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OSCILLATION PROPERTIES OF NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF nTH ORDER

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We consider the nonlinear neutral functional differential equation $[r(t)[x(t)+\int_a^b p(t,\mu)x(\tau(t,\mu))d\mu]^{(n-1)}]'+\delta\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi=0$ with continuous arguments. We will develop oscillatory and asymptotic properties of the solutions.

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1. Introduction. Recently, several authors [2, 3, 4, 5, 6, 7, 12, 13, 14] have studied the oscillation theory of second-order and higher-order neutral functional differential equations, in which the highest-order derivative of the unknown function is evaluated both at the present state and at one or more past or future states. For some related results, refer to [1, 8, 10, 11].

In this paper, we extend these results to nth-order nonlinear neutral equations with continuous arguments

$$\left[r(t)\left[x(t)+\int_a^b p(t,\mu)x(\tau(t,\mu))d\mu\right]^{(n-1)}\right]'+\delta\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi=0, \tag{1.1}$$

where $\delta=\mp1$, $t\geq 0$, and establish some new oscillatory criteria. Suppose that the following conditions hold:

(a) $r(t)\in C([t_0,\infty),\mathbb{R})$, $r(t)\in C^1$, $r(t)>0$, and $\int^\infty (dt/r(t))=\infty$;

(b) $p(t,\mu)\in C([t_0,\infty)\times[a,b],\mathbb{R})$, $0\leq p(t,\mu)$;

(c) $\tau(t,\mu)\in C([t_0,\infty)\times[a,b],\mathbb{R})$, $\tau(t,\mu)\leq t$ and $\lim_{t\to\infty}\min_{\mu\in[a,b]}\tau(t,\mu)=\infty$;

(d) $q(t,\xi)\in C([t_0,\infty)\times[c,d],\mathbb{R})$ and $q(t,\xi)>0$;

(e) $f(x)\in C(\mathbb{R},\mathbb{R})$ and $xf(x)>0$ for $x\neq 0$;

(f) $\sigma(t,\xi)\in C([t_0,\infty)\times[c,d],\mathbb{R})$, and

$$\lim_{t\to\infty}\min_{\xi\in[c,d]}\sigma(t,\xi)=\infty. \tag{1.2}$$

A solution $x(t)\in C[t_0,\infty)$ of (1.1) is called oscillatory if $x(t)$ has arbitrarily large zeros in $[t_0,\infty)$, $t_0>0$. Otherwise, $x(t)$ is called nonoscillatory.

2. Main results. We will prove the following lemma to be used in Theorem 2.2.

Lemma 2.1. *Let $x(t)$ be a nonoscillatory solution of (1.1) and let $z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu$. Then, the following results hold:*

(i) *there exists a $T > 0$ such that for $\delta = 1$,*

$$z(t)z^{(n-1)}(t) > 0, \quad t \ge T, \tag{2.1}$$

and for $\delta = -1$ either

$$z(t)z^{(n-1)}(t) < 0, \quad t \ge T, \qquad \text{or} \qquad \lim_{t\to\infty} z^{(n-2)}(t) = \infty, \tag{2.2}$$

(ii) *if $r'(t) \ge 0$, then there exists an integer l, $l \in \{0,1,\dots,n\}$ with $(-1)^{n-l-1}\delta = 1$ such that*

$$\begin{gathered} z^{(i)}(t) > 0 \quad \text{on } [T,\infty) \quad \text{for } i = 0,1,2,\dots,l, \\ (-1)^{i-l}z^{(i)}(t) > 0 \quad \text{on } [T,\infty) \quad \text{for } i = l,l+1,\dots,n \end{gathered} \tag{2.3}$$

for some $t \ge T$.

Proof. Let $x(t)$ be an eventually positive solution of (1.1), say $x(t) > 0$ for $t \ge t_0$. Then, there exits a $t_1 \ge t_0$ such that $x(\tau(t,\mu))$ and $x(\sigma(t,\xi))$ are also eventually positive for $t \ge t_1$, $\xi \in [c,d]$, and $\mu \in [a,b]$. Since $x(t)$ is eventually positive and $p(t,\mu)$ is nonnegative, we have

$$z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu > 0 \quad \text{for } t \ge t_1. \tag{2.4}$$

(i) From (1.1), we have

$$\delta[r(t)z^{(n-1)}(t)]' = -\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi. \tag{2.5}$$

Since $q(t,\xi) > 0$ and f is positive for $t \ge t_1$, we have $\delta[r(t)z^{(n-1)}(t)]' < 0$. For $\delta = 1$, $r(t)z^{(n-1)}(t)$ is a decreasing function for $t \ge t_1$. Hence, we can have either

$$r(t)z^{(n-1)}(t) > 0 \quad \text{for } t \ge t_1 \tag{2.6}$$

or

$$r(t)z^{(n-1)}(t) < 0 \quad \text{for } t \ge t_2 \ge t_1. \tag{2.7}$$

We claim that (2.6) is satisfied for $\delta = 1$. Suppose this is not the case, then we have (2.7). Since $r(t)z^{(n-1)}(t)$ is decreasing,

$$r(t)z^{(n-1)}(t) \le r(t_2)z^{(n-1)}(t_2) < 0 \quad \text{for } t \ge t_2. \tag{2.8}$$

Divide both sides of the last inequality by $r(t)$ and integrate from t_2 to t, respectively, then we obtain

$$z^{(n-2)}(t) - z^{(n-2)}(t_2) \le r(t_2)z^{(n-1)}(t_2)\int_{t_2}^t \frac{dt}{r(t)} < 0 \quad \text{for } t \ge t_2. \tag{2.9}$$

Now, taking condition (a) into account we can see that $z^{(n-2)}(t)-z^{(n-2)}(t_2)\to-\infty$ as $t\to\infty$. That implies $z(t)\to-\infty$, but this is a contradiction to $z(t)>0$. Therefore, for $\delta=1$,

$$r(t)z^{(n-1)}(t)>0 \quad \text{for } t\ge t_1. \tag{2.10}$$

Since both $z(t)$ and $r(t)$ are positive, we conclude that

$$z(t)z^{(n-1)}(t)>0. \tag{2.11}$$

For $\delta=-1$, $r(t)z^{(n-1)}(t)$ is increasing. Hence, either

$$r(t)z^{(n-1)}(t)<0 \quad \text{for } t\ge t_1, \tag{2.12}$$

or

$$r(t)z^{(n-1)}(t)>0 \quad \text{for } t\ge t_2\ge t_1. \tag{2.13}$$

If (2.12) holds, we replace $z(t)$ for $r(t)$ to get

$$z(t)z^{(n-1)}(t)<0. \tag{2.14}$$

If (2.13) holds, using the increasing nature of $r(t)z^{(n-1)}(t)$, we obtain

$$r(t)z^{(n-1)}(t)\ge r(t_2)z^{(n-1)}(t_2)>0 \quad \text{for } t\ge t_2. \tag{2.15}$$

Divide both sides of (2.15) by $r(t)$ and integrate from t_2 to t, then we get

$$z^{(n-2)}(t)-z^{(n-2)}(t_2)\ge r(t_2)z^{(n-1)}(t_2)\int_{t_2}^{t}\frac{dt}{r(t)}>0 \quad \text{for } t\ge t_2. \tag{2.16}$$

Taking condition (a) into account, it is not difficult to see that $z^{(n-2)}(t)\to\infty$ as $t\to\infty$. Hence, for $\delta=-1$, either (2.14) holds or $\lim_{t\to\infty}z^{(n-2)}(t)=\infty$.

(ii) From (1.1), we can see that

$$\delta\left[r'(t)z^{(n-1)}(t)+r(t)z^{(n)}(t)\right]=-\int_c^d q(t,\xi)f(x(\sigma(t,\xi)))d\xi, \tag{2.17}$$

and then

$$\delta z^{(n)}(t)=-\frac{\delta r'(t)z^{(n-1)}(t)}{r(t)}-\int_c^d\frac{q(t,\xi)f(x(\sigma(t,\xi)))d\xi}{r(t)}. \tag{2.18}$$

Using (i) and (2.18), we obtain

$$\delta z^{(n)}(t)<0. \tag{2.19}$$

Suppose that $\lim_{t\to\infty}z^{(n-2)}(t)\ne\infty$ when $\delta=-1$. Thus, because of the positive nature of $z(t)$ and (2.19), there exists an integer l, $l\in\{0,1,\dots,n\}$ with $(-1)^{n-l-1}\delta=1$ by

Kiguradze's lemma [9] such that

$$\begin{aligned} &z^{(i)}(t)>0 \quad \text{on } [T,\infty) \quad \text{for } i=0,1,2,\dots,l,\\ &(-1)^{i-l}z^{(i)}(t)>0 \quad \text{on } [T,\infty) \quad \text{for } i=l,l+1,\dots,n \end{aligned} \tag{2.20}$$

for some $t \ge T$.

If $\lim_{t\to\infty} z^{(n-2)}(t) = \infty$ and $\delta = -1$, $z^{(n-1)}(t)$ is eventually positive. Moreover, $z^{(n)}(t)$ is also eventually positive by (2.19). But, this is the case $l = n$ in (2.20). Thus, the proof is complete. □

Theorem 2.2. *Let $P(t) = \int_a^b p(t,\mu)d\mu < 1$. Suppose that f is increasing and for all constant $k > 0$,*

$$\int^{\infty}\int_c^d q(s,\xi)f((1-P(\sigma(s,\xi)))k)d\xi\, ds = \infty. \tag{2.21}$$

(i) *If $\delta = 1$, then every solution $x(t)$ of (1.1) is oscillatory when n is even, and every solution $x(t)$ of (1.1) is either oscillatory or satisfies*

$$\liminf_{t\to\infty} |x(t)| = 0 \tag{2.22}$$

when n is odd.

(ii) *If $\delta = -1$, then every solution $x(t)$ of (1.1) is either oscillatory or else*

$$\lim_{t\to\infty} |x(t)| = \infty \quad \text{or} \quad \liminf_{t\to\infty} |x(t)| = 0 \tag{2.23}$$

when n is even, and every solution $x(t)$ of (1.1) is either oscillatory or else

$$\lim_{t\to\infty} |x(t)| = \infty \tag{2.24}$$

when n is odd.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1), say $x(t) > 0$ for $t \ge t_0$. Let $z(t)$ be a function defined by

$$z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu. \tag{2.25}$$

Recall from Lemma 2.1, if $\delta = 1$, then (2.1) holds and if $\delta = -1$, either $z(t)z^{(n-1)}(t) < 0$ for $t \ge T$ or $\lim_{t\to\infty} z^{(n-2)}(t) = \infty$.

Suppose that $\lim_{t\to\infty} z^{(n-2)}(t) \ne \infty$ for $\delta = -1$. Thus, there exist a $t_1 \ge T$ and an integer $l \in \{0,1,\dots,n-1\}$ with $(-1)^{n-l-1}\delta = 1$ such that

$$\begin{aligned} &z^{(i)}(t)>0, \quad i=0,1,2,\dots,l,\\ &(-1)^{i-l}z^{(i)}(t)>0, \quad i=l,l+1,\dots,n,\ t\ge t_1, \end{aligned} \tag{2.26}$$

by Kiguradze's lemma [9].

Let n be even and $\delta = 1$, or n be odd and $\delta = -1$. Since $(-1)^{n-l-1}\delta = (-1)^{-l-1} = 1$, then l is odd. Now, $z(t)$ is increasing by (2.26). Therefore, we have

$$z(t) = x(t) + \int_a^b p(t,\mu)x(\tau(t,\mu))d\mu \le x(t) + \int_a^b p(t,\mu)z(\tau(t,\mu))d\mu, \tag{2.27}$$

since $x(t) \le z(t)$. Since $z(t)$ is increasing and $\tau(t,\mu) < t$, this will imply that

$$z(t) \le x(t) + P(t)z(t). \tag{2.28}$$

Thus, we have

$$(1 - P(t))z(t) \le x(t). \tag{2.29}$$

On the other hand, we have $z(t)$ positive and increasing with $\lim_{t\to\infty} \min_{\xi\in[a,b]} \sigma(t,\xi) = \infty$. These imply that there exist a $k > 0$ and a $t_2 \ge t_1$ such that

$$z(\sigma(t,\xi)) \ge k \quad \text{for } t \ge t_2. \tag{2.30}$$

Integrating (1.1) from t_2 to t, then we have

$$\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_c^d q(s,\xi)f(x(\sigma(s,\xi)))d\xi ds = 0. \tag{2.31}$$

By (2.29), (2.30), and increasing nature of f, we obtain

$$f(x(\sigma(t,\xi))) \ge f((1 - P(\sigma(t,\xi)))k) \quad \text{for } t \ge t_2. \tag{2.32}$$

Substituting (2.32) into (2.31), we get

$$\delta r(t)z^{(n-1)}(t) - \delta r(t_2)z^{(n-1)}(t_2) + \int_{t_2}^t \int_c^d q(s,\xi)f((1 - P(\sigma(s,\xi)))k)d\xi ds \le 0. \tag{2.33}$$

From (2.21) and (2.33), we can conclude that $\delta r(t)z^{(n-1)}(t) \to -\infty$ as $t \to \infty$. This contradicts the following:

$$\begin{aligned} z^{(n-1)}(t) > 0 \quad & \text{for } \delta = 1, \\ z^{(n-1)}(t) < 0 \quad & \text{for } \delta = -1. \end{aligned} \tag{2.34}$$

Thus, this proves that $x(t)$ is oscillatory when $\delta = 1$ and n is even, or $x(t)$ is either oscillatory or $\lim_{t\to\infty} z^{(n-2)}(t) = \infty$ when $\delta = -1$ and n is odd. Obviously, if $\lim_{t\to\infty} z^{(n-2)}(t) = \infty$, then $\lim_{t\to\infty} x(t) = \infty$.

Let n be odd and $\delta = 1$, or n be even and $\delta = -1$. If the integer $l > 0$, then we can find the same conclusion as above. Let $l = 0$. Since

$$\begin{gathered} \int^\infty \int_c^d q(s,\xi)d\xi ds = \infty, \\ \lim_{t\to\infty} \delta r(t)z^{(n-1)}(t) = L \ge 0, \end{gathered} \tag{2.35}$$

and by using these two in (2.31), then it is easy to see that

$$\liminf_{t\to\infty} f(x(t)) = 0 \quad \text{or} \quad \liminf_{t\to\infty} x(t) = 0. \tag{2.36}$$

This completes the proof. □

Example 2.3. Consider the following functional differential equation:

$$\left[e^{-t/2}\left[x(t)+\int_1^2 (1-e^{-t-\mu})x(t-\mu)d\mu\right]''\right]' - \int_3^5 \frac{(e^2+e-1)(e^{(t+\xi)/3})}{4e^{7/2}(e-1)} x\left(\frac{t+\xi}{6}\right)d\xi = 0 \tag{2.37}$$

so that $\delta=-1$, $n=3$, $r(t)=e^{-t/2}$, $p(t,\mu)=1-e^{-t-\mu}$, $\tau(t,\mu)=t-\mu$, $q(t,\xi)=(e^2+e-1)(e^{(t+\xi)/3})/4e^{7/2}(e-1)$, $f(x)=x$, $\sigma(t,\xi)=(t+\xi)/6$ in (1.1).

We can easily see that the conditions of Theorem 2.2 are satisfied. Then, all solutions of this problem are either oscillatory or tends to infinity as t goes to infinity. It is easy to verify that $x(t)=e^t$ is a solution of this problem.

Theorem 2.4. *Let $P(t)=\int_a^b p(t,\mu)d\mu < 1$, and let f be increasing and $r(t)=1$. Suppose that*

$$\int^{\infty}\int_c^d s^{n-1}q(s,\xi)f((1-P(\sigma(s,\xi)))k)d\xi\, ds = \infty \tag{2.38}$$

for every constant $k>0$. Then, every bounded solution $x(t)$ of (1.1) is oscillatory when $(-1)^n\delta = 1$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t)>0$ for $t\ge t_0$. Then, obviously there exists a $t_1\ge t_0$ such that $x(t)$, $x(\tau(t,\mu))$, and $x(\sigma(t,\xi))$ are positive for $t\ge t_1$, $\mu\in[a,b]$, and $\xi\in[c,d]$. Let $z(t)=x(t)+\int_a^b p(t,\mu)x(\tau(t,\mu))d\mu$, then from (1.1), $\delta z^{(n)}(t)<0$ for $t\ge t_1$. Hence, for $\delta=1$, $z^{(n-1)}(t)$ is decreasing and for $\delta=-1$, $z^{(n-1)}(t)$ is increasing.

Since $z^{(n)}(t)<0$ for $\delta=1$, by Kiguradze's lemma [9] there exists an integer l, $0\le l\le n-1$ with $n-l$ is odd and for $t\ge t_1$ such that

$$\begin{gathered} z^{(i)}(t)>0, \quad i=0,1,\dots,l, \\ (-1)^{n-i-1}z^{(i)}(t)>0, \quad i=l,l+1,\dots,n-1. \end{gathered} \tag{2.39}$$

For $\delta=-1$, $z^{(n)}(t)>0$, by Kiguradze's lemma [9] either

$$z^{(i)}(t)>0, \quad i=0,1,\dots,n-1, \tag{2.40}$$

or there exists an integer l, $0\le l\le n-2$ with $n-l$ is even and for $t\ge t_1$ such that

$$\begin{gathered} z^{(i)}(t)>0, \quad i=0,1,\dots,l, \\ (-1)^{n-i}z^{(i)}(t)>0, \quad i=l,l+1,\dots,n-1. \end{gathered} \tag{2.41}$$

Since $z(t)$ is bounded, l cannot be 2 for both cases. Then for $(-1)^n\delta=1$, we have

$$(-1)^{i-1}z^{(i)}(t)>0, \quad i=1,2,\dots,n-1. \tag{2.42}$$

This shows that

$$\lim_{t\to\infty} z^{(i)}(t) = 0 \quad \text{for } i = 1,2,\dots,n-1. \tag{2.43}$$

Using (2.43) and integrating (1.1) n times from t to ∞ to find

$$(-1)^n \delta[z(\infty) - z(t)] = \frac{1}{(n-1)!} \int_t^\infty \int_c^d (s-t)^{n-1} q(s,\xi) f(x(\sigma(s,\xi))) d\xi\, ds, \tag{2.44}$$

where $z(\infty) = \lim_{t\to\infty} z(t)$. On the other hand, from (2.42), $z(t)$ is increasing for large t and $z(t)$ is positive, so we have

$$f(x(\sigma(t,\xi))) \geq f((1-P(\sigma(t,\xi)))k) \quad \text{for } t \geq t_1,\ k > 0 \tag{2.45}$$

as in the proof of Theorem 2.2. Thus, from (2.44) and (2.45), we have

$$z(\infty) - z(t_1) \geq \frac{1}{(n-1)!} \int_{t_1}^\infty \int_c^d (s-t)^{n-1} q(s,\xi) f((1-P(\sigma(s,\xi)))k) d\xi\, ds. \tag{2.46}$$

By (2.38), the right-hand side of the above inequality is ∞, therefore $z(\infty) = \infty$ and this contradicts the boundedness of $z(t)$. Thus, every bounded solution $x(t)$ of (1.1) is oscillatory when $(-1)^n\delta = 1$. □

Example 2.5. Consider the following functional differential equation:

$$\left[x(t) + \int_\pi^{2\pi} \frac{(1-e^{-t})}{4} x\left(t - \frac{\mu}{2}\right) d\mu\right]'' + \int_\pi^{5\pi/2} \left(\frac{1}{2} - e^{-t}\right) x(t+\xi) d\xi = 0, \quad t > -\ln\left(\frac{1}{2}\right) \tag{2.47}$$

so that $\delta = 1$, $n = 2$, $r(t) = 1$, $p(t,\mu) = (1-e^{-t})/4$, $\tau(t,\mu) = t - \mu/2$, $q(t,\xi) = 1/2 - e^{-t}$, $f(x) = x$, $\sigma(t,\xi) = (t+\xi)$ in (1.1).

We can easily see that the conditions of Theorem 2.4 are satisfied. Then, all bounded solutions of this problem are oscillatory. It is easy to verify that $x(t) = \sin t$ is a solution of this problem.

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