EXTENSIONS OF BAILEY'S TRANSFORM AND APPLICATIONS

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The well-known Bailey's transform is extended. Using the extended transform, we derive hitherto undiscovered ordinary and q-hypergeometric identities and discuss their particular cases of importance, namely, two new q-sums for Saalschützian ${}_4\Phi_3$, new double series Rogers-Ramanujan-type identities of modulo 81, discrete extension of the q-analogs of two quadratic transformations of ${}_2F_1$, and two new quadratic-cubic transformations of ${}_3F_2$.

1. Introduction

The well-known transform was discovered by Bailey [12] in 1947 and is being used, since then, to obtain various ordinary and q-hypergeometric identities and Rogers-Ramanujan-type identities.

It states that if

$$\begin{aligned}\beta_n &= \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r},\\ \gamma_n &= \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n},\end{aligned} \tag{1.1}$$

then, subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.2}$$

Making use of this remarkably simple transformation, Bailey [12, 13] outlined a technique for obtaining transformations of both ordinary and q-hypergeometric series. He also used these transformations to obtain a number of identities of Rogers-Ramanujan type. Subsequently, Slater [27, 28] gave a very exhaustive list of 130 identities of the Rogers-Ramanujan type derived by her, using (1.2). For further details on (1.2), see [29].

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Andrews [4, 5, 10] exploited Bailey's transform in the form of Bailey pair and Bailey chains to show that all of the 130 identities given by Slater [27, 28] can be embedded in infinite families of multiple-series Rogers-Ramanujan-type identities.

Using (1.2), Bressoud [14] found finite forms of Rogers-Ramanujan type identities and further, recently, Bressoud et al. [15] introduced the concept of change of base in Bailey pairs and derived many new multiple-series Rogers-Ramanujan-type identities.

Verma and Jain [31, 32] and Jain [20, 22] also used (1.2) to derive a number of q-hypergeometric transformations and identities.

In brief, after Bailey and Slater, a large number of mathematicians have used Bailey's transform (1.2) to make applications in the theory of generalized hypergeometric series, number theory, partition theory, combinatorics, physics, and computer algebra (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 20, 25, 26, 31, 32]).

In this paper, we extend such a fundamental and useful transform of Bailey. The extensions are discussed in Section 2 as Theorems 2.1 and 2.2. Using the extended Bailey's transform, we derive five new q-hypergeometric identities in Section 3. The identities (3.1) and (3.3) both convert a Saalschützian ${}_4\Phi_3$ of base q^2 into a Saalschützian ${}_4\Phi_3$ of base q, like Singh's quadratic transformation, [19, Appendix, III.21]. The identity (3.2) converts a ${}_4\Phi_3$ of base q^2 into a ${}_4\Phi_2$ of base q. The identity (3.4) provides a transformation of a very well-poised ${}_{12}\Phi_{11}$ with base q^2 into a very well-poised ${}_{10}\Phi_9$ with base q. The identity (3.5) is a transformation of a very well-poised ${}_{12}\Phi_{11}$ with base q^3 into a Saalschützian ${}_6\Phi_5$ with base q.

In Section 4, seven new and interesting ordinary hypergeometric identities (4.1) to (4.7) are derived. *An interesting fact about these identities is that they will have no exact q-analogs, unlike other results of ordinary hypergeometric series, until the required sums of q-hypergeometric series, having parameters of different bases and needed arguments, are investigated.* This fact is illustrated for the result (4.1) in Section 5. In this way, an open problem of investigating the q-analogs of (4.1) to (4.7), by any other method, arises naturally in the study of extended Bailey's transform.

The particular cases and applications of some of the results of Sections 3 and 4 are discussed in Section 6. They include two new q-sums for Saalschützian ${}_4\Phi_3$, derivations of double-series Rogers-Ramanujan-type identities of modulo 81, discrete extensions of the q-analogs of two quadratic transformations of ${}_2F_1(z)$, and derivations of two new quadratic-cubic transformations for ${}_3F_2(z)$.

We have followed the definitions and notations from [19, 29].

2. Extended Bailey's transform

Theorem 2.1. *If*

$$\begin{aligned}\beta_n &= \sum_{r=0}^{[n/p]} \alpha_r u_{n-pr} v_{n+r} t_{n-r} w_{n+pr},\\ \gamma_n &= \sum_{r=pn}^{\infty} \delta_r u_{r-pn} v_{r+n} t_{r-n} w_{r+pn},\end{aligned} \tag{2.1}$$

then, subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.2}$$

where p is any integer and α_r, δ_r, u_r, v_r, t_r, and w_r are any functions of r only. Obviously, the $p = 1$ case is the original Bailey's transform.

Proof. Observe that

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \sum_{r=pn}^{\infty} \alpha_n \delta_r u_{r-pn} v_{r+n} t_{r-n} w_{r+pn}. \tag{2.3}$$

If this double series is convergent, then using [30, page 10, Lemma 3], namely,

$$\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} A(r,n) = \sum_{r=0}^{\infty} \sum_{n=0}^{[r/p]} A(r-np,n), \tag{2.4}$$

in (2.3) after the replacement of r by $r + pn$, we get

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{r=0}^{\infty} \sum_{n=0}^{[r/p]} \alpha_n \delta_r u_{r-pn} v_{r+n} t_{r-n} w_{r+pn} = \sum_{r=0}^{\infty} \beta_r \delta_r. \tag{2.5}$$

□

Theorem 2.2. *If*

$$\begin{aligned} \beta_n &= \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} t_{n+2r} w_{pn-r} z_{p'n+r}, \\ \gamma_n &= \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} t_{r+2n} w_{pr-n} z_{p'r+n}, \end{aligned} \tag{2.6}$$

then, subject to convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \tag{2.7}$$

where, α_r, δ_r, u_r, v_r, t_r, w_r, and z_r are any functions of r only, and p and p' are any integers. Obviously, when $t_r = 1$ and $p = p' = 1$, it will yield the original Bailey's transform.

Proof. Proceeding as in the proof of Theorem 2.1 and using [24, page 56, Lemma 10] in place of (2.4), we get (2.7). □

Here, it may be noted that Theorems 2.1 and 2.2 convert into original Bailey's transform when $p = 1$ and $t_r = 1$, $p = p' = 1$, respectively. Hence, only those cases in which the choice for parameters α_r, u_r, p, p', and so forth does not involve the conditions, $p = 1$ for Theorem 2.1 and $t_r = 1$, $p = p' = 1$ for Theorem 2.2, will be the cases not contained

in original Bailey's transform. In fact, Theorems 2.1 and 2.2 can be combined to form a single theorem, but such a presentation has been avoided for easier visualization of the extensions and application process.

The two theorems suggest that various new sequences of other combinations of involved summation indices may be introduced in their β_n and γ_n. Further, it may be possible in many cases to decide the new expressions for α_r, δ_r, u_r, v_r, t_r, w_r, and z_r, which yield closed forms for β_n and γ_n, using one or two of the known summation theorems. Thus, many more new results may be discovered. But we have discussed certain cases which use the well-known classical summation theorems [19, Appendix, II.5,6,12,21,22] and their ordinary hypergeometric analogs only. It may be emphasized that there may be other various appropriate cases to have the closed forms for β_n and γ_n in Theorems 2.1 and 2.2.

3. New q-hypergeometric identities

In the extensions of Bailey's transform, discussed in the previous section as Theorems 2.1 and 2.2, we observed five expressions for α_r, δ_r, u_r, v_r, t_r, w_r, and z_r, which yield closed forms for β_n and γ_n and lead to five new q-hypergeometric identities. The identities are as follows:

$$
{}_4\Phi_3\left[\begin{matrix} a^2, b^2, q^{-N+1}, q^{-N} \\ a^2b^2q, c^2, \dfrac{q^{2-2N}}{c^2} \end{matrix}; q^2, q^2\right] = \frac{(ac, -c/a; q)_N}{(c^2; q^2)_N} \cdot {}_4\Phi_3\left[\begin{matrix} a^2, ab, -ab, q^{-N} \\ a^2b^2, ac, \dfrac{-aq^{1-N}}{c} \end{matrix}; q, q\right], \tag{3.1}
$$

$$
{}_4\Phi_3\left[\begin{matrix} a, aq, q^{-N+1}, q^{-N} \\ b^2q, \dfrac{aq^{-N+1}}{c}, \dfrac{aq^{-N+2}}{c} \end{matrix}; q^2, q^2\right] = \frac{(c; q)_N}{(c/a; q)_N} \cdot {}_4\Phi_2\left[\begin{matrix} a, b, -b, q^{-N} \\ b^2, c \end{matrix}; q, \frac{-cq^N}{a}\right], \tag{3.2}
$$

$$
{}_4\Phi_3\left[\begin{matrix} a, b, -q^{-N}, q^{-N} \\ -ab, c, \dfrac{q^{1-2N}}{c} \end{matrix}; q, q\right] = \frac{(caq, c/a; q^2)_N}{(c; q)_{2N}} \cdot {}_4\Phi_3\left[\begin{matrix} a^2, ab, abq, q^{-2N} \\ acq, a^2b^2, \dfrac{aq^{2-2N}}{c} \end{matrix}; q^2, q^2\right], \tag{3.3}
$$

$$
\begin{aligned}
&{}_{12}W_{11}(a; a^2/b^2, c, cq, d, dq, abq^{1+N}/cd, abq^{2+N}/cd, q^{-N+1}, q^{-N}; q^2, q^2) \\
&= \frac{(aq, bq/c, bq/d, aq/cd; q)_N}{(bq, aq/c, aq/d, bq/cd; q)_N} \cdot {}_{10}W_9\left(b; c, d, a/b, b\sqrt{\frac{q}{a}}, -b\sqrt{\frac{q}{a}}, abq^{1+N}/cd, q^{-N}; q, \frac{bq}{a}\right)
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&{}_{12}W_{11}\left(a^3; b, bq, bq^2, c, cq, cq^2, q^{-N+2}, q^{-N+1}, q^{-N}; q^3, \left(\frac{a^4q^{1+N}}{bc}\right)^3\right) \\
&\quad = \frac{(a^3q, a^3q/bc; q)_N}{(a^3q/b, a^3q/c; q)_N} \cdot {}_6\Phi_5\left[\begin{matrix} a, a\omega, a\omega^2, b, c, q^{-N} \\ a^{3/2}\sqrt{q}, -a^{3/2}\sqrt{q}, a^{3/2}, -a^{3/2}, bcq^{-N}/a^3 \end{matrix}; q, q\right],
\end{aligned} \tag{3.5}
$$

where ω is the cube root of unity.

Proof of (3.1). Choosing

$$\alpha_r = \frac{(b^2;q^2)_r(-a^2)^r q^{r^2+2r}}{(a^2b^2q,q^2;q^2)_r}, \qquad u_r = \frac{q^r}{(q;q)_r}, \qquad t_r = (a^2;q^2)_r,$$
$$\delta_r = \frac{(q^{-N};q)_r}{(ac,-aq^{1-N}/c;q)_r}, \tag{3.6}$$

$v_r = w_r = 1$, and $p = 2$ in (2.1), and making use of the q-Pfaff-Saalschütz sum [19, Appendix, II.12], we get

$$\beta_n = \frac{(a^2b^2;q^2)_n(a^2;q)_n q^n}{(a^2b^2;q)_n(q;q)_n},$$
$$\gamma_n = \frac{(c^2;q^2)_N(a^2;q^2)_n(q^{-N};q)_{2n}q^{-n^2}}{(ac,-c/a;q)_N(c^2;q^2)_n(q^{2-2N}/c^2;q^2)_n(-a^2)^n}. \tag{3.7}$$

Putting these values in (2.2), we obtain the result (3.1). □

Proof of (3.2). Choosing

$$\alpha_r = \frac{q^{(1+2N)r-r^2}(c/a)^{2r}}{(b^2q;q^2)_r(q^2;q^2)_r}, \qquad u_r = \frac{q^{Nr+r^2}(c/a)^r}{(q;q)_r}, \qquad t_r = q^{-r^2},$$
$$\delta_r = \frac{(a,q^{-N};q)_r}{(c;q)_r}, \tag{3.8}$$

$v_r = w_r = 1$, and $p = 2$ in (2.1), and making use of the q-Chu-Vandermonde sums [19, Appendix, II.6,7], we get

$$\beta_n = \frac{(b^2;q^2)_n q^{-n(n-1)/2}(cq^N/a)^n}{(b^2;q)_n(q;q)_n},$$
$$\gamma_n = \frac{(c/a;q)_N(a,q^{-N};q)_{2n}q^{n^2+n(1-2N)}}{(c;q)_N(aq^{-N+1}/c;q)_{2n}}. \tag{3.9}$$

Putting these values in (2.2), we obtain the result (3.2). □

Proof of (3.3). Choosing

$$\alpha_r = \frac{(b;q)_r q^{(r^2+3r)/2}\,a^r}{(-ab;q)_r(q;q)_r}, \qquad u_r = \frac{q^{2r}}{(q^2;q^2)_r}, \qquad w_r = (a;q)_r,$$
$$\delta_r = \frac{(q^{-2N};q^2)_r}{(acq,aq^{2-2N}/c;q^2)_r}, \tag{3.10}$$

$v_r = t_r = z_r = 1$, and $p = 2$ in (2.6), and making use of the q-Pfaff-Saalschütz sum [19, Appendix, II.12], we get

$$\begin{aligned}\beta_n &= \frac{(a^2;q^2)_n (ab;q)_{2n} q^{2n}}{(a^2b^2;q^2)_n (q^2;q^2)_n},\\ \gamma_n &= \frac{(c;q)_{2N}(a;q)_n (q^{-2N};q^2)_n q^{-n(n+1)/2}}{(acq,c/a;q^2)_N (c;q)_n (q^{1-2N}/c;q^2)_n (a^n)}.\end{aligned} \tag{3.11}$$

Putting these values in (2.7), we obtain the result (3.3). □

Proof of (3.4). Choosing

$$\begin{aligned}\alpha_r &= \frac{(a, q^2\sqrt{a}, -q^2\sqrt{a}, a^2/b^2; q^2)_r q^{2r} (b/a)^{2r}}{(\sqrt{a}, -\sqrt{a}, q^2b^2/a, q^2; q^2)_r}, \qquad u_r = \frac{(b/a;q)_r q^r}{(q;q)_r}, \qquad w_r = \frac{(b;q)_r}{(aq;q)_r},\\ \delta_r &= \frac{(q\sqrt{b}, -q\sqrt{b}, c, d, abq^{1+N}/cd, q^{-N};q)_r}{(\sqrt{b}, -\sqrt{b}, qb/c, qb/d, cdq^{-N}/a, bq^{1+N};q)_r},\end{aligned} \tag{3.12}$$

$v_r = t_r = 1$, and $p = 2$ in (2.1), and making use of the Jackson's ${}_8\Phi_7$ sum [19, Appendix, II.22], we get

$$\begin{aligned}\beta_n &= \frac{(a^2/b^2, b;q)_n (b^2q/a;q^2)_{2n} (bq/a)^n}{(aq;q^2)_n (qb^2/a, q;q)_n},\\ \gamma_n &= \frac{(bq, aq/c, aq/d, bq/cd;q)_N (c, d, abq^{1+N}/cd, q^{-N};q)_{2n} (a/b)^{2n}}{(a^2q, bq/c, bq/d, a^2q/cd;q)_N (qa^2/c, qa^2/d, cdq^{-N}/b, aq^{1+N};q)_{2n}}.\end{aligned} \tag{3.13}$$

Putting these values in (2.2), we obtain the result (3.4). □

Proof of (3.5). Choosing

$$\begin{aligned}\alpha_r &= \frac{(a^3, q^3\sqrt{a^3}, -q^3\sqrt{a^3}; q^3)_r q^{3r(3r+1)/2} (-a^3)^r}{(\sqrt{a^3}, -\sqrt{a^3}, q^3; q^3)_r}, \qquad u_r = \frac{q^r}{(q;q)_r}, \qquad w_r = \frac{1}{(a^3q;q)_r},\\ \delta_r &= \frac{(b, c, q^{-N};q)_r}{(bcq^{-N}/a^3;q)_r},\end{aligned} \tag{3.14}$$

$v_r = t_r = 1$, and $p = 3$ in (2.1), and making use of the ${}_6\Phi_5$ sum and q-Pfaff-Saalschütz sum [19, Appendix, II.20,12], respectively, we get

$$\begin{aligned}\beta_n &= \frac{(a^3;q^3)_n q^n}{(a^3;q)_{2n} (q;q)_n},\\ \gamma_n &= \frac{(a^3q/b, a^3q/c;q)_N (b, c, q^{-N};q)_{3n} q^{-3n(3n-1)/2} (-1)^n}{(a^3q, a^3q/bc;q)_N (a^3q/b, a^3q/c, a^3q^{1+N};q)_{3n}} \left(\frac{a^3q^N}{bc}\right)^{3n}.\end{aligned} \tag{3.15}$$

Putting these values in (2.2), we obtain the result (3.5). □

4. New ordinary hypergeometric identities

In this section, seven new and interesting ordinary hypergeometric identities are obtained, from the extended Bailey's transform quoted in Section 2. The seven transformations are as follows:

$$
{}_6F_5\left[\begin{matrix} a, a+(1/3), a+(2/3), \triangle(2;1-c+3a), -N \\ d, (3/2)+3a-d, \triangle(3;1+3a-c-N) \end{matrix};1\right]
= \frac{(c)_N}{(c-3a)_N}\cdot {}_4F_3\left[\begin{matrix} 3a, 2+3a-2d, 2d-3a-1, -N \\ c, d, 3a-d+(3/2) \end{matrix};\frac{1}{4}\right], \tag{4.1}
$$

$$
{}_5F_4\left[\begin{matrix} w, \triangle(2;f-a), \triangle(2;w+a) \\ w+a, f-a, \triangle(2;f) \end{matrix};1\right]
= \frac{\Gamma(f)\Gamma(f-w-1/2)}{\Gamma(f-w/2)\Gamma(f-w/2-1/2)}\cdot {}_4F_3\left[\begin{matrix} a, f-w-a, w, f-w-\frac{1}{2} \\ \triangle(3;2f-w-1) \end{matrix};\frac{2}{27}\right], \tag{4.2}
$$

$$
\begin{aligned}
&{}_{11}F_{10}\left[\begin{matrix} v, 1+\frac{v}{2}, d, w, \frac{1}{2}+2v-w-d+N, \triangle(2;1+v-w), \\ \frac{v}{2}, 1+v-d, 1+v-w, \frac{1}{2}-v+w+d-N, \triangle(2;v+w), \\ \triangle(3;v+w), \quad -N \\ \triangle(3;1+2v-w), 1+v+N \end{matrix};1\right] \\
&\quad = \frac{(1+v)_N(1/2+v-w)_N(1+2v-2d-w)_{2N}}{(1+v-d)_N(1/2+v-d-w)_N(1+2v-w)_{2N}} \\
&\qquad \cdot {}_5F_4\left[\begin{matrix} 1+v-2w, d, w, \frac{1}{2}+2v-w-d+N, -N \\ \frac{1}{2}+v-w, w-2v+2d-2N, 1+2v-2d-w, 1+2v-w+2N \end{matrix};1\right],
\end{aligned} \tag{4.3}
$$

$$
{}_4F_3\left[\begin{matrix} d, \triangle(2;f), -N \\ \triangle(3;2f) \end{matrix};\frac{32}{27}\right]
= \frac{(f-d+1/2)_N}{(f+1/2)_N}\cdot {}_4F_3\left[\begin{matrix} \triangle(2;d), \triangle(2;-N) \\ f-d+\frac{1}{2}, f+N+\frac{1}{2}, d-f+N+\frac{1}{2} \end{matrix};4\right], \tag{4.4}
$$

$$
{}_6F_5\left[\begin{matrix} 1+v-g, \triangle(3;v), \triangle(2;-N) \\ h, \frac{3}{2}+v-h, \triangle(3;1+v-g-N) \end{matrix};1\right] = \frac{(g)_N}{(g-v)_N}\cdot {}_4F_3\left[\begin{matrix} v, h-\frac{1}{2}, 1+v-h, -N \\ g, 2h-1, 2+2v-2h \end{matrix};4\right], \tag{4.5}
$$

$$
{}_{11}F_{10}\left[\begin{matrix} a,1+\dfrac{a}{2},b,\dfrac{1}{2}+2a-b-d,d-a,1+3a-d+N,\\ \dfrac{a}{2},1+a-b,\dfrac{1}{2}+b+d-a,1+2a-d,d-2a-N,\end{matrix}\right.
$$

$$
\left.\begin{matrix} \triangle(3;d), & \triangle(2;-N)\\ \triangle(3;1+3a-d), & \triangle(2;1+2a+N)\end{matrix};-1\right]
$$

$$
=\frac{(1+2a)_N(1+3a-2d)_N}{(1+2a-d)_N(1+3a-d)_N}
\cdot {}_5F_4\left[\begin{matrix} b,b+d-a,\dfrac{1}{2}+a-b,2d-2a,-N\\ \dfrac{1}{2}+a,1+2a-2b,2d+2b-2a,2d-3a-N\end{matrix};1\right], \tag{4.6}
$$

$$
{}_4F_3\left[\begin{matrix} a,\dfrac{3a}{2}-1,\dfrac{3a}{2}-\dfrac{3}{2},-N\\ a-1,3a-2,b\end{matrix};\dfrac{4}{3}\right]
$$

$$
=\frac{(b-a)_N}{(b)_N}\cdot {}_5F_4\left[\begin{matrix} \triangle(2;a),\triangle(3;-N)\\ a,b-a,\triangle(2;1+a-b-N)\end{matrix};1\right]. \tag{4.7}
$$

Proof of (4.1). Choosing $\alpha_r=(-1/4)^r/(d)_r(3/2+3a-d)_r r!$, $u_r=1/r!$, $\delta_r=(-N)_r/(c)_r$, $t_r=(3a)_r$, and $v_r=w_r=z_r=1$ in (2.6), and using the Vandermonde and Saalschütz sum [29, Appendix, III.4,2] to simplify the γ_n and β_n, and putting these in (2.7), we obtain (4.1). □

Proof of (4.2). Choosing $\alpha_r=(a)_r(f-w-a)_r(1/4)^r/r!$, $u_r=(1/4)^r/r!$, $v_r=1/(f)_r$, $w_r=(w)_r$, $\delta_r=t_r=z_r=1$ and $p=2$ in (2.6), and using the Saalschütz and the Gauss sum [29, Appendix, III.2,3] to simplify the γ_n and β_n, and putting these in (2.7), we obtain (4.2). □

Proof of (4.3). Choosing $\alpha_r=(1+\nu-2w)_r/r!$, $u_r=1/r!$, $\delta_r=(1+\nu/2)_r(d)_r(1/2+2\nu-w-d+N)_r(-N)_r/(\nu/2)_r(1+\nu-d)_r(1/2-\nu+w+d-N)_r(1+\nu+N)_r$, $w_r=(w)_r$, $v_r=(\nu)_r$, $z_r=1/(1+2\nu-w)_r$, $t_r=1$, and $p=p'=2$ in (2.6), and using the Saalschütz and Dougall sum [29, Appendix, III.2,14] to simplify the γ_n and β_n, and putting these in (2.7), we obtain (4.3). □

Proof of (4.4). Choosing $\alpha_r=(1/4)^r/r!$, $u_r=1/r!$, $\delta_r=(d)_r(-N)_r$, $v_r=1/(f+1/2)_r$, $t_r=w_r=1$ and $p=2$ in (2.1), and using the Vandermonde sum [29, Appendix, III.4] to simplify the γ_n and β_n, and putting these in (2.2), we obtain (4.4). □

Proof of (4.5). Choosing $\alpha_r=(1/4)^r/(h)_r(3/2+\nu-h)_r r!$, $u_r=1/r!$, $\delta_r=(-N)_r$, $v_r=(\nu)_r$, $t_r=w_r=1$, and $p=2$ in (2.1), and using the Saalschütz and Vandermonde sum [29, Appendix, III.2,4] to simplify the γ_n and β_n, and putting these in (2.2), we obtain (4.5). □

Proof of (4.6). Choosing $\alpha_r = (a)_r(1 + a/2)(b)_r(1/2 + 2a - b - d)_r/(a/2)_r(1 + a - b)_r(1/2 - a + b + d)_r r!$, $u_r = 1/r!$, $\delta_r = (-N)_r/(2d - 3a - N)_r$, $v_r = (d)_r$, $t_r = (d - a)_r$, $w_r = 1/(1 + 2a)_r$, and $p = 2$ in (2.1), and using the Dougall and Vandermonde sum [29, Appendix, III.14,4] to simplify the γ_n and β_n, and putting these in (2.2), we obtain (4.6). □

Proof of (4.7). Choosing $\alpha_r = (1/27)^r/(a)_r r!$, $u_r = 1/r!$, $\delta_r = (-N)_r/(b)_r$, $t_r = (a)_r$, $v_r = w_r = 1$, and $p = 3$ in (2.1), and using the Saalschütz and Vandermonde sum [29, Appendix, III.2,4] to simplify the γ_n and β_n, and putting these in (2.2), we obtain (4.7). □

5. A note on q-analogs of (4.1) to (4.7)

As mentioned in introduction, now we will illustrate the fact about q-analogs of (4.1).

We choose

$$\alpha_r = \frac{q^{(r^2+3r)/2}(-1)^r}{(a^6q^3/d^2, d^2, q^2; q^2)_r}, \qquad u_r = \frac{q^r}{(q;q)_r}, \qquad \delta_r = \frac{(q^{-N};q)_r}{(c;q)_r}, \qquad t_r = (a^3;q)_r \tag{5.1}$$

and $v_r = w_r = z_r = 1$. Using these in (2.6), we get

$$\begin{aligned} \beta_n &= \sum_{r=0}^{n} \frac{(a^3;q)_{2r+n}(q^{-n};q)_r q^{nr+r+n}}{(a^6q^3/d^2, d^2, q^2; q^2)_r (c, q; q)_n}, \\ \gamma_n &= \sum_{r=0}^{N} \frac{(a^3;q)_{3n+r}(q^{-N};q)_{n+r} q^r}{(c;q)_{n+r}(q;q)_r}. \end{aligned} \tag{5.2}$$

Now γ_n can be simplified by the Vandermonde sum, but to simplify β_n, we will have to sum the following ${}_3\Phi_2$ q-hypergeometric series, having one parameter of base q and remaining parameters of base q^2 with a special argument, namely,

$$\sum_{r=0}^{n} \frac{(a^3q^n;q^2)_r (a^3q^{n+1};q^2)_r (q^{-n};q)_r q^{nr+r}}{(a^6q^3/d^2, d^2, q^2; q^2)_r}, \tag{5.3}$$

which is not possible from available summation theorems. Similar difficulty arises with all the other choices of α_r, δ_r, u_r, v_r, and so forth, selected in order to obtain the q-analogs of identities (4.1) to (4.7).

6. Particular cases and applications of investigated identities

(i) As $q \to 1$ in (3.1) to (3.5), one obtains corresponding new ordinary hypergeometric identities (6.1) to (6.5), noted as below:

$${}_4F_3\left[\begin{matrix} a, b, \triangle(2; -N) \\ \frac{1}{2} + a + b, c, 1 - c - N \end{matrix}; 1\right] = \frac{(a + c)_N}{(c)_N} \cdot {}_3F_2\left[\begin{matrix} a + b, 2a, -N \\ 2(a + b), a + c \end{matrix}; 1\right], \tag{6.1}$$

$$
{}_4F_3\left[\begin{matrix}\triangle(2;a),\triangle(2;-N)\\ \dfrac{1}{2}+b,\triangle(2;1+a-c-N)\end{matrix};1\right]=\frac{(c)_N}{(c-a)_N}\cdot{}_3F_2\left[\begin{matrix}a,b,-N\\ 2b,c\end{matrix};2\right], \tag{6.2}
$$

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}a,b,-N\\ c,1-c-2N\end{matrix};1\right]&=\frac{2^{2N}\big((1+a+c)/2\big)_N\big((c-a)/2\big)_N}{(c)_{2N}}\\
&\quad\cdot{}_4F_3\left[\begin{matrix}a,\triangle(2;a+b),-N\\ a+b,\dfrac{1+a+c}{2},1+\dfrac{a-c}{2}-N\end{matrix};1\right],
\end{aligned} \tag{6.3}
$$

$$
\begin{aligned}
&{}_{11}F_{10}\left[\begin{matrix}a,1+\dfrac{a}{2},2a-b,\triangle(2;c),\triangle(2;d),\\ \dfrac{a}{2},1+a-b,\triangle(2;1+2a-c),\triangle(2;1+2a-d),\\ \triangle(2;1+2a-b-c-d+N),\quad \triangle(2;-N)\\ \triangle(2;c+d+b-N),\quad \triangle(2;1+2a+N)\end{matrix};1\right]\\
&=\frac{(1+2a)_N(1+b-c)_N(1+b-d)_N(1+2a-c-d)_N}{(1+b)_N(1+2a-d)_N(1+2a-c)_N(1+b-c-d)_N}\\
&\quad\cdot{}_7F_6\left[\begin{matrix}b,1+\dfrac{b}{2},c,d,1+2a+b-c-d,\dfrac{1}{2}+b-a,-N\\ \dfrac{b}{2},1+b-c,1+b-d,c+d-2a-N,1+2b-2a,\dfrac{1}{2}+a,1+b+N\end{matrix};1\right],
\end{aligned} \tag{6.4}
$$

$$
\begin{aligned}
&{}_{11}F_{10}\left[\begin{matrix}a,1+\dfrac{a}{2},\triangle(3;b),\triangle(3;c),\triangle(3;-N)\\ \dfrac{a}{2},\triangle(3;1+3a-b),\triangle(3;1+3a-c),\triangle(3;1+3a+N)\end{matrix};1\right]\\
&=\frac{(1+3a)_N(1+3a-b-c)_N}{(1+3a-b)_N(1+3a-c)_N}\cdot{}_4F_3\left[\begin{matrix}a,b,c,-N\\ \dfrac{3a}{2},\dfrac{1+3a}{2},b+c-3a-N\end{matrix};\frac{3}{4}\right].
\end{aligned} \tag{6.5}
$$

(ii) When $b=1/\sqrt{q}$ in (3.1), the ${}_4\Phi_3(q^2)$ on the left becomes ${}_3\Phi_2$ and can be summed by q-Pfaff-Saalschütz sum [19, Appendix, II.2] to have

$$
{}_4\Phi_3\left[\begin{matrix}\dfrac{a}{\sqrt{q}},-\dfrac{a}{\sqrt{q}},a^2,q^{-N}\\ \dfrac{a^2}{q},ac,-\dfrac{aq^{1-N}}{c}\end{matrix};q,q\right]=\frac{(c^2;q)_N(c^2/q;q^2)_N}{(c^2/q,ac,-c/a;q)_N}. \tag{6.6}
$$

And when $c = b/q$ in (3.3), the ${}_4\Phi_3(q^2)$ on the right gets converted into ${}_3\Phi_2$ and can be summed by q-Pfaff-Saalschütz sum [19, Appendix, II.2] to have

$$ {}_4\Phi_3\left[\begin{matrix} a, b, -q^{-N}, q^{-N} \\ ab, \dfrac{b}{q}, \dfrac{q^{2-2N}}{b} \end{matrix}; q, q\right] = \frac{(ab, ab/q, b^2; q^2)_N}{(b, b/q, a^2b^2; q^2)_N}. \tag{6.7} $$

Equations (6.6) and (6.7) provide two new q-summation theorems for a Saalschützian ${}_4\Phi_3$. Further, by applying Sear's ${}_4\Phi_3$ transformation [19, Appendix, III.15] and Watson's ${}_8\Phi_7$ transformation [19, Appendix, III.17], on the ${}_4\Phi_3$'s of (6.6) and (6.7), one can develop a number of summation formulae.

(iii) In this section we will obtain the new double-series Rogers-Ramanujan-type identities of modulo 81, from our investigated q-hypergeometric transformation (3.5), which connect a very well-poised ${}_{12}\Phi_{11}$ of base q^3 with a Saalschützian ${}_6\Phi_5$ of base q.

Jain [22] also derived such double-series Rogers-Ramanujan-type identities of other moduli using his own investigated q-hypergeometric transformation connecting a ${}_8\Phi_7$ of base q with a ${}_4\Phi_3$ of base q^2 and the classical Bailey's transform. However, we will derive our identities on the line of Jain [22] but will make use of the q-hypergeometric transformation (3.5) and the extended Bailey's transform given in Theorem 2.1 of this paper. Incidently, the investigation of these identities matches with the intuitive feelings of Agarwal mentioned in a presidential address [1, page 7].

First, we will derive a transformation (6.8) using (3.5) and Theorem 2.1. This transformation (6.8) on specialization yields the double-series Rogers-Ramanujan-type identities of modulo 81:

$$ \begin{aligned} &(a^9q^3;q^3)_\infty \sum_{N=0}^{\infty} \sum_{k=0}^{[N/3]} \frac{(a^3q;q)_{3N}(a^3;q^3)_k(q^{-3N};q^3)_k q^{3N(N-1)+3N(1-p)+k} a^{9N}}{(q^3;q^3)_N (a^9q^3;q^3)_{2N}(a^3;q^2)_{2k}(q^{-3N}/a^3;q)_k (q^3;q^3)_k} \\ &\quad = \sum_{N=0}^{\infty} \sum_{s=0}^{p} \frac{(a^3;q^3)_N (1-a^3q^{6N})(q^{-3p};q^3)_s a^{39N+9s} q^{18Ns+3s^2/2+3s/2+81N^2/2-3N/2-9Np}}{(q^3;q^3)_N (1-a^3)(q^3;q^3)_s (-1)^{N+s}} \end{aligned} \tag{6.8} $$

Proof of (6.8). Setting $b = \omega q^{-N}$, $c = \omega^2 q^{-N}$ in (3.5), we get

$$ \begin{aligned} &\sum_{k=0}^{[N/3]} \frac{(a^3;q^3)_k (1-a^3q^{6k}) a^{27k} q^{27k^2/2-3k/2}}{(1-a^3)(a^9q^3;q^3)_{N+3k}(q^3;q^3)_{N-3k}} \\ &\quad = \frac{(a^3q;q)_{3N}}{(q^3;q^3)_N (a^9q^3;q^3)_{2N}} \sum_{k=0}^{\infty} \frac{(a^3;q^3)_k (q^{-3N};q^3)_k q^k}{(a^3;q^2)_{2k}(q^{-3N}/a^3;q)_k (q^3;q^3)_k}. \end{aligned} \tag{6.9} $$

Now in extended Bailey's transform, that is, Theorem 2.1, choosing

$$u_k = \frac{1}{(q^3;q^3)_k}, \qquad w_k = \frac{1}{(a^9q^3)_k}, \qquad \alpha_k = \frac{(a^3;q^3)_k(1-a^3q^{6k})a^{12k}q^{27k^2/2-3k/2}}{(q^3;q^3)_k(1-a^3)},$$
$$\delta_k = \frac{(x;q^3)_k(y;q^3)_k}{x^ky^k}a^{9k}q^{3k(1-p)} \tag{6.10}$$

and evaluating β_N and γ_N by using (6.9) and the following transformation formula [22, equation (3.11)]

$${}_2\Phi_1\left[\begin{matrix}a,b\\e\end{matrix};q,\frac{ec}{ab}\right] = \frac{(e/a,e/b;q)_\infty}{(e,e/ab;q)_\infty}{}_3\Phi_2\left[\begin{matrix}a,b,c\\\dfrac{abq}{e},0\end{matrix};q,q\right] \tag{6.11}$$

(where, either a, b, or c is of the form q^{-p}, p a nonnegative integer. In case only c is of the form q^{-p} then (6.11) is valid only if $|\,ec/ab\,|<1$), we get (6.8) on letting $x, y \to \infty$. □

6.1. The double series Rogers-Ramanujan type identities of modulo 81. Now (6.8) for $a = 1$, $p = 0$ yields

$$(q^3;q^3)_\infty \sum_{N,k=0}^{\infty} \frac{(q;q)_{3N+8k}q^{3N^2+18Nk+28k^2}}{(q^2;q^2)_{2k-1}(q^3;q^3)_{N+2k}(1-q^{3k})(q^3;q^3)_{2N+6k}}$$
$$= \prod_{N=1}^{\infty}(1-q^{81N-42})(1-q^{81N})(1-q^{81N-39}). \tag{6.12}$$

But (6.8), for $a = 1$, $p = 1$, gives

$$(q^3;q^3)_\infty \sum_{N,k=0}^{\infty} \frac{(q;q)_{3N+8k}q^{3N^2+18Nk+28k^2-3N-9k}}{(q^2;q^2)_{2k-1}(q^3;q^3)_{N+2k}(1-q^{3k})(q^3;q^3)_{2N+6k}}$$
$$= \prod_{N=1}^{\infty}(1-q^{81N-51})(1-q^{81N})(1-q^{81N-30})$$
$$+ \prod_{N=1}^{\infty}(1-q^{81N-48})(1-q^{81N})(1-q^{81N-33}). \tag{6.13}$$

On the other hand, (6.8), for $a = q$, $p = 0$, reduces to

$$(q^{12};q^3)_\infty \sum_{N,k=0}^{\infty} \frac{(q^4;q)_{3N+8k}q^{3N^2+18Nk+28k^2+9N+27k}}{(q^3;q^2)_{2k}(q^3;q^3)_{N+2k}(q^{12};q^3)_{2N+6k}}$$
$$= \prod_{N=1}^{\infty}(1-q^{81N-3})(1-q^{81N})(1-q^{81N-78}). \tag{6.14}$$

The Rogers-Ramanujan-type identities of modulo 27 [29, equations (7.3.1.20), (7.3.1.17)] also follow directly from our cubic q-transformation (3.5) by letting $b \to \infty$, $c \to \infty$, $N \to \infty$ in it and using Jacobi triple product identity, after setting $a = 1$ and $a = q$.

(iv) When in (3.1), we replace c by ax, it becomes (6.15) and when in (3.2), we select $c = ax$, it becomes (6.16) as given below:

$$
\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} a^2, b^2, q^{-N+1}, q^{-N} \\ a^2b^2q, a^2x^2, \dfrac{q^{2-2N}}{a^2x^2} \end{matrix}; q^2, q^2\right] \\
&\quad = \frac{(-a^2x, x; q)_N}{(a^2x^2; q^2)_N} \cdot {}_4\Phi_3\left[\begin{matrix} a^2, ab, -ab, q^{-N} \\ a^2b^2, a^2x, \dfrac{-q^{1-N}}{x} \end{matrix}; q, q\right]
\end{aligned} \tag{6.15}
$$

$$
\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} a, aq, q^{-N+1}, q^{-N} \\ b^2q, \dfrac{q^{-N+1}}{x}, \dfrac{q^{-N+2}}{x} \end{matrix}; q^2, q^2\right] \\
&\quad = \frac{(ax; q)_N}{(x; q)_N} \cdot {}_4\Phi_2\left[\begin{matrix} a, b, -b, q^{-N} \\ b^2, ax \end{matrix}; q, -xq^N\right].
\end{aligned} \tag{6.16}
$$

The results (6.15) and (6.16) provide generalizations or discrete extensions of the results of Jain [20, equation (3.7.1.3)] and [18, equation (3.7.1.5), page 88], which are the q-analogs of the quadratic transformations of Gauss for ${}_2F_1$.

The identity (4.1) is a discrete extension of Bailey's cubic transformation for a ${}_3F_2(z)$ [17, page 190, equation (2)], which follows from (4.1), on replacing N by Nz and c by $-N$ and letting $Nz \to +\infty$ through integer values of Nz with z fixed and $0 < z < 1$, and then using analytic continuation with respect to z. Further, in the same way, two of our ordinary hypergeometric identities (4.5) and (4.7) lead to two new quadratic-cubic transformations (6.17) and (6.18), respectively, as described below.

Let $v = a$, $h = b + (1/2)$, replace N by Nz and g by $-N$ in (4.5), and then letting $Nz \to +\infty$ through integer values of Nz with z fixed and $0 < z < 1$, and then using analytic continuation with respect to z, we get

$$
(1-z)^{-a}\,{}_3F_2\left[\begin{matrix} \dfrac{a}{3}, \dfrac{a+1}{3}, \dfrac{a+2}{3} \\ b + \dfrac{1}{2}, 1 + a - b \end{matrix}; \frac{27z^2}{4(1-z)^3}\right] = {}_3F_2\left[\begin{matrix} a, b, \dfrac{1}{2} + a - b \\ 2b, 1 + 2a - 2b \end{matrix}; 4z\right]. \tag{6.17}
$$

Similarly, replacing N by Nz and b by $-N$ in (4.7), and letting $Nz \to +\infty$ through integer values of Nz, and proceeding as above, we get

$$
(1-z)^{-a}\,{}_2F_1\left[\begin{matrix} \dfrac{a}{2}, \dfrac{a+1}{2} \\ a \end{matrix}; \frac{4z^3}{27(1-z)^2}\right] = {}_3F_2\left[\begin{matrix} a, \dfrac{3a}{2} - 1, \dfrac{3a-3}{2} \\ a - 1, 3a - 2 \end{matrix}; z/3\right]. \tag{6.18}
$$

Further, (6.18) may be used, with an algebraic expression for a particular ${}_2F_1$ [24, page 70, Example 10], to have an algebraic expression for a particular ${}_3F_2$ as below:

$$
{}_3F_2\left[\begin{matrix} a, 3a - 1, 3a - \dfrac{3}{2} \\ 2a - 1, 6a - 2 \end{matrix}; y\right] = \frac{1}{(1-y)\sqrt{1-4y}} \cdot \left(\frac{2}{1 - 3y + (1-y)\sqrt{1-4y}}\right)^{2a-1}. \tag{6.19}
$$

Equation (6.19) would be as useful in obtaining generating functions as the formula in [24, page 70, Example 10], see [24, page 137] and also [23, page 85–88].

7. Conclusion

In conclusion, this paper illustrates the concept of extensions of Bailey's transform and its use in obtaining hitherto undiscovered ordinary and q-hypergeometric identities with their particular cases of interests. This is not the end because these extensions will, certainly, explore many more new and useful results. For example, the study of these extensions on the line of Andrews [10] and Bressoud et al. [15] and, further, additional results from these extensions will form the subject matter of our subsequent communications.

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References

[1] R. P. Agarwal, *Special functions and their applications: keynote address*, Proceedings of National Symposium on Special Functions and Their Applications (Gorakhpur, 1986) (U. P. Singh and R. Y. Denis, eds.), University Gorakhpur, Gorakhpur, 1986, pp. 1–13.

[2] ———, *Resonance of Ramanujan's Mathematics. Vol. I*, New Age International Publishers, New Delhi, 1996.

[3] A. K. Agarwal, G. E. Andrews, and D. M. Bressoud, *The Bailey lattice*, J. Indian Math. Soc. (N.S.) **51** (1987), 57–73.

[4] G. E. Andrews, *A general theory of identities of the Rogers-Ramanujan type*, Bull. Amer. Math. Soc. **80** (1974), 1033–1052.

[5] ———, *An analytic generalization of the Rogers-Ramanujan identities for odd moduli*, Proc. Natl. Acad. Sci. USA **71** (1974), 4082–4085.

[6] ———, *Problems and prospects for basic hypergeometric functions*, Theory and Application of Special Functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975) (R. Askey, ed.), Math. Res. Center, Univ. Wisconsin, Publ. no. 35, Academic Press, New York, 1975, pp. 191–224.

[7] ———, *The theory of partitions*, Encyclopedia of Mathematics and Its Applications, vol. 2, Addison-Wesley Publishing, Massachusetts, 1976.

[8] ———, *Partitions and Durfee dissection*, Amer. J. Math. **101** (1979), no. 3, 735–742.

[9] ———, *The hard-hexagon model and Rogers-Ramanujan type identities*, Proc. Natl. Acad. Sci. USA **78** (1981), no. 9, part 1, 5290–5292.

[10] ———, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984), no. 2, 267–283.

[11] ———, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, CBMS Regional Conference Series in Mathematics, vol. 66, the American Mathematical Society, Rhode Island, 1986.

[12] W. N. Bailey, *Some identities in combinatory analysis*, Proc. London Math. Soc. (2) **49** (1947), 421–425.

[13] _____, *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **50** (1948), 1–10.
[14] D. M. Bressoud, *Some identities for terminating q-series*, Math. Proc. Cambridge Philos. Soc. **89** (1981), no. 2, 211–223.
[15] D. M. Bressoud, M. E. H. Ismail, and D. Stanton, *Change of base in Bailey pairs*, Ramanujan J. **4** (2000), no. 4, 435–453.
[16] R. Y. Denis, S. N. Singh, and S. P. Singh, *On certain transformation formulae for abnormal q-series*, South East Asian J. Math. Math. Sci. **1** (2003), no. 3, 7–19.
[17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions. Vols. I, II*, McGraw-Hill, New York, 1953.
[18] H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, 1983.
[19] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Encyclopedia of Mathematics and Its Applications, vol. 35, Cambridge University Press, Cambridge, 1990.
[20] V. K. Jain, *A study of certain hypergeometric identities*, Ph.D. thesis, University of Roorkee, Roorkee.
[21] _____, *Some expansions involving basic hypergeometric functions of two variables*, Pacific J. Math. **91** (1980), no. 2, 349–361.
[22] _____, *Certain transformations of basic hypergeometric series and their applications*, Pacific J. Math. **101** (1982), no. 2, 333–349.
[23] E. B. McBride, *Obtaining Generating Functions*, Springer Tracts in Natural Philosophy, vol. 21, Springer, New York, 1971.
[24] E. D. Rainville, *Special Functions*, Macmillan, New York, 1967.
[25] U. B. Singh, *A note on a transformation of Bailey*, Quart. J. Math. Oxford Ser. (2) **45** (1994), no. 177, 111–116.
[26] S. P. Singh, *Certain transformation formulae for q-series*, Indian J. Pure Appl. Math. **31** (2000), no. 10, 1369–1377.
[27] L. J. Slater, *A new proof of Rogers transformations of infinite series*, Proc. London Math. Soc. (2) **53** (1951), 460–475.
[28] _____, *Further identies of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147–167.
[29] _____, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.
[30] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, 1984.
[31] A. Verma and V. K. Jain, *Transformations between basic hypergeometric series on different bases and identities of Rogers-Ramanujan type*, J. Math. Anal. Appl. **76** (1980), no. 1, 230–269.
[32] _____, *Transformations of nonterminating basic hypergeometric series, their contour integrals and applications to Rogers-Ramanujan identities*, J. Math. Anal. Appl. **87** (1982), no. 1, 9–44.

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