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A SUMMABILITY FACTOR THEOREM FOR ABSOLUTE SUMMABILITY INVOLVING ALMOST INCREASING SEQUENCES

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We obtain sufficient conditions for the series $\sum a_n\lambda_n$ to be absolutely summable of order k by a triangular matrix.

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A triangle is a lower triangular matrix with all principal diagonal entriesbeing nonzero. Given an almost increasing sequence $\{X_n\}$ and a sequence $\{\lambda_n\}$ satisfying certain conditions, we obtain sufficient conditions for the series $\sum a_n\lambda_n$ to be absolutely summable of order $k \geq 1$ by a triangle T. As a corollary we obtain the corresponding result when T is a weighted mean matrix.

Theorem 1 of this paper is an example of a summability factor theorem. There is a large literature dealing with summability factor theorems. For example, MathSciNet lists over 500 papers dealing with this topic. For some other papers treating absolute summability factor theorems (of order $k \geq 1$) the reader may wish to consult [3, 4, 5].

Let T be a lower triangular matrix,and $\{s_n\}$ a sequence. Then

$$T_n := \sum_{\nu=0}^{n} t_{n\nu}s_\nu. \tag{1}$$

A series $\sum a_n$, with partial sums s_n, is said to be summable $|T|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}\left|T_n - T_{n-1}\right|^k < \infty. \tag{2}$$

We may associate with T two lower triangular matrices $\overline{T}$ and $\hat{T}$ as follows:

$$\begin{aligned} \bar{t}_{n\nu} &= \sum_{r=\nu}^{n} t_{nr} \quad n,\nu = 0,1,2,\ldots, \\ \hat{t}_{n\nu} &= \bar{t}_{n\nu} - \bar{t}_{n-1,\nu} \quad n = 1,2,3,\ldots. \end{aligned} \tag{3}$$

We may write

$$T_n = \sum_{\nu=0}^{n} t_{n\nu}\sum_{i=0}^{\nu} a_i\lambda_i = \sum_{i=0}^{n} a_i\lambda_i\sum_{\nu=i}^{n} t_{n\nu} = \sum_{i=0}^{n} \bar{t}_{ni}a_i\lambda_i. \tag{4}$$

Thus

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i \\
&= \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n} \bar{t}_{n-1,i} a_i \lambda_i \\
&= \sum_{i=0}^{n} (\bar{t}_{ni} - \bar{t}_{n-1,i}) a_i \lambda_i \\
&= \sum_{i=0}^{n} \hat{t}_{ni} a_i \lambda_i = \sum_{i=0}^{n} \hat{t}_{ni} \lambda_i (s_i - s_{i-1}) \\
&= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i - \sum_{i=0}^{n} \hat{t}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i + \hat{t}_{nn} \lambda_n s_n - \sum_{i=0}^{n} \hat{t}_{ni} \lambda_i s_{i-1} \\
&= \sum_{i=0}^{n-1} \hat{t}_{ni} \lambda_i s_i + t_{nn} \lambda_n s_n - \sum_{i=0}^{n-1} \hat{t}_{n,i+1} \lambda_{i+1} s_i \\
&= \sum_{i=0}^{n} (\hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1}) s_i + t_{nn} \lambda_n s_n.
\end{aligned} \tag{5}$$

We may write

$$\begin{aligned}
(\hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1}) &= \hat{t}_{ni} \lambda_i - \hat{t}_{n,i+1} \lambda_{i+1} - t_{n,i+1} \lambda_i + t_{n,i+1} \lambda_i \\
&= (\hat{t}_{ni} - \hat{t}_{n,i+1}) \lambda_i + t_{n,i+1} (\lambda_i - \lambda_{i+1}) \\
&= \lambda_i \Delta_i \hat{t}_{ni} + \hat{t}_{n,i+1} \Delta \lambda_i.
\end{aligned} \tag{6}$$

Therefore

$$\begin{aligned}
T_n - T_{n-1} &= \sum_{i=0}^{n-1} \Delta_i \hat{t}_{ni} \lambda_i s_i + \sum_{i=0}^{n-1} \hat{t}_{n,i+1} \Delta \lambda_i s_i + t_{nn} \lambda_n s_n \\
&= T_{n1} + T_{n2} + T_{n3}.
\end{aligned} \tag{7}$$

A positive sequence $\{b_n\}$ is said to be almost increasing if there exists a positive increasing sequence $\{c_n\}$ and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$ for each n.

Theorem 1. *Let $\{X_n\}$ be an almost increasing sequence and let $\{\beta_n\}$ and $\{\lambda_n\}$ be sequences such that*

(i) $|\Delta\lambda_n| \le \beta_n$,

(ii) $\lim \beta_n = 0$,

(iii) $\sum_{n=1}^{\infty} n |\Delta\beta_n| X_n < \infty$,

(iv) $|\lambda_n| X_n = O(1)$

are satisfied.

Let T be a triangle satisfying the following:

(v) $nt_{nn} = O(1)$,

(vi) $t_{n-1,\nu} \geq t_{n\nu}$ *for* $n \geq \nu + 1$,

(vii) $\bar{t}_{n0} = 1$ *for all* n,

(viii) *if* $\sum_{n=1}^{m} (1/n)|s_n|^k = O(X_m)$, *then the series* $\sum a_n \lambda_n$ *is summable* $|T|_k$, $k \geq 1$.

Proof. To prove the theorem it will be sufficient to show that

$$\sum_{n=1}^{\infty} n^{k-1} \left| T_{nr} \right|^k < \infty \quad \text{for } r = 1, 2, 3. \tag{8}$$

From [2, page 86] it follows that (vi) and (vii) imply that

$$\sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| = O(t_{nn}). \tag{9}$$

From (iv) and the fact that $\{X_n\}$ is almost increasing, it follows that $\lambda_n = O(1)$. Using Hölder's inequality, (9), and (v),

$$\begin{aligned}
\sum_{n=1}^{m+1} n^{k-1} \left| T_{n1} \right|^k &\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| \left| \lambda_i \right| \left| s_i \right| \right)^k \\
&\leq \sum_{n=1}^{m+1} n^{k-1} \left(\sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| \left| \lambda_i \right|^k \left| s_i \right|^k \right) \left(\sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| \right)^{k-1} \\
&= O(1) \sum_{n=1}^{m+1} (nt_{nn})^{k-1} \sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| \left| \lambda_i \right|^k \left| s_i \right|^k \\
&= O(1) \sum_{n=1}^{m+1} (nt_{nn})^{k-1} \sum_{i=0}^{n-1} \left| \Delta_i \hat{t}_{ni} \right| \left| \lambda_i \right| \left| \lambda_i \right|^{k-1} \left| s_i \right|^k \\
&= O(1) \sum_{i=0}^{m} \left| \lambda_i \right| \left| s_i \right|^k \sum_{n=i+1}^{m+1} (nt_{nn})^{k-1} \left| \Delta_i \hat{t}_{ni} \right| \qquad (10) \\
&= O(1) \sum_{i=0}^{m} \left| \lambda_i \right| \left| s_i \right|^k t_{ii} \\
&= O(1) \sum_{i=0}^{m} \left| \lambda_i \right| \left(\sum_{r=0}^{i} \left| s_r \right|^k t_{rr} - \sum_{r=0}^{i-1} \left| s_r \right|^k t_{rr} \right) \\
&= O(1) \left[\sum_{i=0}^{m} \left| \lambda_i \right| \sum_{r=0}^{i} \left| s_r \right|^k t_{rr} - \sum_{j=0}^{m-1} \left| \lambda_{j+1} \right| \sum_{r=0}^{j} \left| s_r \right|^k t_{rr} \right] \\
&= O(1) \left[\sum_{i=0}^{m-1} \Delta \left| \lambda_i \right| \sum_{r=0}^{i} \left| s_r \right|^k t_{rr} + \left| \lambda_m \right| \sum_{r=0}^{m} \left| s_r \right|^k t_{rr} \right].
\end{aligned}$$

If we define

$$X_i = \sum_{r=0}^{i} \left| s_r \right|^k t_{rr}, \tag{11}$$

then $\{X_i\}$ is an almost increasing sequence. Using (iv) and (i),

$$\sum_{n=1}^{m+1} n^{k-1}|T_{n1}|^k = O(1)\sum_{i=0}^{m-1}\beta_i X_i + O(1) = O(1), \tag{12}$$

using the result of [1, Lemma 3].

Using (i), Hölder's inequality, (ii), (v), (vi), and (vii),

$$\begin{aligned}
\sum_{n=1}^{m+1} n^{k-1}|T_{n2}|^k &\le \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{i=0}^{n-1}\hat{t}_{n,i+1}s_i\Delta\lambda_i\right|^k \\
&\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\hat{t}_{n,i+1}||\Delta\lambda_i||s_i|\right)^k \\
&\le \sum_{n=1}^{m+1} n^{k-1}\left(\sum_{i=0}^{n-1}|\hat{t}_{n,i+1}||s_i|\beta_i\right)^k \\
&\le \sum_{n=1}^{m+1} n^{k-1}\sum_{i=0}^{n-1}|\hat{t}_{n,i+1}||s_i|^k\beta_i\left(\sum_{i=0}^{n-1}|\hat{t}_{n,i+1}|\beta_i\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m+1}(nt_{nn})^{k-1}\sum_{i=0}^{n-1}|\hat{t}_{n,i+1}||s_i|^k\beta_i \\
&= O(1)\sum_{i=1}^{m}\beta_i|s_i|^k\sum_{n=i+1}^{m+1}|\hat{t}_{n,i+1}| \\
&= O(1)\sum_{i=1}^{m}\beta_i|s_i|^k = O(1)\sum_{i=1}^{m} i\beta_i\frac{1}{i}|s_i|^k \\
&= O(1)\sum_{i=1}^{m}\beta_i\left[\sum_{r=1}^{i}\frac{|s_r|^k}{r} - \sum_{r=1}^{i-1}\frac{|s_r|^k}{r}\right] \\
&= O(1)\left[\sum_{i=1}^{m} i\beta_i\sum_{r=1}^{i}\frac{|s_r|^k}{r} - \sum_{j=1}^{m-1}(j+1)\beta_{j+1}\sum_{r=1}^{j}\frac{|s_r|^k}{r}\right] \\
&= O(1)\sum_{i=1}^{m-1}\Delta(i\beta_i)\sum_{r=1}^{i}\frac{1}{r}|s_r|^k + O(1)m\beta_m\sum_{i=0}^{m}\frac{1}{i}|s_i|^k \\
&= O(1)\sum_{i=1}^{m-1}|\Delta(i\beta_i)|X_i + O(1)m\beta_m X_m \\
&= O(1)\sum_{i=0}^{m-1} i|\Delta(\beta_i)|X_i + O(1)\sum_{i=0}^{m-1}\beta_i X_i + O(1)m\beta_m X_m = O(1),
\end{aligned} \tag{13}$$

again using [1, Lemma 3].

Using (v),

$$\begin{aligned}\sum_{n=1}^{m+1} n^{k-1}|T_{n3}|^k &\le \sum_{n=1}^{m+1} n^{k-1}|t_{nn}\lambda_n s_n|^k \\ &= O(1)\sum_{n=1}^{m}(nt_{nn})^{k-1}t_{nn}|\lambda_n|^k|s_n|^k \qquad (14)\\ &= O(1)\sum_{n=1}^{m} t_{nn}|\lambda_n|^{k-1}|\lambda_n||s_n|^k = O(1),\end{aligned}$$

as in the proof of T_{n1}. □

A weighted mean matrix, denoted by $(\overline{N}, p_n)$, is a lower triangular matrix with nonzero entries p_k/P_n, where $\{p_n\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{i=0}^{n} p_i \to \infty$ as $n \to \infty$.

Corollary 2. *Let $\{X_n\}$ be an almost increasing sequence and let condition (viii) of Theorem 1 be satisfied. If $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy conditions (i)–(iv) of Theorem 1 and if $\{p_n\}$ is a sequence such that*

(i) $np_n/P_n = O(1)$,

then the series $\sum a_n\lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \ge 1$.

Proof. With $T = (\overline{N}, p_n)$, condition (v) of Theorem 1 reduces to condition (i). Conditions (vi) and (vii) of Theorem 1 are automatically satisfied. □

It should be noted that, in [1], an incorrect definition of absolute summability was used (see, e.g., [2]). Corollary 2 gives the correct version of Bor's theorem.

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References

[1] H. Bor, *An application of almost increasing sequences*, Int. J. Math. Math. Sci. **23** (2000), no. 12, 859–863.

[2] B. E. Rhoades, *Inclusion theorems for absolute matrix summability methods*, J. Math. Anal. Appl. **238** (1999), no. 1, 82–90.

[3] B. E. Rhoades and E. Savaş, *Some necessary conditions for absolute matrix summability factors*, Indian J. Pure Appl. Math. **33** (2002), no. 7, 1003–1009.

[4] ______, *Some summability factor theorems for absolute summability*, Analysis (Munich) **22** (2002), no. 2, 163–174.

[5] ______, *A summability type factor theorem*, Tamkang J. Math. **34** (2003), no. 4, 395–401.

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