INEQUALITY FOR RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN LOCALLY CONFORMAL ALMOST COSYMPLECTIC MANIFOLDS

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We establish inequalities between the Ricci curvature and the squared mean curvature, and also between the k-Ricci curvature and the scalar curvature for a slant, semi-slant, and bi-slant submanifold in a locally conformal almost cosymplectic manifold with arbitrary codimension.

1. Preliminaries

Let $\widetilde{M}$ be a $(2m+1)$-dimensional almost contact manifold with almost contact structure (φ,ξ,η), that is, a global vector field ξ, a $(1,1)$ tensor field φ, and a 1-form η on $\widetilde{M}$ such that $\varphi^2X = -X+\eta(X)\xi$, $\eta(\xi) = 1$ for any vector field X on $\widetilde{M}$. We consider a product manifold $\widetilde{M}\times\mathbb{R}$, where $\mathbb{R}$ denotes a real line. Then a vector field on $\widetilde{M}\times\mathbb{R}$ is given by $(X, f(d/dt))$, where X is a vector field tangent to $\widetilde{M}$, t the coordinate of $\mathbb{R}$, and f a function on $\widetilde{M}\times\mathbb{R}$. We define a linear map J on the tangent space of $\widetilde{M}\times\mathbb{R}$ by $J(X, f(d/dt)) = (\varphi X - f\xi, \eta(X)(d/dt))$. Then we have $J^2 = -I$, and hence J is an almost complex structure on $\widetilde{M}\times\mathbb{R}$. The manifold $\widetilde{M}$ is said to be *normal* (see [6]) if the almost complex structure J is integrable (i.e., J arises from a complex structure on $\widetilde{M}\times\mathbb{R}$). Let g be a Riemannian metric on $\widetilde{M}$ compatible with (φ,ξ,η), that is, $g(\varphi X,\varphi Y) = g(X,Y)-\eta(X)\eta(Y)$ for any vector fields X and Y tangent to $\widetilde{M}$. Thus, the manifold $\widetilde{M}$ is almost contact metric, and (φ,ξ,η,g) is its almost contact metric structure. Clearly, we have $\eta(X) = g(X,\xi)$ for any vector field X tangent to $\widetilde{M}$. Let Φ denote the fundamental 2-form of $\widetilde{M}$ defined by $\Phi(X,Y) = g(\varphi X,Y)$ for any vector fields X and Y tangent to $\widetilde{M}$. The manifold $\widetilde{M}$ is said to be *almost cosymplectic* if the forms η and Φ are closed. That is, $d\eta = 0$ and $d\Phi = 0$, where d is the operator of exterior differentiation. If $\widetilde{M}$ is almost cosymplectic and normal, then it is called *cosymplectic* (see[1]). It is well known that the almost contact metric manifold is cosymplectic if and only if $\tilde{\nabla}\varphi$ vanishes identically, where $\tilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$. An almost contact metric manifold $\widetilde{M}$ is a locally conformal almost cosymplectic manifold if and only if there exists a 1-form ω such that $d\Phi = 2\omega\wedge\Phi$, $d\eta = \omega\wedge\eta$, and $d\omega = 0$.

On the other hand, it is wellknown that the Riemannian curvature tensor $\tilde{R}$ on a locally conformal almost cosymplectic manifold $\widetilde{M}$ $(m \geq 2)$ of pointwise constant φ-sectional

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curvature c satisfies (see[6])

$$
\begin{aligned}
g(\tilde{R}(X,Y)Z,W)
&= \frac{c-3f^2}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\}\\
&\quad+\frac{c+f^2}{4}\{g(X,\varphi W)g(Y,\varphi Z)-g(X,\varphi Z)g(Y,\varphi W)-2g(X,\varphi Y)g(Z,\varphi W)\}\\
&\quad-\left(\frac{c+f^2}{4}+f'\right)\{g(X,W)\eta(Y)\eta(Z)-g(X,Z)\eta(Y)\eta(W)+g(Y,Z)\eta(X)\eta(W)\\
&\qquad -g(Y,W)\eta(X)\eta(Z)\},\quad X,Y,Z,W\in T_pM,
\end{aligned}
\tag{1.1}
$$

where f is the function such that $\omega = f\eta$, $f' = \xi f$.

In [5], Lotta has introduced the following notion of slant submanifolds into almost contact metric manifolds. A submanifold M tangent to ξ in locally conformal almost cosymplectic manifold $\widetilde{M}$ is said to be *slant* if for any $p \in M$ and any $X \in T_pM$, linearly independent of ξ, the angle between φX and T_pM is a constant $\theta \in [0,\pi/2]$, called the *slant angle* of M in $\widetilde{M}$. Invariant and anti-invariant submanifolds of $\widetilde{M}$ are slant submanifolds with slant angles $\theta = 0$ and $\theta = \pi/2$, respectively.

We say that a submanifold M tangent to ξ is a *bi-slant* submanifold in $\widetilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(1) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$;

(2) for any $i = 1,2$, $\mathcal{D}_i$ is slant distribution with slant angle θ_i.

On the other hand, CR-submanifolds of $\widetilde{M}$ are bi-slant submanifolds with $\theta_1 = 0$, $\theta_2 = \pi/2$.

Let $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

Remark 1.1. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

A submanifold M tangent to ξ is called a *semi-slant* submanifold in $\widetilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that

(1) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$;

(2) the distribution $\mathcal{D}_1$ is an invariant distribution, that is, $\varphi(\mathcal{D}_1) = \mathcal{D}_1$;

(3) the distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

Remark 1.2. The invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds.

(1) If $d_2 = 0$, then M is an invariant submanifold.

(2) If $d_1 = 0$ and $\theta = \pi/2$, then M is an anti-invariant submanifold.

For the other properties and examples of slant, bi-slant, and semi-slant submanifolds in an almost contact metric manifold, we refer to [2, 3].

Let M be an n-dimensional submanifold of a locally conformal almost cosymplectic manifold $\widetilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{1.2}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇, and $\nabla^{\perp}$ are the Riemannian, induced Riemannian, and induced normal connections in $\widetilde{M}$, M, and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form related to the shape operator A by $g(h(X,Y),N) = g(A_N X, Y)$. Also, let R be the Riemannian curvature tensor of M. Then the equation of Gauss is given by

$$\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + g\big(h(X,W),h(Y,Z)\big) - g\big(h(X,Z),h(Y,W)\big), \tag{1.3}$$

for any vectors X, Y, Z, W tangent to M.

For any vector X tangent to M, we put $\varphi X = PX + FX$, where PX and FX are the tangential and the normal components of φX, respectively. Given an orthonormal basis $\{e_1,\dots,e_n\}$ of M, we define the squared norm of P by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j) \tag{1.4}$$

and the mean curvature vector $H(p)$ at $p \in M$ is given by $H = (1/n)\sum_{i=1}^{n} h(e_i,e_i)$.

We put

$$h_{ij}^r = g\big(h(e_i,e_j),e_r\big), \qquad \|h\|^2 = \sum_{i,j=1}^{n} g\big(h(e_i,e_j),h(e_i,e_j)\big), \tag{1.5}$$

where $\{e_{n+1},\dots,e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$ and $r = n+1,\dots,2m+1$. A submanifold M in $\widetilde{M}$ is called *totally geodesic* if the second fundamental form vanishes identically and *totally umbilical* if there is a real number λ such that $h(X,Y) = \lambda g(X,Y)H$ for any tangent vectors X, Y on M.

For an n-dimensional Riemannian manifold M, we denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For an orthonormal basis $\{e_1,\dots,e_n\}$ of the tangent space T_pM, the scalar curvature τ is defined by

$$\tau = \sum_{i<j} K_{ij}, \tag{1.6}$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j.

Suppose that L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1,\dots,e_k\}$ of L such that $e_1 = X$. Define the Ricci curvature Ric_L of L at X by

$$\mathrm{Ric}_L(X) = K_{12} + \cdots + K_{1k}. \tag{1.7}$$

We simply called such a curvature a *k-Ricci curvature*. The scalar curvature τ of the k-plane section L is given by

$$\tau(L) = \sum_{1\leq i<j\leq k} K_{ij}. \tag{1.8}$$

For each integer k, $2 \leq k \leq n$, the Riemannain invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1}\inf_{L,X} \operatorname{Ric}_L(X), \quad p \in M, \tag{1.9}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannain manifold, the relative null space of M at a point $p \in M$ is defined by

$$N_p = \{X \in T_pM \mid h(X,Y) = 0\ \forall Y \in T_pM\}. \tag{1.10}$$

2. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [4]). We prove similar inequalities for slant, bi-slant, and semi-slant submanifolds in a locally conformal almost cosymplectic manifold $\widetilde{M}$. We consider submanifolds M tangent to ξ.

Theorem 2.1. *Let M be an n-dimensional θ-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, the following hold.*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ,*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c-3f^2) + \frac{3}{2}(c+f^2)\cos^2\theta - 4\left(\frac{c+f^2}{4} + f'\right) + n^2\|H\|^2\right\}. \tag{2.1}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.1) if and only if $X \in N_p$.*

(3) *The equality case of (2.1) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. (1) Let $X \in T_pM$ be a unit tangent vector at p orthogonal to ξ. We choose an orthonormal basis $e_1,\dots,e_n = \xi, e_{n+1},\dots,e_{2m+1}$, such that $e_1,\dots,e_n$ are tangent to M at p with $e_1 = X$. Then, from the equation of Gauss, we have

$$\begin{aligned} n^2\|H\|^2 = 2\tau + \|h\|^2 - \frac{n(n-1)(c-3f^2)}{4} \\ - \frac{3(n-1)(c+f^2)}{4}\cos^2\theta + 2(n-1)\left(\frac{c+f^2}{4} + f'\right). \end{aligned} \tag{2.2}$$

From (2.2), we get

$$
\begin{aligned}
n^2\|H\|^2 = {}& 2\tau + \sum_{r=n+1}^{2m+1}\left[\left(h_{11}^r\right)^2 + \left(h_{22}^r + \cdots + h_{nn}^r\right)^2 + 2\sum_{1\le i<j\le n}\left(h_{ij}^r\right)^2\right] \\
& - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - \frac{n(n-1)(c-3f^2)}{4} \\
& - \frac{3(n-1)(c+f^2)}{4}\cos^2\theta + 2(n-1)\left(\frac{c+f^2}{4} + f'\right) \\
= {}& 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}\left[\left(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r\right)^2 + \left(h_{11}^r - h_{22}^r - \cdots - h_{nn}^r\right)^2\right] \\
& + 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}\left(h_{ij}^r\right)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r \\
& - \frac{n(n-1)(c-3f^2)}{4} - \frac{3(n-1)(c+f^2)}{4}\cos^2\theta + 2(n-1)\left(\frac{c+f^2}{4} + f'\right).
\end{aligned}
\tag{2.3}
$$

By using the equation of Gauss, we have

$$
\begin{aligned}
\sum_{2\le i<j\le n} K_{ij} = {}& \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}\left[h_{ii}^r h_{jj}^r - \left(h_{ij}^r\right)^2\right] + \frac{(n-1)(n-2)(c-3f^2)}{8} \\
& + \frac{3(n-2)(c+f^2)}{8}\cos^2\theta + \frac{1}{2}\left(\frac{c+f^2}{4} + f'\right)(-2n+4).
\end{aligned}
\tag{2.4}
$$

Substituting (2.4) in (2.3), we get

$$
\frac{1}{2}n^2\|H\|^2 \ge 2\,\mathrm{Ric}(X) - \frac{(n-1)(c-3f^2)}{2} - \frac{3(c+f^2)}{4}\cos^2\theta + 2\left(\frac{c+f^2}{4} + f'\right), \tag{2.5}
$$

or equivalently (2.1).

(2) Assume that $H(P) = 0$. Equality holds in (2.1) if and only if

$$
\begin{gathered}
h_{12}^r = \cdots = h_{1n}^r = 0, \\
h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, \quad r \in \{n+1,\dots,2m+1\}.
\end{gathered}
\tag{2.6}
$$

Then $h_{1j}^r = 0$ for all $j \in \{1,\dots,n\}$, $r \in \{n+1,\dots,2m+1\}$, that is, $X \in N_p$.

(3) Then equality case of (2.1) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$
\begin{gathered}
h_{ij}^r = 0, \quad i \ne j, r \in \{n+1,\dots,2m+1\}, \\
h_{11}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1,\dots,n\},\ r \in \{n+1,\dots,2m+1\}.
\end{gathered}
\tag{2.7}
$$

In this case, it follows that p is a totally geodesic point. The converse is trivial. □

THEOREM 2.2. *Let M be an n-dimensional bi-slant submanifold satisfying $g(X,\varphi Y)=0$, for any $X\in\mathfrak{D}_1$ and any $Y\in\mathfrak{D}_2$, tangent to ξ in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, the following hold.*

(1) *For each unit vector $X\in T_pM$ orthogonal to ξ and if*

(i) *X is tangent to $\mathfrak{D}_1$,*

$$\operatorname{Ric}(X)\leq\frac{1}{4}\left\{(n-1)(c-3f^2)+\frac{3}{2}(c+f^2)\cos^2\theta_1-4\left(\frac{c+f^2}{4}+f'\right)+n^2\|H\|^2\right\},\tag{2.8}$$

and if

(ii) *X is tangent to $\mathfrak{D}_2$,*

$$\operatorname{Ric}(X)\leq\frac{1}{4}\left\{(n-1)(c-3f^2)+\frac{3}{2}(c+f^2)\cos^2\theta_2-4\left(\frac{c+f^2}{4}+f'\right)+n^2\|H\|^2\right\}.\tag{2.9}$$

(2) *If $H(p)=0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.8) and (2.9) if and only if $X\in N_p$.*

(3) *The equality case of (2.8) and (2.9) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. (1) Let $X\in T_pM$ be a unit tangent vector at p orthogonal to ξ. We choose an othonormal basis $e_1,\dots,e_n=\xi,e_{n+1},\dots,e_{2m+1}$ such that $e_1,\dots,e_n$ are tangent to M at p with $e_1=X$. Then, from the equation of Gauss, we have

$$\begin{aligned}n^2\|H\|^2&=2\tau+\|h\|^2-\frac{n(n-1)(c-3f^2)}{4}\\&\quad-\frac{6(c+f^2)}{4}(d_1\cos^2\theta_1+d_2\cos^2\theta_2)+2(n-1)\left(\frac{c+f^2}{4}+f'\right),\end{aligned}\tag{2.10}$$

where $2d_1=\dim\mathfrak{D}_1$ and $2d_2=\dim\mathfrak{D}_2$.

From (2.10), we get

$$\begin{aligned}n^2\|H\|^2&=2\tau+\sum_{r=n+1}^{2m+1}\left[(h_{11}^r)^2+(h_{22}^r+\cdots+h_{nn}^r)^2+2\sum_{1\leq i<j\leq n}(h_{ij}^r)^2\right]\\&\quad-2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n}h_{ii}^rh_{jj}^r-\frac{n(n-1)(c-3f^2)}{4}\\&\quad-\frac{6(c+f^2)}{4}(d_1\cos^2\theta_1+d_2\cos^2\theta_2)+2(n-1)\left(\frac{c+f^2}{4}+f'\right)\end{aligned}$$

$$
\begin{aligned}
&= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}\Big[(h_{11}^r + h_{22}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2\Big] \\
&\quad + 2\sum_{r=n+1}^{2m+1}\sum_{1\le i<j\le n}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}h_{ii}^r h_{jj}^r - \frac{n(n-1)(c-3f^2)}{4} \\
&\quad - \frac{6(c+f^2)}{4}(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) + 2(n-1)\left(\frac{c+f^2}{4} + f'\right).
\end{aligned} \tag{2.11}
$$

We distinguish two cases.

(i) If X is tangent to $\mathfrak{D}_1$, then we have

$$
\begin{aligned}
\sum_{2\le i<j\le n}K_{ij} &= \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}\Big[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\Big] + \frac{(n-1)(n-2)(c-3f^2)}{8} \\
&\quad + \frac{c+f^2}{8}\big[6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_1\big] + \frac{1}{2}\left(\frac{c+f^2}{4} + f'\right)(-2n+4).
\end{aligned} \tag{2.12}
$$

Substituting (2.12) in (2.11), one gets

$$
\frac{1}{2}n^2\|H\|^2 \ge 2\,\mathrm{Ric}(X) - \frac{(n-1)(c-3f^2)}{2} - \frac{3(c+f^2)}{4}\cos^2\theta_1 + 2\left(\frac{c+f^2}{4} + f'\right), \tag{2.13}
$$

which is equivalent to (2.8).

(ii) If X is tangent to $\mathfrak{D}_2$, then we have

$$
\begin{aligned}
\sum_{2\le i<j\le n}K_{ij} &= \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}\Big[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2\Big] + \frac{(n-1)(n-2)(c-3f^2)}{8} \\
&\quad + \frac{c+f^2}{8}\big[6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_2\big] + \frac{1}{2}\left(\frac{c+f^2}{4} + f'\right)(-2n+4).
\end{aligned} \tag{2.14}
$$

Substituting (2.14) in (2.11), one gets

$$
\frac{1}{2}n^2\|H\|^2 \ge 2\,\mathrm{Ric}(X) - \frac{(n-1)(c-3f^2)}{2} - \frac{3(c+f^2)}{4}\cos^2\theta_2 + 2\left(\frac{c+f^2}{4} + f'\right), \tag{2.15}
$$

which is equivalent to (2.9).

(2) Assume that $H(p) = 0$. Equality holds in (2.8) and (2.9) if and only if

$$
\begin{gathered}
h_{12}^r = \cdots = h_{1n}^r = 0, \\
h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, \quad r \in \{n+1,\dots,2m+1\}.
\end{gathered} \tag{2.16}
$$

Then $h_{1j}^r = 0$ for all $j \in \{1,\dots,n\}$, $r \in \{n+1,\dots,2m+1\}$, that is, $X \in N_p$.

(3) Then equality case of (2.8) and (2.9) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$\begin{gathered} h^r_{ij} = 0, \quad i \neq j,\ r \in \{n+1,\dots,2m+1\}, \\ h^r_{11} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad i \in \{1,\dots,n\},\ r \in \{n+1,\dots,2m+1\}. \end{gathered} \tag{2.17}$$

In this case, it follows that p is a totally geodesic point. The converse is trivial. □

Corollary 2.3. *Let M be an n-dimensional semi-slant submanifold in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, the following hold.*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ and if*

(i) *X is tangent to $\mathfrak{D}_1$,*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c-3f^2) - 4\left(\frac{c+f^2}{4} + f'\right) + n^2\|H\|^2\right\}, \tag{2.18}$$

and if

(ii) *X is tangent to $\mathfrak{D}_2$,*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c-3f^2) + \frac{3}{2}(c+f^2)\cos^2\theta - 4\left(\frac{c+f^2}{4} + f'\right) + n^2\|H\|^2\right\}. \tag{2.19}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.18) and (2.19) if and only if $X \in N_p$.*

(3) *The equality case of (2.18) and (2.19) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.4. *Let M be an n-dimensional invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then, the following hold.*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ,*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c-3f^2) + \frac{3}{2}(c+f^2) - 4\left(\frac{c+f^2}{4} + f'\right) + n^2\|H\|^2\right\}. \tag{2.20}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.20) if and only if $X \in N_p$.*

(3) *The equality case of (2.20) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.5. *Let M be an n-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional cosymplectic space form $\tilde{M}(c)$. Then, the following hold.*

(1) *For each unit vector $X \in T_pM$ orthogonal to ξ,*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\left\{(n-1)(c-3f^2) - 4\left(\frac{c+f^2}{4} + f'\right) + n^2\|H\|^2\right\}. \tag{2.21}$$

(2) *If $H(p) = 0$, then a unit tangent vector X orthogonal to ξ at p satisfies the equality case of (2.21) if and only if $X \in N_p$.*

(3) *The equality case of (2.21) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

3. k-Ricci curvature and squared mean curvature

In this section, we prove relationship between the k-Ricci curvature and the squared mean curvature for slant, bi-slant, and semi-slant submanifolds in a locally conformal almost cosymplectic manifold $\widetilde{M}$. We state an inequality between the scalar curvature and the squared mean curvature for submanifolds M tangent to the vector field ξ.

Theorem 3.1. *Let M be an n-dimensional θ-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then,*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{1}{4n}\left[n(c-3f^2)+3(c+f^2)\cos^2\theta - 8\left(\frac{c+f^2}{4}+f'\right)\right], \tag{3.1}$$

equality holding at a point $p \in M$ if and only if p is a totally umbilical point.

Proof. Let p be a point of M. We choose an orthonormal basis $\{e_1, e_2, \dots, e_n = \xi\}$ for the tangent space T_pM and $\{e_{n+1}, \dots, e_{2m+1}\}$ for the normal space $T_p^\perp M$ at p such that the normal vector e_{n+1} is in the direction of the mean curvature vector and $e_1, e_2, \dots, e_n$ diagonalize the shape operator A_{n+1}. Then, we have

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix},$$
$$A_r = (h_{ij}^r), \quad \sum_{i=1}^n h_{ii}^r = 0, \quad n+2 \leq r \leq 2m+1. \tag{3.2}$$

From the equation of Gauss,

$$\begin{aligned} n^2\|H\|^2 = 2\tau + \sum_{i=1}^n a_i^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^n (h_{ij}^r)^2 - \frac{n(n-1)(c-3f^2)}{4} \\ - \frac{3(n-1)(c+f^2)}{4}\cos^2\theta + 2(n-1)\left(\frac{c+f^2}{4}+f'\right). \end{aligned} \tag{3.3}$$

On the other hand,

$$\sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_{i=1}^n a_i^2 - 2\sum_{i<j} a_i a_j. \tag{3.4}$$

Therefore, from the above equation, we have

$$n^2\|H\|^2 = \left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \le n\sum_{i=1}^{n} a_i^2. \tag{3.5}$$

Combining (3.3) and (3.5),

$$\begin{aligned} n(n-1)\|H\|^2 \ge 2\tau + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - \frac{n(n-1)(c-3f^2)}{4} \\ - \frac{3(n-1)(c+f^2)}{4}\cos^2\theta + 2(n-1)\left(\frac{c+f^2}{4}+f'\right), \end{aligned} \tag{3.6}$$

which implies inequality (3.1). If the equality sign of (3.1) holds at a point $p \in M$, then from (3.4) and (3.6) we get $A_r = 0$ $(r = n+2,\dots,2m+1)$ and $a_1 = \cdots = a_n$. Consequently, p is a totally umbilical point. The converse is trivial. □

THEOREM 3.2. *Let M be an n-dimensional bi-slant submanifold satisfying $g(X,\varphi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $Y \in \mathcal{D}_2$, tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then,*

$$\begin{aligned} \|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{1}{4n(n-1)}\Big[n(n-1)(c-3f^2) + 6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)(c+f^2) \\ - 8(n-1)\left(\frac{c+f^2}{4}+f'\right)\Big], \end{aligned} \tag{3.7}$$

where $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

THEOREM 3.3. *Let M be an n-dimensional semi-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then,*

$$\begin{aligned} \|H\|^2 \ge \frac{2\tau}{n(n-1)} - \frac{1}{4n(n-1)}\Big[n(n-1)(c-3f^2) + 6(d_1 + d_2\cos^2\theta)(c+f^2) \\ - 8(n-1)\left(\frac{c+f^2}{4}+f'\right)\Big], \end{aligned} \tag{3.8}$$

where $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

THEOREM 3.4. *Let M be an n-dimensional θ-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2 \le k \le n)$ and any point $p \in M$,*

$$\|H\|^2 \ge \Theta_k(p) - \frac{1}{4n}\left[n(c-3f^2) + 3(c+f^2)\cos^2\theta - 8\left(\frac{c+f^2}{4}+f'\right)\right]. \tag{3.9}$$

Proof. Let $\{e_1,\ldots,e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1\cdots i_k}$ the k-plane section spanned by $e_{i_1},\ldots,e_{i_k}$. It follows from (1.7) and (1.8) that

$$\begin{aligned}\tau(L_{i_1\cdots i_k}) &= \frac{1}{2}\sum_{i\in\{i_1,\ldots,i_k\}}\operatorname{Ric}_{L_{i_1\cdots i_k}}(e_i),\\ \tau(p) &= \frac{1}{\binom{n-2}{k-2}}\sum_{1\le i_1<\cdots<i_k\le n}\tau(L_{i_1\cdots i_k}).\end{aligned}\tag{3.10}$$

Combining (1.9) and (3.10), we obtain

$$\tau(p)\ge\frac{n(n-1)}{2}\Theta_k(p).\tag{3.11}$$

Therefore, by using (3.1) and (3.11), we can obtain the inequality in Theorem 3.4. □

Theorem 3.5. *Let M be an n-dimensional bi-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2\le k\le n)$ and any point $p\in M$,*

$$\begin{aligned}\|H\|^2\ge\Theta_k(p)-\frac{1}{4n(n-1)}\Big[&n(n-1)(c-3f^2)+6(d_1\cos^2\theta_1+d_2\cos^2\theta_2)(c+f^2)\\&-8(n-1)\Big(\frac{c+f^2}{4}+f'\Big)\Big],\end{aligned}\tag{3.12}$$

where $2d_1=\dim\mathfrak{D}_1$ and $2d_2=\dim\mathfrak{D}_2$.

Theorem 3.6. *Let M be an n-dimensional semi-slant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2\le k\le n)$ and any point $p\in M$,*

$$\begin{aligned}\|H\|^2\ge\Theta_k(p)-\frac{1}{4n(n-1)}\Big[&n(n-1)(c-3f^2)+6(d_1+d_2\cos^2\theta)(c+f^2)\\&-8(n-1)\Big(\frac{c+f^2}{4}+f'\Big)\Big],\end{aligned}\tag{3.13}$$

where $2d_1=\dim\mathfrak{D}_1$ and $2d_2=\dim\mathfrak{D}_2$.

Corollary 3.7. *Let M be an n-dimensional invariant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2\le k\le n)$ and any point $p\in M$,*

$$\|H\|^2\ge\Theta_k(p)-\frac{1}{4n}\Big[n(c-3f^2)+3(c+f^2)-8\Big(\frac{c+f^2}{4}+f'\Big)\Big].\tag{3.14}$$

Corollary 3.8. *Let M be an n-dimensional anti-invariant submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2 \le k \le n)$ and any point $p \in M$,*

$$\|H\|^2 \ge \Theta_k(p) - \frac{1}{4n}\left[n(c-3f^2) - 8\left(\frac{c+f^2}{4} + f'\right)\right]. \tag{3.15}$$

Corollary 3.9. *Let M be an n-dimensional contact CR-submanifold tangent to ξ into a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\widetilde{M}$. Then, for any integer k $(2 \le k \le n)$ and any point $p \in M$,*

$$\|H\|^2 \ge \Theta_k(p) - \frac{1}{4n(n-1)}\left[n(n-1)(c-3f^2) + 6d_1(c+f^2) - 8(n-1)\left(\frac{c+f^2}{4} + f'\right)\right]. \tag{3.16}$$

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