EXISTENCE AND ALGORITHM OF SOLUTIONS FOR GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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We introduce and study a new class of generalized nonlinear variational-like inequalities. Under suitable conditions, we prove the existence of solutions for the class of generalized nonlinear variational-like inequalities. A new iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality is given and the convergence of the algorithm is also proved. The results presented in this paper improve and generalize some results in recent literature.

1. Introduction

Variational-like inequalities are a useful and important generalization of variational inequalities [3, 8, 26]. They have potential and significant applications in optimization theory, structural analysis, and economics, see [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. Some mixed variational-like inequalities have been studied by Parida and Sen [26], Tian [27], and Yao [29] by using the Berge maximum theorem in finite- and infinite-dimensional spaces. Huang and Deng [10] extended the auxiliary principle technique to study the existence of solutions for a class of generalized strongly nonlinear mixed variational-like inequalities. By using the minimax inequality technique, Ding [5, 6] studied some classes of nonlinear variational-like inequalities in reflexive Banach spaces.

The purpose of this paper is to introduce and study a new class of generalized nonlinear variational-like inequalities, which includes several kinds of variational-like inequalities as special cases. A few existence results of solutions for the generalized nonlinear variational-like inequality are established. We construct an iterative algorithm for finding the approximate solutions of the generalized nonlinear variational-like inequality and obtain the convergence of the algorithm under certain conditions.

2. Preliminaries

Let H be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H, let $A, C, F : K \to H$, $N : H \times H \to H$, and $\eta : K \times K \to H$ be mappings, and let $f : K \to (-\infty, \infty]$ be a real functional.

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Suppose that $a: H \times H \to (-\infty, \infty)$ is a coercive continuous bilinear form, that is, there exist positive constants c and d such that

(C1) $a(v,v) \geq c\|v\|^2$, for all $v \in H$;

(C2) $a(u,v) \leq d\|u\|\|v\|$, for all $u,v \in H$.

Clearly, $c \leq d$.

We consider the following generalized nonlinear variational-like inequality problem.

Find $u \in K$ such that

$$a(u,v-u) + f(v) - f(u) \geq \langle N(Au,Cu) + Fu, \eta(v,u)\rangle, \quad \forall v \in K. \tag{2.1}$$

Special cases. (A) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$ and $Fu = 0$ for all $u,v \in K$, then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, \eta(v,u)\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.2}$$

which was introduced and studied by Ding [5].

(B) If $N(Au,Cu) = Au - Cu$, $a(u,v) = 0$ and $\eta(u,v) = gu - gv$ for all $u,v \in K$, then the generalized nonlinear variational-like inequality problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Cu - Au, gv - gu\rangle \geq f(u) - f(v), \quad \forall v \in K, \tag{2.3}$$

which was studied by Yao [29].

Definition 2.1. Let $A, C: K \to H$, $N: H \times H \to H$ and $\eta: K \times K \to H$ be mappings.

(1) A is said to be *Lipschitz continuous* with constant α if there exists a constant $\alpha > 0$ such that

$$\|Au - Av\| \leq \alpha\|u - v\|, \quad \forall u,v \in K. \tag{2.4}$$

(2) N is said to be *Lipschitz continuous* with constant β in the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u,w) - N(v,w)\| \leq \beta\|u - v\|, \quad \forall u,v,w \in H. \tag{2.5}$$

(3) N is said to be *η-antimonotone* with respect to A in the first argument if

$$\langle N(Au,w) - N(Av,w), \eta(u,v)\rangle \leq 0, \quad \forall u,v \in K,\ w \in H. \tag{2.6}$$

(4) N is said to be *η-relaxed Lipschitz* with constant γ with respect to C in the second argument if there exists a constant $\gamma > 0$ such that

$$\langle N(w,Cu) - N(w,Cv), \eta(u,v)\rangle \leq -\gamma\|u - v\|^2, \quad \forall u,v \in K,\ w \in H. \tag{2.7}$$

(5) η is said to be *Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$\|\eta(u,v)\| \leq \delta\|u - v\|, \quad \forall u,v \in K. \tag{2.8}$$

Similarly, we can define the Lipschitz continuity of N in the second argument.

Definition 2.2. Let K be a nonempty closed convex subset of a Hilbert space H and $f : K \to (-\infty,\infty]$ be a real functional.

(1) f is said to be *convex* if for any $u,v \in K$ and for any $\alpha \in [0,1]$,

$$f(\alpha u + (1-\alpha)v) \leq \alpha f(u) + (1-\alpha)f(v). \tag{2.9}$$

(2) f is said to be *lower semicontinuous* on K if for each $\alpha \in (-\infty,\infty]$, the set $\{u \in K : f(u) \leq \alpha\}$ is closed in K.

Lemma 2.3 [1, 2]. *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, and let $\phi,\psi : K \times K \to R$ be mappings satisfying the following conditions:*

(a) *$\psi(x,y) \leq \phi(x,y)$, for all $x,y \in X$, and $\psi(x,x) \geq 0$, for all $x \in X$;*

(b) *for each $x \in X$, $\phi(x,y)$ is upper semicontinuous with respect to y;*

(c) *for each $y \in X$, the set $\{x \in X : \psi(x,y) < 0\}$ is a convex set;*

(d) *there exists a nonempty compact set $K \subset X$ and $x_0 \in K$ such that $\psi(x_0,y) < 0$, for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $\phi(x,\hat{y}) \geq 0$, for all $x \in X$.

3. Existence theorems

In this section, we give four existence theorems of solutions for the generalized nonlinear variational-like inequality (2.1).

Theorem 3.1. *Let K be a nonempty closed convex subset of a Hilbert space H. Let $a : H \times H \to (-\infty,\infty)$ be a coercive continuous bilinear form with (C1) and (C2) and let $f : K \to (-\infty,\infty]$ be a proper convex lower semicontinuous functional with* $\operatorname{int}(\operatorname{dom} f) \cap K \neq \emptyset$. *Suppose that $A,C : K \to H$ and $N : H \times H \to H$ are continuous mappings, $\eta : K \times K \to H$ is Lipschitz continuous with constant δ, for each $v \in K$, $\eta(\cdot,v)$ is continuous and $\eta(v,u) = -\eta(u,v)$ for all $u,v \in K$. Assume that N is η-antimonotone with respect to A in the first argument and η-relaxed Lipschitz with constant ξ with respect to C in the second argument. Suppose that for given $x,y \in H$ and $v \in K$, the mapping $u \mapsto \langle N(x,y),\eta(u,v)\rangle$ is concave and upper semicontinuous. If $F : K \to H$ is completely continuous, then the generalized nonlinear variational-like inequality (2.1) has a solution $u \in K$.*

Proof. We first prove that for each fixed $\hat{u} \in K$, there exists a unique $\hat{w} \in K$ such that

$$a(\hat{w},v-\hat{w}) + f(v) - f(\hat{w}) \geq \langle N(A\hat{w},C\hat{w}) + F\hat{u},\eta(v,\hat{w})\rangle, \quad \forall v \in K. \tag{3.1}$$

Let $\hat{u}$ be in K. Define the functionals ϕ and $\psi : K \times K \to R$ by

$$\begin{aligned}\phi(v,w) &= a(v,v-w) + f(v) - f(w) - \langle N(Av,Cv) + F\hat{u},\eta(v,w)\rangle,\\ \psi(v,w) &= a(w,v-w) + f(v) - f(w) - \langle N(Aw,Cw) + F\hat{u},\eta(v,w)\rangle\end{aligned} \tag{3.2}$$

for all $v,w \in K$.

We check that the functionals ϕ and ψ satisfy all the conditions of Lemma 2.3 in the weak topology. It follows from the definitions of ϕ and ψ that for all $v,w \in K$,

$$\begin{aligned}\phi(v,w) - \psi(v,w) &= a(v-w,v-w) - \langle N(Av,Cv) - N(Aw,Cv),\eta(v,w)\rangle \\ &\quad - \langle N(Aw,Cv) - N(Aw,Cw),\eta(v,w)\rangle \\ &\geq (c+\xi)\|v-w\|^2 \geq 0,\end{aligned} \tag{3.3}$$

which means that ϕ and ψ satisfy the condition (a) of Lemma 2.3. Notice that f is a convex lower semicontinuous functional and for given $x,y \in H$, $v \in K$, the mapping $u \mapsto \langle N(x,y),\eta(u,v)\rangle$ is concave and upper semicontinuous. It follows that $\phi(v,w)$ is weakly upper semicontinuous with respect to w and the set $\{v \in K : \psi(v,w) < 0\}$ is convex for each $w \in K$. Therefore, the conditions (b) and (c) of Lemma 2.3 hold. Since f is proper convex lower semicontinuous, for each $v \in \operatorname{int}(\operatorname{dom} f)$, $\partial f(v) \neq \emptyset$, see Ekeland and Temam [9]. Let v^* be in $\operatorname{int}(\operatorname{dom} f) \cap K$. It follows that

$$f(u) \geq f(v^*) + \langle r,u-v^*\rangle, \quad \forall r \in \partial f(v^*),\ u \in K. \tag{3.4}$$

Put

$$\begin{gathered}D = (c+\xi)^{-1}\left(\|r\| + \delta\|N(Av^*,Cv^*)\| + \delta\|F\hat{u}\|\right), \\ T = \{w \in K : \|w-v^*\| \leq D\}.\end{gathered} \tag{3.5}$$

Obviously, T is a weakly compact subset of K and for any $w \in K \setminus T$,

$$\begin{aligned}\psi(v^*,w) &= a(w-v^*,v^*-w) + f(v^*) - f(w) - \langle N(Aw,Cw) + F\hat{u},\eta(v^*,w)\rangle \\ &\leq -a(w-v^*,w-v^*) - \langle r,w-v^*\rangle \\ &\quad + \langle N(Aw,Cw) - N(Av^*,Cw),\eta(w,v^*)\rangle \\ &\quad + \langle N(Av^*,Cw) - N(Av^*,Cv^*),\eta(w,v^*)\rangle \\ &\quad + \langle N(Av^*,Cv^*),\eta(w,v^*)\rangle + \langle F\hat{u},\eta(w,v^*)\rangle \\ &\leq -\|w-v^*\|\left[(c+\xi)\|w-v^*\| - \|r\| - \delta\|N(Av^*,Cv^*)\| - \delta\|F\hat{u}\|\right] < 0,\end{aligned} \tag{3.6}$$

which yields that the condition (d) of Lemma 2.3 holds. Thus Lemma 2.3 ensures that there exists a $\hat{w} \in K$ such that $\phi(v,\hat{w}) \geq 0$ for all $v \in K$, that is,

$$a(v,v-\hat{w}) + f(v) - f(\hat{w}) \geq \langle N(Av,Cv) + F\hat{u},\eta(v,\hat{w})\rangle, \quad \forall v \in K. \tag{3.7}$$

Let t be in $(0,1]$ and v be in K. Replacing v by $v_t = tv + (1-t)\hat{w}$ in (3.7), we see that

$$a(v_t,t(v-\hat{w})) + f(v_t) - f(\hat{w}) \geq \langle N(Av_t,Cv_t) + F\hat{u},\eta(v_t,\hat{w})\rangle, \quad \forall v \in K. \tag{3.8}$$

Note that a is bilinear and f is convex. From (3.8) we deduce that

$$t\left[a(v_t,v-\hat{w}) + f(v) - f(\hat{w})\right] \geq t\langle N(Av_t,Cv_t) + F\hat{u},\eta(v,\hat{w})\rangle, \quad \forall v \in K, \tag{3.9}$$

which implies that

$$a(v_t, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(Av_t, Cv_t) + F\hat{u}, \eta(v, \hat{w})\rangle, \quad \forall v \in K. \tag{3.10}$$

Letting $t \to 0^+$ in the above inequality, we conclude that

$$a(\hat{w}, v - \hat{w}) + f(v) - f(\hat{w}) \geq \langle N(A\hat{w}, C\hat{w}) + F\hat{u}, \eta(v, \hat{w})\rangle, \quad \forall v \in K. \tag{3.11}$$

That is, $\hat{w}$ is a solution of (3.1). Now we prove the uniqueness. For any two solutions $w_1, w_2 \in K$ of (3.1), we know that

$$\begin{aligned} a(w_1, w_2 - w_1) + f(w_2) - f(w_1) &\geq \langle N(Aw_1, Cw_1) + F\hat{u}, \eta(w_2, w_1)\rangle, \\ a(w_2, w_1 - w_2) + f(w_1) - f(w_2) &\geq \langle N(Aw_2, Cw_2) + F\hat{u}, \eta(w_1, w_2)\rangle. \end{aligned} \tag{3.12}$$

Adding these inequalities, we deduce that

$$\begin{aligned} c\|w_1 - w_2\|^2 &\leq a(w_1 - w_2, w_1 - w_2) \\ &\leq \langle N(Aw_1, Cw_1) - N(Aw_2, Cw_1), \eta(w_1, w_2)\rangle \\ &\quad + \langle N(Aw_2, Cw_1) - N(Aw_2, Cw_2), \eta(w_1, w_2)\rangle \\ &\leq -\xi\|w_1 - w_2\|^2, \end{aligned} \tag{3.13}$$

which yields that $w_1 = w_2$. That is, $\hat{w}$ is a unique solution of (3.1). This means that there exists a mapping $G : K \to K$ satisfying $G(\hat{u}) = \hat{w}$, where $\hat{w}$ is the unique solution of (3.1) for each $\hat{u} \in K$.

Next we show that G is a completely continuous mapping. Let u_1 and u_2 be arbitrary elements in K. Using (3.1), we get that

$$\begin{aligned} a(Gu_1, Gu_2 - Gu_1) + f(Gu_2) - f(Gu_1) &\geq \langle N(A(Gu_1), C(Gu_1)) + Fu_1, \eta(Gu_2, Gu_1)\rangle, \\ a(Gu_2, Gu_1 - Gu_2) + f(Gu_1) - f(Gu_2) &\geq \langle N(A(Gu_2), C(Gu_2)) + Fu_2, \eta(Gu_1, Gu_2)\rangle. \end{aligned} \tag{3.14}$$

Adding (3.14), we arrive at

$$\begin{aligned} c\|Gu_1 - Gu_2\|^2 &\leq a(Gu_1 - Gu_2, Gu_1 - Gu_2) \\ &\leq \langle N(A(Gu_1), C(Gu_1)) - N(A(Gu_2), C(Gu_1)), \eta(Gu_1, Gu_2)\rangle \\ &\quad + \langle N(A(Gu_2), C(Gu_1)) - N(A(Gu_2), C(Gu_2)), \eta(Gu_1, Gu_2)\rangle \\ &\quad + \langle Fu_1 - Fu_2, \eta(Gu_1, Gu_2)\rangle \\ &\leq -\xi\|Gu_1 - Gu_2\|^2 + \delta\|Fu_1 - Fu_2\|\|Gu_1 - Gu_2\|, \end{aligned} \tag{3.15}$$

that is,

$$\|Gu_1 - Gu_2\| \leq \frac{\delta}{c + \xi}\|Fu_1 - Fu_2\|. \tag{3.16}$$

Since F is completely continuous, it follows from (3.16) that $G : K \to K$ is a completely continuous mapping. Hence the Schauder fixed point theorem guarantees that G has a fixed point $u \in K$, which means that u is a solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

Theorem 3.2. *Let a, f, C, N, F, and η be as in Theorem 3.1 and let N be Lipschitz continuous with constant ζ in the first argument. Suppose that $A : K \to H$ is Lipschitz continuous with constant ρ. If $c + \xi > \delta\zeta\rho$, then the generalized nonlinear variational-like inequality (2.1) has a solution $u \in K$.*

Proof. Put

$$\begin{gathered} D = (c+\xi-\delta\zeta\rho)^{-1}\left(\|r\| + \delta\|N(Av^*,Cv^*)\| + \delta\|F\hat{u}\|\right), \\ T = \{w \in K : \|w - v^*\| \le D\}. \end{gathered} \tag{3.17}$$

As in the proof of Theorem 3.1, we conclude that

$$\begin{aligned} \psi(v^*,w) \le {} & -a(w-v^*,w-v^*) - \langle r, w-v^*\rangle \\ & + \langle N(Aw,Cw) - N(Av^*,Cw), \eta(w,v^*)\rangle \\ & + \langle N(Av^*,Cw) - N(Av^*,Cv^*), \eta(w,v^*)\rangle \\ & + \langle N(Av^*,Cv^*), \eta(w,v^*)\rangle + \langle F\hat{u}, \eta(w,v^*)\rangle \\ \le {} & -\|w-v^*\|\big[(c+\xi-\delta\zeta\rho)\|w-v^*\| \\ & - \|r\| - \delta\|N(Av^*,Cv^*)\| - \delta\|F\hat{u}\|\big] < 0 \end{aligned} \tag{3.18}$$

for any $w \in K \setminus T$. The rest of the argument is now essentially the same as in the proof of Theorem 3.1 and therefore is omitted. □

Theorem 3.3. *Let a, f, A, C, N, and η be as in Theorem 3.1. Suppose that $F : K \to H$ is Lipschitz continuous with constant l. If $\delta l/(c+\xi) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

Proof. Let u_1 and u_2 be arbitrary elements in K. As in the proof of Theorem 3.1, we deduce that

$$\|Gu_1 - Gu_2\| \le \frac{\delta}{c+\xi}\|Fu_1 - Fu_2\| \le \frac{\delta l}{c+\xi}\|u_1 - u_2\|, \quad \forall u_1, u_2 \in K, \tag{3.19}$$

which yields that $G : K \to K$ is a contraction mapping and hence it has a unique fixed point $u \in K$, which is a unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

The following theorem follows from the arguments of Theorems 3.1, 3.2 and, 3.3.

Theorem 3.4. *Let a, f, A, C, N, and η be as in Theorem 3.2. Suppose that $F : K \to H$ is Lipschitz continuous with constant l. If $0 < \delta l/(c+\xi-\delta\zeta\rho) < 1$, then the generalized nonlinear variational-like inequality (2.1) has a unique solution $u \in K$.*

4. Algorithm and convergence theorems

Based on Theorem 3.1, we suggest the following iterative algorithm.

Algorithm 4.1. *Let $A,C,F : K \to H$, $N : H \times H \to H$, and $\eta : K \times K \to H$ be mappings, and let $f : K \to (-\infty,\infty]$ be a real functional. For any given $u_0 \in K$, compute sequence $\{u_n\}_{n\geq 0}$ by the iterative scheme*

$$a(u_{n+1}, v-u_{n+1}) + f(v) - f(u_{n+1}) \geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(v,u_{n+1})\rangle, \tag{4.1}$$

for all $v \in K$ and $n \geq 0$.

Theorem 4.2. *Let a, f, F, N, A, C, and η be as in Theorem 3.3. If $\delta l/(c+\xi) < 1$, then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to the unique solution.*

Proof. Using Algorithm 4.1, we obtain that

$$\begin{aligned} a(u_{n+1}, u_n - u_{n+1}) + f(u_n) - f(u_{n+1}) &\geq \langle N(Au_{n+1}, Cu_{n+1}) + Fu_n, \eta(u_n, u_{n+1})\rangle, \\ a(u_n, u_{n+1} - u_n) + f(u_{n+1}) - f(u_n) &\geq \langle N(Au_n, Cu_n) + Fu_{n-1}, \eta(u_{n+1}, u_n)\rangle, \end{aligned} \tag{4.2}$$

for all $n \geq 1$. Adding (4.2), we get that

$$\begin{aligned} c\|u_{n+1}-u_n\|^2 &\leq a(u_{n+1}-u_n, u_{n+1}-u_n) \\ &\leq \langle N(Au_{n+1}, Cu_{n+1}) - N(Au_n, Cu_{n+1}), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \langle N(Au_n, Cu_{n+1}) - N(Au_n, Cu_n), \eta(u_{n+1}, u_n)\rangle \\ &\quad + \langle Fu_n - Fu_{n-1}, \eta(u_{n+1}, u_n)\rangle \\ &\leq -\xi\|u_{n+1}-u_n\|^2 + \delta l\|u_n - u_{n-1}\|\|u_{n+1}-u_n\|, \end{aligned} \tag{4.3}$$

that is,

$$\|u_{n+1}-u_n\| \leq \frac{\delta l}{c+\xi}\|u_n - u_{n-1}\|, \quad \forall n \geq 1, \tag{4.4}$$

which yields that $\{u_n\}_{n\geq 0}$ is a Cauchy sequence by $\delta l/(c+\xi) < 1$. Consequently, $\{u_n\}_{n\geq 0}$ converges to some element u in K. Letting $n \to \infty$ in (4.1), we infer that

$$a(u, v-u) + f(v) - f(u) \geq \langle N(Au, Cu) + Fu, \eta(v,u)\rangle, \quad \forall v \in K. \tag{4.5}$$

Hence u is a solution of the generalized nonlinear variational-like inequality (2.1). It follows from Theorem 3.3 that u is the unique solution of the generalized nonlinear variational-like inequality (2.1). This completes the proof. □

Similarly we have the following result.

THEOREM 4.3. *Let a, f, F, N, A, C, and η be as in Theorem 3.4. If $0 < \delta l/(c+\xi-\delta\zeta\rho) < 1$, then the generalized nonlinear variational-like inequality (2.1) possesses a unique solution and the iterative sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 4.1 converges strongly to the unique solution.*

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