

CONVERGENCE OF IMPLICIT ITERATES WITH ERRORS FOR MAPPINGS WITH UNBOUNDED DOMAIN IN BANACH SPACES

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We prove that an implicit iterative process with errors converges weakly and strongly to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings on unbounded sets in a uniformly convex Banach space. Our results generalize and improve upon, among others, the corresponding recent results of Sun (2003) in the following two different directions: (i) domain of the mappings is unbounded, (ii) the iterative sequence contains an error term.

1. Introduction

In 1967, Browder [1] studied the iterative construction for fixed points of nonexpansive mappings on closed and convex subsets of a Hilbert space (see also [2]). The Ishikawa iteration process in the context of nonexpansive mappings on bounded closed convex subsets of a Banach space has been considered by a number of authors (see, e.g., Tan and Xu [16] and the references therein).

Ghosh and Debnath [4], in 1997, established a necessary and sufficient condition (in Theorem 3.1) for convergence of the Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset C of a Banach space. Qihou [11] has extended this result (in Theorem 1) for Ishikawa iterates with errors, in the sense of Liu [10], of an asymptotically quasi-nonexpansive mapping on C .

Fixed point results for asymptotically nonexpansive mappings have been obtained by Kirk and Ray [9] on unbounded sets in a Banach space. Recently, Hussain and Khan [7] have constructed (in Theorem 3.8) approximating sequences to fixed points of a class of mappings, containing nonexpansive mappings as a subclass, on closed convex unbounded subsets of a Hilbert space (see [13, 14] as well).

Sun [15] has recently extended an implicit iteration process for a finite family of nonexpansive mappings due to Xu and Ori [20] to the case of asymptotically quasi-nonexpansive mappings.

The aim of this paper is to prove the weak convergence and strong convergence of an implicit iterative process with errors, in the sense of Xu [19], for a finite family of asymptotically quasi-nonexpansive mappings on a closed convex unbounded set in a real

uniformly convex Banach space. Our results unify, improve, and generalize the corresponding results of Sun [15], Wittmann [17], and Xu and Ori [20].

2. Preliminaries and lemmas

In 1967, Diaz and Metcalf [2] introduced the notion of a quasi-nonexpansive mapping and Goebel and Kirk [5] introduced, in 1972, the concept of an asymptotically nonexpansive mapping.

Let C be a nonempty subset of a normed space E and $T : C \rightarrow C$ a given mapping. We assume that the set of fixed points of T , $F(T) = \{x \in C : T(x) = x\}$ is nonempty. The mapping T is said to be (1) nonexpansive provided $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; (2) quasi-nonexpansive if $\|Tx - p\| \leq \|x - p\|$ for all $x \in C, p \in F(T)$; (3) asymptotically nonexpansive if there is a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\|$ for all $x, y \in C$ and for all $n \geq 1$; (4) asymptotically quasi-nonexpansive if there is a sequence $\{u_n\} \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} u_n = 0$ such that $\|T^n x - p\| \leq (1 + u_n)\|x - p\|$ for all $x \in C, p \in F(T)$ and for all $n \geq 1$, and (5) uniformly L -Lipschitzian if for some $L > 0, \|T^n x - T^n y\| \leq L\|x - y\|$ holds for all $x, y \in C$ and for all $n \geq 1$.

Denote the indexing set $\{1, 2, 3, \dots, N\}$ by I . We say that a finite family $\{T_i : i \in I\}$ of N self-mappings on C is

(i) asymptotically nonexpansive if it satisfies

$$\|T^n x_i - T^n y_i\| \leq (1 + u_{in})\|x - y\| \quad \forall x, y \in C, \forall n \geq 1, \tag{2.1}$$

(ii) asymptotically quasi-nonexpansive provided

$$\|T^n x_i - q_i\| \leq (1 + u_{in})\|x - q_i\| \quad \forall x \in C, q_i \in F(T_i), \forall n \geq 1, \tag{2.2}$$

where $\{u_{in}\}$ is the sequence of reals as in (3) and $i \in I$.

It is obvious, in view of these definitions, that a nonexpansive mapping is asymptotically nonexpansive, nonexpansive mapping with the nonempty fixed point set is quasi-nonexpansive, and an asymptotically nonexpansive mapping is uniformly L -Lipschitzian with $L = \sup\{1 + u_n : n \geq 1\}$. However, the converses of these claims are not true in general.

The Mann and Ishikawa iteration processes have been used by a number of authors to approximate the fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, and quasi-nonexpansive mappings on Banach spaces (see, e.g., [4, 8, 10, 12, 14, 19]).

For a nonempty subset C of a normed space E and $T : C \rightarrow E$, Liu [10] introduced in 1995 the concept of Ishikawa iteration process with errors by the iterative sequence $\{x_n\}$ defined as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \quad n \geq 1, \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ satisfying appropriate conditions and $\sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty$. If $\beta_n = 0, v_n = 0$ for all $n \geq 1$, then this process becomes the Mann iteration process with errors.

The above definitions of Liu depend on the convergence of the error terms u_n and v_n . The occurrence of errors is random and so the conditions imposed on the error terms are unreasonable. Moreover, there is no assurance that the iterates defined by Liu will fall within the domain under consideration.

In 1998, Xu [19] gave the following new definitions in place of these noncompatible ones.

For a nonempty convex subset C of a normed space E and $T : C \rightarrow C$, the Ishikawa iteration process with errors is the iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\ y_n &= a_n x_n + \beta_n T^n x_n + \gamma_n v_n, \quad n \geq 1, \end{aligned} \tag{2.4}$$

with $\{u_n\}, \{v_n\}$ bounded sequences in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}, \{\gamma'_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$.

It reduces to the Mann iteration process with errors when $\beta'_n = 0 = \gamma'_n$ for all $n \geq 1$.

Clearly, the normal Ishikawa and Mann iteration processes are special cases of the Ishikawa iteration process with errors.

Huang [6] has computed fixed points of asymptotically nonexpansive mappings while Fukhar and Khan [3] have approximated common fixed points of two asymptotically nonexpansive mappings, using iteration process with errors in the sense of Liu [10] (see also [12]).

Let $\{T_i : i \in I\}$ be a finite family of asymptotically quasi-nonexpansive self-mappings on a convex subset C of a normed space E . The implicit iterative process of Sun [15] with an error term, in the sense of Xu [19], and with an initial value $x_0 \in C$, is defined as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\ x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\ x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\ x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\ x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \end{aligned} \tag{2.5}$$

where $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

The above table in compact form is

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n \tag{2.6}$$

with $n \geq 1$ and $n = (k - 1)N + i, i \in I$.

In the sequel, we assume that the sequence $\{x_n\}$, defined by the implicit iteration process with errors (2.6), exists and the set $F = \bigcap_{i=1}^N F(T_i)$, $i \in I$, is nonempty.

The distance between a point x and a set C and closed ball with centre zero and radius r in E are, respectively, defined by

$$d(x, C) = \inf_{y \in C} \|x - y\|, \quad B_r(0) = \{x \in E : \|x\| \leq r\}. \tag{2.7}$$

Definition 2.1 (see [15]). Let C be a closed subset of a normed space E and let $T : C \rightarrow C$ be a mapping. Then T is said to be semicompact if for any bounded sequence $\{x_n\}$ in C with $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightarrow x^* \in C$ as $n_i \rightarrow \infty$.

Definition 2.2 (see [8]). A normed space E is said to satisfy the Opial condition if for any sequence $\{x_n\}$ in E , x_n converging weakly to x implies that $\limsup_{n \rightarrow \infty} \|x_n - x\| < \limsup_{n \rightarrow \infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

LEMMA 2.3 (see [3]). *Let $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ be three nonnegative sequences satisfying the following condition:*

$$r_{n+1} \leq (1 + s_n)r_n + t_n \quad \forall n \geq 1. \tag{2.8}$$

If $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \rightarrow \infty} r_n$ exists.

LEMMA 2.4 (see [18]). *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda) \|y\|^p - w_p(\lambda)g(\|x - y\|) \tag{2.9}$$

for all $x, y \in B_r(0)$ where $0 \leq \lambda \leq 1$ and $w_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p$.

LEMMA 2.5 (see [8]). *Let E be a uniformly convex Banach space satisfying the Opial condition and let C be a nonempty closed convex subset of E . Let T be an asymptotically nonexpansive mapping of C into itself. Then $I - T$ is demiclosed at 0 (i.e., for any sequence $\{x_n\}$ in C , the conditions x_n converge weakly to x_0 and $x_n - Tx_n \rightarrow 0$ imply $x_0 - Tx_0 = 0$).*

3. An implicit iterative process with errors

We begin with a necessary and sufficient condition for convergence of $\{x_n\}$ generated by the implicit iteration process with errors in (2.6) to a point of F ; for this we follow the arguments of Sun (see [15, Theorem 3.1]) and use Lemma 2.3 to prove the following general result.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a Banach space E . Let $\{T_i : i \in I\}$ be N asymptotically quasi-nonexpansive mappings of C , with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a bounded sequence in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the iterative sequence $\{x_n\}$ generated by (2.6) converges to a point in F if and only if $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$.*

Proof. The necessity is obvious. We prove the sufficiency of the conditions. Let $p \in F$. Using $x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n$, where $n = (k - 1)N + i$, it follows that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\| \\ &= \|\alpha_n(x_{n-1} - p) + \beta_n(T_i^k x_n - p) + \gamma_n(u_n - p)\| \\ &\leq \alpha_n \|x_{n-1} - p\| + \beta_n(1 + u_{ik}) \|x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + u_{ik}) \|x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + u_{ik}) \|x_n - p\| + \gamma_n \|u_n - p\|. \end{aligned} \tag{3.1}$$

This implies that

$$\alpha_n \|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + u_{ik} \|x_n - p\| + \gamma_n \|u_n - p\|. \tag{3.2}$$

Now using the information that $0 < s < \alpha_n < 1 - s < 1$, we have that

$$\left(\frac{s - u_{ik}}{s}\right) \|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\gamma_n}{s} \|u_n - p\|. \tag{3.3}$$

That is,

$$\|x_n - p\| \leq \left(1 + \frac{u_{ik}}{s - u_{ik}}\right) \|x_{n-1} - p\| + \frac{\gamma_n}{s - u_{ik}} \|u_n - p\|. \tag{3.4}$$

Since $\sum_{k=1}^\infty u_{ik} < \infty$ for all $i \in I$, therefore $\lim_{k \rightarrow \infty} u_{ik} = 0$ for all $i \in I$. This gives that there exists a natural number n_0 (as $k > (n_0/N) + 1$) such that $s - u_{ik} > 0$ and $u_{ik} < s/2$ for all $n > n_0$.

Now the above inequality, with $M = \sup_{n \geq 1} \|u_n - p\|$, becomes

$$\|x_n - p\| \leq \left(1 + \frac{2}{s} u_{ik}\right) \|x_{n-1} - p\| + \frac{2M}{s} \gamma_n. \tag{3.5}$$

This further implies that

$$d(x_n, F) \leq \left(1 + \frac{2}{s} u_{ik}\right) d(x_{n-1}, F) + \frac{2M}{s} \gamma_n. \tag{3.6}$$

Applying Lemma 2.3 to the inequalities (3.5) and (3.6), we conclude that both $\lim_{n \rightarrow \infty} \|x_n - p\|$ and $\lim_{n \rightarrow \infty} d(x_n, F)$ exist. Since $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$, therefore $\lim_{n \rightarrow \infty} d(x_n, F) = 0$. Hereafter, we show that $\{x_n\}$ is a Cauchy sequence. Note that when

$x > 0, 1 + x \leq e^x$, and hence by (3.4), we have

$$\begin{aligned} & \|x_{n+m} - p\| \\ & \leq \exp \left[\sum_{i=1}^N \sum_{k=1}^{\infty} \left(\frac{u_{ik}}{s - u_{ik}} \right) \right] \|x_n - p\| + \frac{2}{s} M \exp \left[\sum_{i=1}^N \sum_{k=1}^{\infty} \left(\frac{u_{ik}}{s - u_{ik}} \right) \right] \sum_{n=1}^{\infty} \gamma_n \quad (3.7) \\ & < Q \|x_n - p\| + R \quad \forall m \geq 1, n \geq 1, \end{aligned}$$

where

$$\begin{aligned} Q &= \exp \left[\sum_{i=1}^N \sum_{k=1}^{\infty} \left(\frac{u_{ik}}{s - u_{ik}} \right) \right] + 1, \\ R &= \frac{2}{s} M \exp \left[\sum_{i=1}^N \sum_{k=1}^{\infty} \left(\frac{u_{ik}}{s - u_{ik}} \right) \right] \sum_{n=1}^{\infty} \gamma_n. \end{aligned} \quad (3.8)$$

Let $\epsilon > 0$. As $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists, therefore for $\epsilon > R$, there exists a natural number n_1 such that $\|x_n - p\| < (\epsilon - R)/(1 + Q)$ for all $n \geq n_1$. Now it follows from the above inequality that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p\| + \|x_n - p\| \\ &< Q \|x_n - p\| + R + \|x_n - p\| \\ &= (1 + Q) \|x_n - p\| + R < \epsilon \quad \forall m \geq 1, n \geq n_1. \end{aligned} \quad (3.9)$$

This proves that $\{x_n\}$ is a Cauchy sequence in E and so it must converge. Let $\lim_{n \rightarrow \infty} x_n = q$ (say). As $\lim_{n \rightarrow \infty} d(x_n, F) = 0$, therefore $d(q, F) = 0$. This implies that there exists $p \in F$ such that $\|q - p\| = 0$. That is, $q = p$. Hence, q is a common fixed point of T_i for all $i \in I$. This completes the proof of the theorem. \square

The above theorem can be restated as follows.

COROLLARY 3.2. *Suppose that all the conditions of Theorem 3.1 hold. Then the implicit iterative sequence $\{x_n\}$ with errors converges to a point $p \in F$ if and only if $\{x_n\}$ has an infinite subsequence $\{x_{n_i}\}$ with limit p .*

We prove a lemma which plays an important role in establishing weak and strong convergence of the implicit iteration process with errors in a uniformly convex Banach space.

LEMMA 3.3. *Let C be a closed convex subset of a real uniformly convex Banach space E . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_n < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then $\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ for all $l \in I$.*

Proof. Set $\sigma_n = \|T_n^k x_n - x_{n-1}\|$, $n = (k - 1)N + i$, $i \in I$. As in the proof of Theorem 3.1, $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$, so $\{x_n - q, T_i^k x_n - q\}$ is a bounded set. Hence, we

can obtain a closed ball $B_r(0) \supset \{x_n - q, T_i^k x_n - q\}$ for some $r > 0$. By Lemma 2.4 and the scheme (2.6), we get

$$\begin{aligned}
 \|x_n - q\|^2 &= \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q) + \gamma_n(u_n - T_i^k x_n)\|^2 \\
 &\leq \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q)\|^2 + \gamma_n M \quad \text{for some } M > 0 \\
 &\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n) \|T_i^k x_n - q\|^2 \\
 &\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n M \\
 &\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)(1 + u_{ik})^2 \|x_n - q\|^2 \\
 &\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n M \\
 &\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n + v_{ik}) \|x_n - q\|^2 \\
 &\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n M,
 \end{aligned}
 \tag{3.10}$$

where $v_{ik} = 2u_{ik} + u_{ik}^2$. Hence, $\sum_{k=1}^\infty v_{ik} < \infty$ for all $i \in I$. Thus from the above inequality and $s < \alpha_n < 1 - s$, we have that

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 + \frac{v_{ik}}{s} \|x_n - q\|^2 - (1 - \alpha_n)g(\sigma_n) + \frac{\gamma_n}{s} M.
 \tag{3.11}$$

Therefore, as in Theorem 3.1, it can be shown that $\lim_{n \rightarrow \infty} \|x_n - q\|^2 = d$ exists. From (3.11), it follows that

$$\begin{aligned}
 (1 - \alpha_n)g(\sigma_n) &\leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{v_{ik}}{s} \|x_n - q\|^2 + \frac{\gamma_n}{s} M \\
 &\leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + v_{ik} Q' + \frac{\gamma_n}{s} M \quad \text{for some } Q' > 0.
 \end{aligned}
 \tag{3.12}$$

From $(1 - \alpha_n) \geq s$, we have

$$sg(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + v_{ik} Q' + \frac{\gamma_n}{s} M.
 \tag{3.13}$$

Let m be a positive integer such that $m \geq n$. Then

$$\sum_{n=1}^m g(\sigma_n) \leq \frac{1}{s^2} \|x_0 - q\|^2 + \frac{Q'}{s^2} \sum_{k=1}^m v_{ik} + \frac{M}{s^3} \sum_{k=1}^m \gamma_n.
 \tag{3.14}$$

When $m \rightarrow \infty$ in (3.14), we have that $\lim_{n \rightarrow \infty} g(\sigma_n) = 0$. Since g is strictly increasing and continuous with $g(0) = 0$, it follows that $\lim_{n \rightarrow \infty} \sigma_n = 0$. Hence,

$$\begin{aligned}
 \|x_n - x_{n-1}\| &\leq \beta_n \|T_n^k x_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\| \\
 &\leq (1 - \alpha_n) \|T_n^k x_n - x_{n-1}\| + \gamma_n R' \quad \text{for some } R' > 0 \\
 &\leq (1 - s) \|T_n^k x_n - x_{n-1}\| + \gamma_n R' \quad \text{for some } R' > 0,
 \end{aligned}
 \tag{3.15}$$

which proves that $\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0$. That is, $\lim_{n \rightarrow \infty} \|x_n - x_{n+l}\| = 0$ for all $l < 2N$. For $n > N$, we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T x_n\| \leq \sigma_n + L \|T_n^{k-1} x_n - x_n\| \\ &\leq \sigma_n + L \left[\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\| \right] \\ &\quad + L \|x_{(n-N)-1} - x_n\|. \end{aligned} \tag{3.16}$$

By $n \equiv (n - N) \pmod{N}$, we get $T_n = T_{n-N}$. Now the above inequality becomes

$$\|x_{n-1} - T x_n\| \leq \sigma_n + L^2 \|x_n - x_{n-N}\| + L \sigma_{n-N} + L \|x_{(n-N)-1} - x_n\|, \tag{3.17}$$

which yields that $\lim_{n \rightarrow \infty} \|x_{n-1} - T x_n\| = 0$. Since

$$\|x_n - T x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T x_n\|, \tag{3.18}$$

so we have that

$$\lim_{n \rightarrow \infty} \|x_n - T x_n\| = 0. \tag{3.19}$$

Hence, for all $l \in I$,

$$\begin{aligned} \|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\ &\leq (1 + L) \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\|, \end{aligned} \tag{3.20}$$

which implies that

$$\lim_{n \rightarrow \infty} \|x_n - T_{n+l} x_n\| = 0 \quad \forall l \in I. \tag{3.21}$$

Thus

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad \forall l \in I. \tag{3.22}$$

□

Now we are in a position to prove our convergence theorems.

THEOREM 3.4. *Let C be a closed convex subset of a real uniformly convex Banach space E satisfying the Opial condition. Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. If for every member T in $\{T_i : i \in I\}$, $I - T$ is demiclosed at 0, then the iterative sequence $\{x_n\}$ generated by (2.6) converges weakly to a point in F .*

Proof. Let $p \in F$. The sequence $\{\|x_n - p\|\}$ is convergent as proved in Lemma 3.3. This gives that $\{x_n\}$ is a bounded sequence and it converges weakly to a point in C . We prove that $\{x_n\}$ has a unique weak subsequential limit in F . In fact, suppose that u and v are weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By Lemma 3.3,

$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0$ for all $l \in I$. Also, $I - T_l$ is demiclosed at 0 for all $l \in I$. Therefore, we obtain that $T_l u = u$ for all $l \in I$. Similarly, we can prove that $T_l v = v$ for all $l \in I$. That is, $u, v \in F$. For the uniqueness, assume that u and v are distinct. Then by the Opial condition,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - u\| &= \lim_{n_i \rightarrow \infty} \|x_{n_i} - u\| \\ &< \lim_{n_i \rightarrow \infty} \|x_{n_i} - v\| \\ &= \lim_{n \rightarrow \infty} \|x_n - v\| \\ &< \lim_{n_j \rightarrow \infty} \|x_{n_j} - u\| \\ &= \lim_{n \rightarrow \infty} \|x_n - u\|, \end{aligned} \tag{3.23}$$

a contradiction. Hence, the proof. □

With the help of Lemma 2.5, the following corollary is an immediate consequence of the above theorem; this result includes, as a special case, Theorem 2 of Xu and Ori [20].

COROLLARY 3.5. *Let C be a closed convex subset of a real uniformly convex Banach space E satisfying the Opial condition. Let $\{T_i : i \in I\}$ be N uniformly asymptotically nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the iterative sequence $\{x_n\}$ generated by (2.6) converges weakly to a point in F .*

Next, we prove strong convergence theorems.

THEOREM 3.6. *Let C be a closed convex subset of a real uniformly convex Banach space E . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C , and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. If at least one member T in $\{T_i : i \in I\}$ is semicompact, then the implicitly defined iterative sequence $\{x_n\}$ in (2.6) converges strongly to a point in F .*

Proof. By Lemma 3.3, it follows that

$$\lim_{n \rightarrow \infty} \|x_n - T_l x_n\| = 0 \quad \forall l \in I. \tag{3.24}$$

Without any loss of generality, assume that T_1 is semicompact. Therefore, by (3.24), it follows that $\lim_{n \rightarrow \infty} \|x_n - T_1 x_n\| = 0$. Since T_1 is semicompact, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightarrow x^* \in C$. Now consider

$$\|x^* - T_l x^*\| = \lim_{n_j \rightarrow \infty} \|x_{n_j} - T_l x_{n_j}\| = 0 \quad \forall l \in I. \tag{3.25}$$

This proves that $x^* \in F$. As $\lim_{n \rightarrow \infty} \|x_n - q\|$ exists for all $q \in F$, therefore x_n converges to $x^* \in F$, and hence the result. □

Remark 3.7. If we take $\gamma_n = 0$, for all $n \geq 1$, then the above theorem becomes Theorem 3.3 due to Sun [15] without the boundedness of C which in turn generalizes Theorem 2 by Wittmann [17] from Hilbert spaces to uniformly convex Banach spaces.

An asymptotically nonexpansive mapping is both uniformly L -Lipschitzian and asymptotically quasi-nonexpansive. Hence, the following generalization of [15, Theorem 3.4] is an immediate consequence of Theorem 3.6.

THEOREM 3.8. *Let C be a closed convex subset of a real uniformly convex Banach space E . Let $\{T_i : i \in I\}$ be N asymptotically nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C , and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. If at least one member T in $\{T_i : i \in I\}$ is semicompact, then the implicitly defined iterative sequence $\{x_n\}$ generated by (2.6) strongly converges to a point in F .*

Definition 3.9 (condition $(*)$). The family $\{T_i : i \in I\}$ of N -self-mappings on a subset C of a normed space E satisfies condition $(*)$ if there exists a nondecreasing function $f : [0, \infty) \rightarrow [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $1/n \sum_{i=1}^N \|x - T_i x\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Note that condition $(*)$ defined above reduces to the [16, condition (A)] if we choose $T_i = T$ (say) for all $i \in I$.

Finally, an application of the convergence criteria established in Theorem 3.1 is given below to obtain yet another strong convergence result in our setting.

THEOREM 3.10. *Let C be a closed convex subset of a real uniformly convex Banach space E . Let $\{T_i : i \in I\}$ be N uniformly L -Lipschitzian asymptotically quasi-nonexpansive mappings of C with $\sum_{n=1}^{\infty} u_{in} < \infty$ for all $i \in I$ and satisfy the condition $(*)$. Suppose that $x_0 \in C$, $\{u_n\}$ is a sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the iterative sequence $\{x_n\}$ generated by (2.6) strongly converges to a point in F .*

Proof. As in the proof of Theorems 3.6, (3.24) holds. Taking \liminf on both sides of condition $(*)$ and using (3.24), we have that $\liminf_{n \rightarrow \infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\liminf_{n \rightarrow \infty} d(x_n, F) = 0$. Now by Theorem 3.1, $x_n \rightarrow p \in F$. \square

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