note that by relaxing the condition, $\Pi_{i=0}^5 q_i \neq 0$. That is, if one $q_i = 0$, then another $q_j = 0$ with $j \neq i$ (see the commentary after (2.3)), then the quartic surfaces obtained are singular along two skew lines. They have already been studied in [1, Proposition 7.2(a)] and in [6, Proposition 5.1].

Remark 2.9. Fix $\lambda > 9$. We will follow the convention previous to Lemma 2.5 in writing the elements of (R,S) in the sequel. Let $\mathcal{T} = \{x \in (R,S) \mid 1 < n < M,\ n \neq (\lambda-1+e)/2\}$ and $\mathcal{U} = \{u \in (R,S) \mid 2 < m < e/3\}$, for $z \in \mathbb{R}$, let $\mathbb{R}_z = \{x \in \mathbb{R} \mid x > z\}$

and

$$\mathcal{I} = \{(r,s) \in \mathbb{R}^3_{>0} \times \mathbb{R}^3_{>0} \mid r = (x,y,z),\ s = (u,v,w), v, y \text{ as in (2.11)}\}, \tag{2.22}$$

and consider $\pi : \mathcal{I} \to \mathcal{T} \times \mathcal{U}$ such that $\pi(r,s) = (x,u)$. The construction of $\mathcal{I}$ implies that π maps bijectively onto $\mathcal{V} = \mathcal{T} \times \mathcal{U}$, hence $\dim(\mathcal{I}) = \dim(\pi(\mathcal{I}))$ by [2, Theorem 2.2.8]. Since each of $\mathcal{T}$, $\mathcal{U}$ is one-dimensional, $\dim(\pi(\mathcal{I})) = \dim(\mathcal{T} \times \mathcal{U})$, hence $\mathcal{I}$ is two-dimensional. In particular, for $\lambda \geq 30.0000$ and $\lambda \neq 104.04$, define a map $\varphi : \mathcal{V} \to \mathbb{CP}^5$ such that $\varphi(x,u) = (x : \eta y : z : \eta u : v : \eta w)$ with $((x,y,z),(u,v,w)) \in \mathcal{I}$. By Proposition 2.7, $\varphi(x,y)$ belongs to M^s and is injective, thus $\varphi(\mathcal{V})$ is a subset of M^s bijective to a two-dimensional variety.

3. The 32 lines on the quartic surface. Fix the three-dimensional real projective space $\mathbb{RP}_3$ with coordinates z_0, z_1, z_2, z_3, the quartic surface $X_f = \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{RP}_3,\ f(z_0,z_1,z_2,z_3) = 0\}$ given by the homogeneous polynomial f of degree 4 in the variables z_0, z_1, z_2, z_3 over the real numbers field $\mathbb{R}$, and a line in $\mathbb{RP}_3$ which is generated by a two-plane in $\mathbb{R}_4$ represented by a 2×4 matrix

$$\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix}. \tag{3.1}$$

One coordinate-free approach characterizes a line as a 2-form $\omega \in \bigwedge^2 \mathbb{R}^4$ and another one is to say that a line is given by a two-dimensional subspace $V \subset \mathbb{R}^4$ which yields a well-defined point in $\mathbb{RP}_5$. Choosing the canonical basis of $\bigwedge^2 \mathbb{R}^4$, one obtains the Plücker coordinates $\{p_{ij}\}$ (or P-coordinates) of the line as follows. Let

$$\Lambda = \begin{pmatrix} z_0 & z_1 & z_2 & z_3 \\ z_0' & z_1' & z_2' & z_3' \end{pmatrix} \tag{3.2}$$

be a 2×4 matrix and the minors of Λ given by

$$p_{i,j} = z_i z_j' - z_j z_i', \quad i,j \in \{0,1,2,3\}, \tag{3.3}$$

where $i \neq j$. If a matrix Λ is a two-plane, this means that $p_{i,j} \neq 0$ for some i,j, and the P-coordinates for this line satisfy the equation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0. \tag{3.4}$$

Conversely, if a point with P-coordinates $\{p_{i,j}\}$ satisfying the above equation is given, then the point representing $\{p_{i,j}\}$ is a two-plane (see, e.g., [3, Chapter 1, Section 5]). Let

H^t be the subgroup of $\mathrm{SL}(4,\mathbb{R})$ spanned by the transformations

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.5}$$

which satisfy the relations

$$\sigma_i^2 = \tau_i^2 = \mathrm{id}, \qquad \sigma_i\tau_i = -\tau_i\sigma_i, \tag{3.6}$$

for $i = 1, 2$. One obtains a central exact sequence of groups:

$$1 \longrightarrow \{\pm 1\} \longrightarrow H^t \longrightarrow G' \longrightarrow 0, \tag{3.7}$$

where $G' \simeq \mathbb{Z}_2^4$. The group H^t is the *Heisenberg group of level two*. The *explicit* action of H^t on f for a polynomial, as above, on the variables z_0, z_1, z_2, z_3 is given by the usual linear action on the polynomials of degree 4; in particular,

$$\sigma(z_0z_1z_2z_3) = z_0z_1z_2z_3 \quad \forall \sigma \in H^t. \tag{3.8}$$

Apply the following coordinate transformation in $\mathbb{C}^6$ to the P-coordinates:

$$\begin{aligned} x_0 &= p_{01} - p_{23}, & x_2 &= p_{02} + p_{13}, \\ x_1 &= \eta(p_{01} + p_{23}), & x_3 &= \eta(p_{02} - p_{13}), \\ x_4 &= p_{03} - p_{12}, & x_5 &= \eta(p_{03} + p_{12}). \end{aligned} \tag{3.9}$$

These are the Klein coordinates (or K-coordinates as a notational convenience) that satisfy

$$0 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = -2(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}) \tag{3.10}$$

which is the equation for Gr, the *Grassmannian variety*, which parametrizes the set of complex lines in $\mathbb{CP}_3$. The K-coordinates are eigenfunctions for the action of H^t on them. They are also very useful in studying properties of hypersurfaces in Gr, known classically as the line complex (c.f. [4, Chapter VIII, Section 130 and Chapter XII, Section 221]). An easy consequence of inverting the transformation given by (3.9) is the following corollary.

Corollary 3.1. *$l = (x_0 : x_1 : x_2 : x_3 : x_4 : x_5)$ with P-coordinates $\{p_{i,j}\}$ defines a real line (i.e., for all i, j, $p_{i,j} \in \mathbb{R}$) if and only if x_0, x_2, x_4 are all positive-real and x_1, x_3, x_5 are purely imaginary.*

In view of the previous corollary, it is quite natural to introduce the notation of (2.4). Using the definition of (2.5), the P-coordinates can be expressed in terms of the $\{q_i\}$-coordinates as

$$\begin{pmatrix} p_{01} \\ p_{03} \\ p_{13} \end{pmatrix} = \begin{pmatrix} \sqrt{q_0}+\sqrt{q_1} \\ \sqrt{q_4}+\sqrt{q_5} \\ \sqrt{q_2}-\sqrt{q_3} \end{pmatrix}, \qquad \begin{pmatrix} p_{02} \\ p_{23} \\ p_{12} \end{pmatrix} = \begin{pmatrix} \sqrt{q_2}+\sqrt{q_3} \\ -\left(\sqrt{q_0}-\sqrt{q_1}\right) \\ -\left(\sqrt{q_4}-\sqrt{q_5}\right) \end{pmatrix}. \tag{3.11}$$

We consider a line l with coordinates $\{p_{i,j}\}$ such that $p_{01} \neq 0$. For example, the line with coordinates

$$\begin{pmatrix} p_a \\ p_b \end{pmatrix} = \begin{pmatrix} 1 & 0 & x & y \\ 0 & 1 & u & v \end{pmatrix} \tag{3.12}$$

in $\mathbb{RP}_3$ is expressed using (3.11) in the $\{q_i\}$-coordinates as

$$\begin{aligned} u = p_{02} &= \sqrt{q_2}+\sqrt{q_3}, & v = p_{03} &= \sqrt{q_4}+\sqrt{q_5}, \\ y = -p_{13} &= -\left(\sqrt{q_2}-\sqrt{q_3}\right), & x = -p_{12} &= \sqrt{q_4}-\sqrt{q_5}. \end{aligned} \tag{3.13}$$

Let W be the complex vector space of quartic forms in the variables z_0, z_1, z_2, z_3 invariant under H^t; then by, for example, [5, Proposition 4.1.1(ii)], it is of dimension five; and let Gr be as before. Let $\mathcal{I}$ be the incidence variety given by

$$\mathcal{I} = \left\{(l,v) \in \mathrm{Gr}\times\mathbb{P}(W) \,\middle|\, f_v\big|_l = 0\right\}. \tag{3.14}$$

If g_0, g_1, g_2, g_3, g_4 is a basis of W, then, for every $v \in W$ with $v = (v_0 : \cdots : v_4)$, let $f_v = \Sigma_{i=0}^4 v_i g_i$ be the associated quartic polynomial. Let π be the projection of $\mathcal{I}$ into Gr. Let $l \in \mathrm{Gr}$ and let $\mathcal{I}_l$ be the fibre under π. For different points of $l \in M$, $\mathcal{I}_l$ has already been calculated in [1, Proposition 7.1] and in [5, Corollary 3.4.3]. What is needed in this situation is an easy extension of [5, Lemma 3.3.1] to the real case and it can be stated as follows.

Lemma 3.2. *Let $l \in M^s$ define a line with real $p_{i,j}$ coordinates. Then $\mathcal{I}_l = (l,v)$ for a unique $v \in \mathbb{RP}_4$.*

Proof. The point v is a solution to a system of nonhomogeneous equations with entries over the real $p_{i,j}$ coordinates in [5, Lemma 3.3.1], hence the claim. □

The proof of the following corollary is a direct consequence of Proposition 2.7 and the calculations are left as an easy verification.

Corollary 3.3. *Let $l = (\sqrt{q_0} : \eta\sqrt{q_1} : \sqrt{q_2} : \eta\sqrt{q_3} : \sqrt{q_4} : \eta\sqrt{q_5})$ satisfy the hypothesis of Proposition 2.1. By Lemma 3.2, the point ν defines a real smooth quartic surface $X_f = \{f_\nu = 0\}$ which contains the G' orbit of l and of a line l' which will be defined as follows. In the K-coordinates $\{x_i\}$, the involution*

$$' : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \longmapsto \left(-\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3} : \frac{1}{x_4} : \frac{1}{x_5}\right), \tag{3.15}$$

which is well defined away from the fourfolds $\{x_i = 0\}$ applied to l, gives a line l'. Writing the P-coordinates associated to this line as $\{p'_{i,j}\}$ (for this, let $q' = -1/\sqrt{q_0} - 1/\sqrt{q_1}$),

$$\begin{gathered}
\begin{pmatrix} p'_{01} \\ q' \cdot p'_{03} \\ q' \cdot p'_{13} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{1}{\sqrt{q_4}} - \dfrac{1}{\sqrt{q_5}} \\ \dfrac{1}{\sqrt{q_2}} + \dfrac{1}{\sqrt{q_3}} \end{pmatrix}, \\
\begin{pmatrix} q' \cdot p'_{02} \\ q' \cdot p'_{23} \\ q' \cdot p'_{12} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\sqrt{q_2}} - \dfrac{1}{\sqrt{q_3}} \\ \dfrac{1}{\sqrt{q_0}} - \dfrac{1}{\sqrt{q_1}} \\ -\dfrac{1}{\sqrt{q_4}} - \dfrac{1}{\sqrt{q_5}} \end{pmatrix}.
\end{gathered} \tag{3.16}$$

Remark 3.4. The two orbits of lines that are in X_f can be grouped as the "even" lines, that is, as those having an even number of minus signs in their K-coordinates, and the "odd" lines as those having an odd number of minus signs in their K-coordinates (in fact, using these K-coordinates, we have studied in [1, Proposition 4.2] group-theoretical properties of the configuration of these lines). It is clear that if X_f contains the double 16, it contains its even and odd lines.

It is now clear from Corollary 3.3 that if X_f is a real H^t-invariant quartic surface defined by l, X_f contains the double 16 of Corollary 3.3. The quartic surface above can contain more than the double 16 of lines. If the quartic surface is the image of a polarized abelian surface of type $(1,3)$ as an irreducible polarized abelian surface, then the image surface contains only the double 16 (c.f. [1, Propositions 6.4 and 6.7]).

4. Description of the program. The program written in Maple 7 (a copy of the program is available upon request) defines a global variable d (this is the value of λ in Proposition 2.1) which has to be given as initial input in the program. Using the subroutines named *Var, Vas*, the values for R, S which are polynomial expressions in terms of λ are calculated. Using the intervals for the solutions given in Proposition 2.1 for q_0 (resp., for q_1), these are calculated by two other routines named *Np, Jb*. In order to evaluate the global variable q_2, one needs to introduce the local variables M, N in terms of d, q_0 and finally evaluate Sq_1. A subroutine then evaluates the positive root of q_2 in terms of the local variables N, Sq_1, and $dq_0 - 1$. q_4 is evaluated introducing the routine rw which uses the equality $q_4 = 1-(q_0+q_2)$. Using the value for q_1, the program uses the routine rz to evaluate q_3, and, applying the routine rw again, it evaluates q_5 in exactly the same way.

In order to draw the lines, one first evaluates the parametric equation for the lines. We introduce the local variables rr, ss in terms of q_4, q_5, q_2, q_3. The last variables are used to give the parametric equation of the line l as given by (3.11). The orbit of l under H^t is evaluated using the procedure *Graf*. The parametric equation of the transversal to l is evaluated using the variables m, n. The procedure *Mpoly* substitutes the variables x, y, u, v for the obtained values rr, ss, zz, ww together with (3.16), which are used to evaluate the values for m and n. The coefficients for the quartic surface are evaluated by means of the nonhomogeneous system of equations described in the proof of Lemma 3.2. Introducing the routine *Mpoly*, the matrix solution is saved as a 4×1 matrix named K. It then defines the H^t quartic invariant polynomial saved as the variable *quar*. The polynomial coefficients of *quar* are defined as the entries of K. This defines a new quartic polynomial in the variables x,y,u,v named quars. By substituting $v=-x-y-u$ in quars, one obtains a new quartic polynomial quart in the variables x, y, u, which gives us the intersection of the quartic surface defined by quars and the special hyperplane $H=\{(x,y,u,v)\in\mathbb{R}^4 \mid v+u+x+y=0\}$. The next routine expands quart and this quartic polynomial is saved as quarn. The program then has to graph $\{\text{quarn}=0\}$ implicitly in terms of a given variable, which we chose as x. For this, we use the *display3d* command of the library plots of Maple. The body of the command consists of the range for x, y, u and the plotting options consist of the grid values, which we chose (unless otherwise specified in Table 4.1) as the default value of 25 for the three variables specified in the range; the style command which specifies how the surface is to be plotted was fixed as *patchnogrid;* the user-defined lighting in Maple is specified by the red, green, and blue components of the *ambientlight* command which was fixed as $[0.6,0.6,0.6]$ (note that the default values are all values set equal to 1), the orientation command has to be specified as a pair (u,v), where u is the horizontal angle, v is the vertical angle, which in the group of test examples we chose to be the values given in the table. In order to visualize the lines, one has to specify the parametric equation of the lines using the variables already calculated, rr, ss, m, n; this is performed by the *spacecurve* command. As optional commands within the last command, the thickness of the lines which was fixed as 1 is given, the color of the lines and the direction from which each line is to be viewed are given again by the orientation command. Also, as part of the *display3d* optional command, one has to specify the values for two directional light sources given by the light command, which consists of a quintuple of values: the first two values are (v,u), where v is the vertical angle and u is the horizontal angle, and the other three values specify the intensities of the red, green, and blue colors. In the test samples given in the table, these were fixed as $[90,-80,0.7,0.6,0.1]$, $[90,80,0.7,0.6,0.1]$. In the image produced by the procedure *Graf* of the Maple program in all the test examples, the line l was chosen in red. The remaining colors for the disjoint lines were chosen as follows: blue, yellow, sienna, cyan, khaki, pink, turquoise, aquamarine, magenta, plum, violet, brown, green, navy, and gold. The ten transversals to l have been drawn in color black.

TABLE 4.1

Description of the configuration of double 16			
Value for d	Angle (u, v)	Compilation time	Number of transversal lines, other visual properties
30.00			
		40″, 1′0.01″,	Six transversals
		1′0.78″,	Six transversals
		1′10.18″,	Six transversals
		1′22.04″, 53.74″	Six transversals
	(−52, 48)		Six transversals
	(−10, 82)		Five transversals
	(−34, 60)		Six transversals
	(−22, 52)		Five transversals
	(−52, 44), (−52, 40), (−56, 62)		Six transversals
	(−56, 66), (−54, 52), (−54, 68)		Six transversals
	(−54, 52), (−54, 68), (−64, 60)		Six transversals
	(−62, 30), (−70, 12)		Six transversals
35.00			
	(42,) full rotation on v angle		Six transversals
37.00			
	(16, 33), (−12, 47)		Six transversals
	(−48, 89)		Six transversals
	(−56, 47)		Seven transversals
40.00	2′33.97″, 1′42.19″		
	(6, 25), (−46, 73)		Six transversals
50.00		1′58.14″, 1′16.57″	
	(−48, 96)		Four transversals
	(−60, 96)		Five transversals
	(−66, 96)		Five transversals
	(−32, 64)		Six transversals
60.00		51.04″, 49.89″	
		40.54″	
	(−171, −54)		Four transversals
	(155, −56)		Five transversals
70.00		1′42.92″, 1′26.87″	
		1′27.74″	
	(−28, 52)		Four transversals
	(−12, 42)		Five transversals
	(128, −88)		Five transversals
	(154, −170)		Five transversals

TABLE 4.1 Continued.

Description of the configuration of double 16			
Value for d	Angle (u,v)	Compilation time	Number of transversal lines, other visual properties
80.00		58.30″, 124.30″	
		122.70″	
	$(-20,44)$		Five transversals
	$(-36,44)$		Five transversals
90.00		43.45″, 40.72″	
	$(166,-30)$		Five transversals
	$(174,-38)$,$(134,-64)$		Five transversals
100.00		1′07.28″	
	$(-24,26)$		Five transversals
	$(144,-60)$		Five transversals
	$(42,22)$		Five transversals
	$(-24,48)$		Five transversals

5. Application of the program. We treat the problem of visualizing examples of real quartic surfaces obtained by intersecting a real Heisenberg invariant quartic surface containing the double 16 configuration of lines with the hyperplane H. To show as clearly as possible the double 16 configuration, we only drew, in each case in the table, the red line on the surface and its ten transversals drawn in black including 15 other transversals with the colors mentioned at the end of Section 4. Another remark concerning the values for the orientation of the image given in the table is that the orientation was found by rotating the image and stopped at the angles (u,v) showing the visual properties of the surface and lines as described in the table. The program was run on a Pentium II. In the test examples given in the table, the following properties were tabulated: the value for d, the orientation of the surface specified by the angle (u,v), the approximate compilation time of the program to calculate the image, and the number of lines of the double 16 visible on the screen. These parameters are tested for each of the examples as given in the table. Note that the first row of the table for each value of d (when recorded) gives the compilation time, and, for this value of d, all other entries do not consider this parameter. As one can see from the table, only at most six of the transversals were clearly visible from the given values of d. It is of interest (although not recorded in the table) to note that at the angle $(42,136)$ without drawing the surface for $d = 37.00$, one can see the red line intersecting the ten transversals drawn in black. See also Figure 5.1. It would be of interest to check if the surfaces considered in the table are actually singular and irreducible.

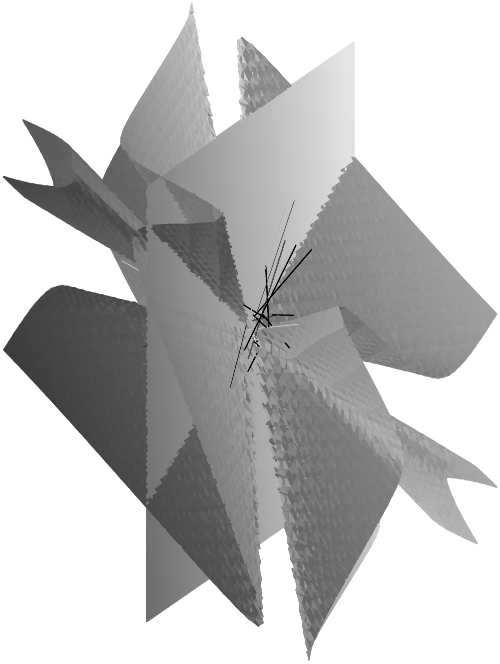

FIGURE 5.1. The quartic surface for $d = 30.00$ at $(-54,68)$.

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