GLEASON-KAHANE-ŻELAZKO THEOREM FOR SPECTRALLY BOUNDED ALGEBRA

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We prove by elementary methods the following generalization of a theorem due to Gleason, Kahane, and Żelazko. Let A be a real algebra with unit 1 such that the spectrum of every element in A is bounded and let $\phi : A \to \mathbb{C}$ be a linear map such that $\phi(1) = 1$ and $(\phi(a))^2 + (\phi(b))^2 \neq 0$ for all a, b in A satisfying $ab = ba$ and $a^2 + b^2$ is invertible. Then $\phi(ab) = \phi(a)\phi(b)$ for all a, b in A. Similar results are proved for real and complex algebras using Ransford's concept of generalized spectrum. With these ideas, a sufficient condition for a linear transformation to be multiplicative is established in terms of generalized spectrum.

1. Introduction

Let A be a real algebra with unit 1 and let $\phi : A \to \mathbb{C}$ be a linear transformation with $\phi(1) = 1$. When is ϕ multiplicative? That is, when is $\phi(ab) = \phi(a)\phi(b)$ for all a, b in A? This question was first answered for the case of a complex Banach algebra by Gleason [3], Kahane and Żelazko [6]. Their result, now known as the Gleason-Kahane-Żelazko theorem, states that, if $\phi(a) \neq 0$ for every invertible element a in A (or equivalently $\phi(a)$ lies in the spectrum of a for every a in A), then ϕ is multiplicative. Subsequently several generalizations of this result were published by many authors. These include

(i) real Banach algebra—Kulkarni [7],

(ii) complex spectrally bounded algebra—Roitman and Sternfeld [10].

The articles by Jarosz [4, 5] and Sourour [11] contain surveys of many of these results.

The aim of the present article is two-fold. First we extend this result to a real spectrally bounded algebra (Theorem 2.9), that is, the algebra in which the spectrum of each element is bounded (Definition 2.6). The result says ϕ is multiplicative if and only if $(\phi(a))^2 + (\phi(b))^2 \neq 0$ for all a, b in A such that $ab = ba$ and $a^2 + b^2$ is invertible. The class of real spectrally bounded algebras includes all the above-mentioned algebras. All these characterizations including the ones to be discussed in this paper are mainly in terms of the spectrum.

Our second aim is to give simple proofs. The classical proofs make use of the tools from the complex function theory, in particular Hadamard's theorem. Our proof uses

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the elementary properties of polynomials, namely, relations between roots and coefficients. The essential ideas are in Lemma 2.5. Similar ideas were used by Roitman and Sternfeld in [10] (see also [8, Theorem 2.4.3]).

In Sections 3 and 4, we attempt to relate these ideas to Ransford's generalized spectrum [9]. In Section 3, it is proved that if for each x in a complex algebra, $\phi(x)$ lies in the generalized spectrum of x, then ϕ is multiplicative. A statement of this theorem was published by Catalin Badea in [1], where it was mentioned that the proof will be published elsewhere, but the proof was not published anywhere. Here is the first instance where a proof is given for that theorem.

In Section 4, the result in Section 3 is extended to a real algebra E in terms of Ransford's spectrum. We have also extended the concept of Ransford's spectrum to the real case. It is shown that if $(\phi(a))^2+(\phi(b))^2 \neq 0$ for all a,b in E such that $ab=ba$ and a^2+b^2 in $\Omega_{\mathbb{R}}$, then ϕ is multiplicative (Theorem 4.8). Examples are given to show that this condition is not necessary.

In the last section, using the sufficient conditions obtained in Sections 3 and 4, we give a sufficient condition for a linear transformation between spectrally bounded, (complex or real) algebras, to be multiplicative.

2. Spectrally bounded real algebra

2.1. Notation. Let A be an algebra with the unit 1. An algebra element $\lambda \cdot 1$ (product of λ and one), where $\lambda \in \mathbb{C}$, will be denoted just as λ. Let $\operatorname{Inv}(A)$ and $\operatorname{Sing}(A)$ denote the set of invertible and singular (noninvertible) elements in A, respectively. For an element a in A the spectrum is denoted by $\operatorname{Sp}(a,A)$. If A is a complex algebra,

$$\operatorname{Sp}(a,A) := \{\lambda \in \mathbb{C} : \lambda - a \in \operatorname{Sing}(A)\}. \tag{2.1}$$

If A is a real algebra,

$$\operatorname{Sp}(a,A) := \{s+it \in \mathbb{C} : (s-a)^2+t^2 \in \operatorname{Sing}(A)\}. \tag{2.2}$$

2.2. Complexification. Complexification of a real algebra A, denoted by $A_{\mathbb{C}}$, is the set $A \times A$ with addition, scalar multiplication, and multiplication are defined in the following way. For every (a,b), (c,d) in $A_{\mathbb{C}}$ and $\alpha+i\beta$ in $\mathbb{C}$,

$$\begin{gathered}(a,b)+(c,d)=(a+c,b+d),\\ (\alpha+i\beta)(a,b)=(\alpha a-\beta b,\alpha b+\beta a),\\ (a,b)(c,d)=(ac-bd,ad+bc).\end{gathered} \tag{2.3}$$

With these operations $A_{\mathbb{C}}$ becomes a complex algebra. Let us recall some results in [2]. These results will be used to prove a lemma.

Proposition 2.1. $a \in \operatorname{Inv}(A)$ *if and only if* $(a,0) \in \operatorname{Inv}(A_{\mathbb{C}})$.

Proposition 2.2. $(a,b) \in \operatorname{Inv}(A_{\mathbb{C}})$ *if and only if* $(a,-b) \in \operatorname{Inv}(A_{\mathbb{C}})$.

Proposition 2.3. $\operatorname{Sp}((a,0),A_{\mathbb{C}}) = \operatorname{Sp}(a,A)$.

Proof.

$$\begin{aligned} s+it\in \mathrm{Sp}(a,A) &\Longleftrightarrow (s-a)^2+t^2\in \mathrm{Sing}(A)\\ &\Longleftrightarrow ((s-a)^2+t^2,0)\in \mathrm{Sing}(A_{\mathbb{C}}) \quad \text{(using Proposition 2.1)}\\ &\Longleftrightarrow (s-a,t)(s-a,-t)\in \mathrm{Sing}(A_{\mathbb{C}})\\ &\Longleftrightarrow (s-a,t)\in \mathrm{Sing}(A_{\mathbb{C}}) \quad \text{(using Proposition 2.2)}\\ &\Longleftrightarrow (s+it)(1,0)-(a,0)\in \mathrm{Sing}(A_{\mathbb{C}})\\ &\Longleftrightarrow s+it\in \mathrm{Sp}((a,0),A_{\mathbb{C}}). \end{aligned} \tag{2.4}$$

□

The following lemma will be used repeatedly.

Lemma 2.4. *Let A be a real algebra and let $\phi : A\to\mathbb{C}$ be a real linear map with $\phi(1)=1$. Assume for all a,b in A, satisfying $ab=ba$ and a^2+b^2 in* $\mathrm{Inv}(A)$,

$$(\phi(a)^2)+(\phi(b))^2\neq 0. \tag{2.5}$$

Now define $F : A_{\mathbb{C}}\to\mathbb{C}$ by

$$F(a,b)=\phi(a)+i\phi(b). \tag{2.6}$$

Then F is a complex linear function. If $F(a,b)=0$ for some $a,b\in A$ with $ab=ba$, then (a,b) is not invertible in $A_{\mathbb{C}}$.

Proof.

$$\begin{aligned} F(a,b)=0 &\Longrightarrow \phi(a)+i\phi(b)=0\\ &\Longrightarrow (\phi(a))^2+(\phi(b))^2=0\\ &\Longrightarrow a^2+b^2\in \mathrm{Sing}(A)\\ &\Longrightarrow (a^2+b^2,0)\in \mathrm{Sing}(A_{\mathbb{C}}) \quad \text{(using Proposition 2.1)}\\ &\Longrightarrow (a,b)(a,-b)\in \mathrm{Sing}(A_{\mathbb{C}}) \quad (\because ab=ba)\\ &\Longrightarrow (a,b)\in \mathrm{Sing}(A_{\mathbb{C}}) \quad \text{(using Proposition 2.2).} \end{aligned} \tag{2.7}$$

□

Lemma 2.5. *Let A be a complex algebra with unit 1, let $\psi : A\to\mathbb{C}$ be a complex linear functional with $\psi(1)=1$. Fix $a\in A$ and define $P:\mathbb{C}\to\mathbb{C}$ by*

$$P(z)=\psi([z-a]^n). \tag{2.8}$$

Let λ_j, $j=1,\dots,n$, be the roots of the polynomial P. Then

$$\psi(a)^2-\psi(a^2)=\frac{\sum_{j=1}^n\lambda_j^2}{n^2}-\frac{1}{n}\psi(a^2). \tag{2.9}$$

Proof. As λ_j, $j=1,\dots,n$, are the roots,

$$P(z)=\prod_{j=1}^n(z-\lambda_j). \tag{2.10}$$

On the other hand by expanding P,

$$\begin{aligned} P(z) &= \psi([z-a]^n) \\ &= \psi\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}z^{n-k}a^k\right) \\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}z^{n-k}\psi(a^k), \end{aligned} \tag{2.11}$$

and comparing the coefficients of like powers of z, we get

$$\sum_{j=1}^{n}\lambda_j = n\psi(a), \qquad \sum_{j<k}\lambda_j\lambda_k = \frac{n(n-1)}{2}\psi(a^2). \tag{2.12}$$

On substituting these values in the equation

$$\left(\sum_{j=1}^{n}\lambda_j\right)^2 = \sum_{j=1}^{n}\lambda_j^2 + 2\sum_{j<k}\lambda_j\lambda_k, \tag{2.13}$$

we get

$$n^2\psi(a)^2 = \sum_{j=1}^{n}\lambda_j^2 + 2\frac{n(n-1)}{2}\psi(a^2). \tag{2.14}$$

Hence

$$\psi(a)^2 - \psi(a^2) = \frac{\sum_{j=1}^{n}\lambda_j^2}{n^2} + \frac{(-1)}{n}\psi(a^2). \tag{2.15}$$

□

Definition 2.6 (spectrally bounded algebra). An algebra A is called *spectrally bounded* if the spectrum of every element in A is bounded.

This means for every a in A, there exist $M_a > 0$ such that $|\lambda| \le M_a$ whenever $\lambda \in \mathrm{Sp}(a,A)$. In other words, if

$$r(a) := \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a,A)\} \tag{2.16}$$

is the *spectral radius*, then, $r(a) \le M_a$. This is a property which we will be using to establish the result.

Definition 2.7 (spectral algebra). A norm which dominates the spectral radius is called a *spectral norm*. A *spectral algebra* is an algebra on which a spectral norm can be defined.

In view of the spectral radius formula, every Banach algebra is a spectral algebra. See [8] for examples of spectral algebras that are not Banach algebras. Also, every spectral algebra is a spectrally bounded algebra. The next example shows that the converse is not true.

Example 2.8. Let $\mathbb{C}(z)$ denote the set of all complex rational functions. Consider the algebra $\mathbb{C} \oplus \mathbb{C}(z)$. Then for an element (λ, f) in the algebra,

$$\operatorname{Sp}(\lambda, f) = \begin{cases} \{\lambda, \mu\} & \text{if } f \equiv \mu, \\ \{\lambda\} & \text{if } f \text{ is not a constant.} \end{cases} \tag{2.17}$$

Hence the algebra is spectrally bounded. But it is not a spectral algebra because in a commutative spectral algebra the spectral radius is subadditive and submultiplicative by [8, Theorem 2.4.11]. Here the spectral radius is neither subadditive nor submultiplicative by the following inequalities:

$$\begin{aligned} r(0,1) &= 1 > 0 + 0 = r(0,z) + r(0,1-z), \\ r(0,1) &= 1 > 0 \cdot 0 = r(0,z) \cdot r\left(0, \frac{1}{z}\right). \end{aligned} \tag{2.18}$$

Theorem 2.9 (compare [8, Theorem 2.4.3]). *Let A be a real unital algebra. Let $\phi : A \to \mathbb{C}$ be linear and unital (i.e., $\phi(1) = 1$). The first four conditions below are equivalent and imply the last two conditions. If A is a spectrally bounded algebra, then all six conditions are equivalent:*

(1) $\phi(a) = i\alpha$ *implies* $\phi(a^2) = -\alpha^2$ *for all* $a \in A$, $\alpha \in \mathbb{R}$;
(2) $\phi(a^2) = (\phi(a))^2$ *for all* $a \in A$;
(3) $\phi(a) = i\alpha$ *implies* $\phi(ab) = i\alpha\phi(b)$ *for all* $a, b \in A$, $\alpha \in \mathbb{R}$;
(4) $\phi(ab) = \phi(a)\phi(b)$ *for* $a, b \in A$;
(5) $(\phi(a))^2 + (\phi(b))^2 \in \operatorname{Sp}(a^2 + b^2, A)$ *for* $a, b \in A$ *such that* $ab = ba$;
(6) $(\phi(a))^2 + (\phi(b))^2 \neq 0$ *for* $a, b \in A$ *such that* $ab = ba$ *and* $a^2 + b^2$ *is invertible.*

Proof. (1)⇒(2) If $\phi(a) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, then $\phi(a - \alpha) = i\beta$. Using (1) we can get

$$\begin{aligned} -\beta^2 &= \phi[(a-\alpha)^2] = \phi[a^2 - 2\alpha a + \alpha^2] \\ &= \phi(a^2) - 2\alpha\phi(a) + \alpha^2 = \phi(a^2) - 2\alpha(\alpha + i\beta) + \alpha^2. \end{aligned} \tag{2.19}$$

Thus

$$\phi(a^2) = \alpha^2 - \beta^2 + i2\alpha\beta = (\alpha + i\beta)^2 = (\phi(a))^2. \tag{2.20}$$

(2)⇒(3) This is [7, Lemma 1].

(3)⇒(4) If $\phi(a) = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$, then $\phi(a - \alpha) = i\beta$. Now using (3), we get

$$\phi((a - \alpha)b) = i\beta\phi(b), \tag{2.21}$$

which implies $\phi(ab) = (\alpha + i\beta)\phi(b) = \phi(a)\phi(b)$.

(4)⇒(1) The implication is trivial. This shows that the first four conditions are equivalent.

(4)⇒(5) Suppose $(\phi(a))^2 + (\phi(b))^2 = s + it \notin \operatorname{Sp}(a^2 + b^2, A)$, $s, t \in \mathbb{R}$, that is, $(a^2 + b^2 - s)^2 + t^2 \in \operatorname{Inv}(A)$. Then there exist $c \in A$ such that $c((a^2 + b^2 - s)^2 + t^2) = 1$. Applying ϕ

on both sides of this equation, we get a contradiction as follows:

$$1 = \phi(1) = \phi\big(c\big((a^2+b^2-s)^2+t^2\big)\big) = \phi(c)\big(\big[(\phi(a))^2+(\phi(b))^2-s\big]^2+t^2\big) = 0. \quad (2.22)$$

(5)⇒(6) As a^2+b^2 is invertible, $0 \notin \mathrm{Sp}(a^2+b^2,A)$. But $(\phi(a))^2+(\phi(b))^2 \in \mathrm{Sp}(a^2+b^2,A)$, so $(\phi(a))^2+(\phi(b))^2 \neq 0$.

(6)⇒(2) The implication holds for a spectrally bounded algebra A.

Assume A is spectrally bounded. Fix $a \in A$, $n \in \mathbb{N}$ and define $P : \mathbb{C} \to \mathbb{C}$ as

$$P(z) = F\Big(\big[z(1,0)-(a,0)\big]^n\Big), \quad (2.23)$$

where F is as in Lemma 2.4. Consider the roots λ_j for $1 \le j \le n$ of the polynomial P, then,

$$P(\lambda_j) = 0, \quad (2.24)$$

that is,

$$F\Big(\big[\lambda_j(1,0)-(a,0)\big]^n\Big) = 0. \quad (2.25)$$

If we write $[\lambda_j(1,0)-(a,0)]^n = (c,d)$, then $cd = dc$, so $[\lambda_j(1,0)-(a,0)]^n$ is not invertible in $A_{\mathbb{C}}$, by Lemma 2.4. Hence $\lambda_j(1,0)-(a,0)$ is also not invertible. That is, $\lambda_j \in \mathrm{Sp}((a,0),A_{\mathbb{C}})$, which is equivalent to $\lambda_j \in \mathrm{Sp}(a,A)$ for $1 \le j \le n$, by Proposition 2.3. Also by Lemma 2.4, we get

$$F(a,0)^2 - F\big((a,0)^2\big) = \frac{\sum_{j=1}^n \lambda_j^2}{n^2} + \frac{(-1)}{n}F\big((a,0)^2\big). \quad (2.26)$$

Hence

$$\big(\phi(a)\big)^2 - \phi(a^2) = \frac{\sum_{j=1}^n \lambda_j^2}{n^2} + \frac{(-1)}{n}\phi(a^2). \quad (2.27)$$

Since $n \in \mathbb{N}$ is arbitrary and A is spectrally bounded, letting $n \to \infty$ and noting $|\sum_{j=1}^n \lambda_j^2| \le n{M_a}^2$ gives

$$\big(\phi(a)\big)^2 = \phi(a^2). \quad (2.28)$$

□

The above proof is along the lines of the proof of [8, Theorem 2.4.3]. Next we show that [8, Theorem 2.4.3] for complex spectrally bounded algebras follows from our Theorem 2.9. In [8], this theorem is stated for complex spectral algebras. But the proof given there works also for spectrally bounded algebras.

Corollary 2.10. *Let A be a complex unital algebra and let $\phi : A \to \mathbb{C}$ be complex linear and unital. Then the first four conditions are equivalent and imply the last two equivalent conditions. If A is a spectrally bounded algebra, then all six conditions are equivalent:*

(1) $\phi(a) = 0$ *implies* $\phi(a^2) = 0$ *for all* $a \in A$;

(2) $\phi(a^2) = (\phi(a))^2$ *for all* $a \in A$;

(3) $\phi(a) = 0$ *implies* $\phi(ab) = 0$ *for all* $a, b \in A$*;*
(4) $\phi(ab) = \phi(a)\phi(b)$ *for all* $a, b \in A$*;*
(5) *for each* $a \in A$*,* $\phi(a) \in \mathrm{Sp}(a)$*;*
(6) $\phi(a) \neq 0$ *for all invertible* a *in* A.

Proof. (1)⇒(2) Let $a \in A$, $\phi(a) = i\alpha$ for some $\alpha \in \mathbb{R}$. Since ϕ is complex linear, we have $\phi(a - i\alpha) = 0$. Hence $0 = \phi((a - i\alpha)^2) = \phi(a^2) - 2i\alpha\phi(a) - \alpha^2$. This implies $\phi(a^2) = -\alpha^2$. Now (2) follows by Theorem 2.9.

(2)⇒(3) The implication follows from Theorem 2.9.

(3)⇒(4) Let $a \in A$ and $\phi(a) = i\alpha$ for some $\alpha \in \mathbb{R}$. Then $\phi(a - i\alpha) = 0$. Hence by (3) $\phi(ab) = i\alpha\phi(b)$ for all $b \in A$. This implies (4) by Theorem 2.9.

(4) obviously implies (1). This establishes equivalence of the first four statements.

(4)⇒(5) Suppose $a \in A$ is invertible. Then $1 = \phi(1) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$. This shows that $\phi(a)$ can not be zero.

(5)⇒(6) Let $a \in A$ be invertible. Then $0 \notin \mathrm{Sp}(a)$. Since $\phi(a) \in \mathrm{Sp}(a)$, $\phi(a) \neq 0$.

(6)⇒(5) For each $a \in A$, $\phi(a - \phi(a)) = 0$. Hence by (6), $a - \phi(a)$ is not invertible, thus $\phi(a) \in \mathrm{Sp}(a)$.

(6)⇒(2) The implication holds when A is spectrally bounded algebra.

Let $a, b \in A$ be such that $ab = ba$ and $a^2 + b^2$ is invertible. Then, since $a^2 + b^2 = (a + ib)(a - ib)$, both $a + ib$ and $a - ib$ are invertible. Now, by (6), $0 \neq \phi(a + ib) = \phi(a) + i\phi(b)$ and $0 \neq \phi(a - ib) = \phi(a) - i\phi(b)$. Hence $(\phi(a))^2 + (\phi(b))^2 = (\phi(a) + i\phi(b))(\phi(a) - i\phi(b)) \neq 0$. Now the conclusion follows from Theorem 2.9. □

The following example shows that the condition (1) of Corollary 2.10 does not imply condition (2) when A is a real algebra.

Define $\phi : \mathbb{C} \to \mathbb{C}$ as $\phi(x + iy) = x - y + iy$. Then ϕ is real linear and $\phi(1) = 1$. If $\phi(x + iy) = 0$, then $x = y = 0$, so $\phi[(x + iy)^2] = 0$. But ϕ is not Jordan as we can see $[\phi(i)]^2 = -2i$ whereas $\phi(i^2) = -1$. Hence the condition (1) in Theorem 2.9 cannot be replaced by the condition $\phi(a) = 0$ implies $\phi(a^2) = 0$. However, if A is a complex algebra and ϕ is a complex linear function, then condition (1) is equivalent to $\phi(a) = 0$ implies $\phi(a^2) = 0$ for a in A.

3. Ransford spectrum in a complex algebra

Ransford extended the concept of spectrum for a general complex normed linear space in [9] by replacing the set of all invertible elements with a set, denoted as Ω, satisfying some properties as follows. Let X be a complex linear space and 1 a fixed nonzero element in X. Let Ω be a subset of X such that

(1) $0 \notin \Omega$,
(2) $1 \in \Omega$,
(3) $\mathbb{C}^*\Omega \subseteq \Omega$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Then, for every $x \in X$, Ransford's Ω spectrum of X is given by

$$\mathrm{Sp}^{\Omega}(x) := \{\lambda \in \mathbb{C} : x - \lambda \notin \Omega\}. \tag{3.1}$$

It is proved in [9] that if X is a normed linear space and Ω an open subset of X, then

$\mathrm{Sp}^{\Omega}(x)$ is bounded for every $x \in X$. That is, if $\lambda \in \mathrm{Sp}^{\Omega}(x)$ then $|\lambda| \leq M_x$ for some $M_x > 0$. In fact, it is proved in [9] that $\mathrm{Sp}^{\Omega}(x)$ is a nonempty compact subset of $\mathbb{C}$ for every x in X. He also proved an analog of the spectral radius formula using a property called pseudoconvexity. When X is an algebra, we assume another property for the set Ω in terms of multiplication as follows.

(4) There is an increasing sequence $\{n_j\}$ (i.e., $n_1 < n_2 < n_3 \cdots$ where $n_j \in \mathbb{N}$ for $j = 1, 2, 3, \dots$) such that

$$x \in \Omega \Longrightarrow x^{n_j} \in \Omega \tag{3.2}$$

holds true for all $j \in \mathbb{N}$.

The statement of the following theorem, with slight modifications, was given in [1] but the proof is not published anywhere.

Theorem 3.1. *Let X be a complex algebra with unit 1 and let Ω be a subset of X which satisfies* (1), (2), (3), (4), *and* $\mathrm{Sp}^{\Omega}(x)$ *is bounded for every $x \in X$. Let $\phi : X \to \mathbb{C}$ be a linear functional satisfying $\phi(1) = 1$. Then first two of the following conditions are equivalent and imply the third:*

(1) $\phi(x) \neq 0$ *for all* $x \in \Omega$,

(2) $\phi(x) \in \mathrm{Sp}^{\Omega}(x)$ *for all* $x \in X$,

(3) $\phi(ab) = \phi(a)\phi(b)$ *for all* $a, b \in X$.

Proof. Suppose (1) holds. Then for $x \in X$, $\phi(x - \phi(x)) = 0$. Hence $x - \phi(x) \notin \Omega$, that is, $\phi(x) \in \mathrm{Sp}^{\Omega}(x)$.

Conversely, suppose (2) holds and let $x \in \Omega$. Then $0 \notin \mathrm{Sp}^{\Omega}(x)$. On the other hand, $x \in \mathrm{Sp}^{\Omega}(x)$. Hence $\phi(x) \neq 0$. This shows (1) and (2) are equivalent.

Next we prove that (1) implies (3). Fix $x \in X$ and $n_j \in \mathbb{N}$. Define $P : \mathbb{C} \to \mathbb{C}$ as follows:

$$P(z) = \phi\left([z - x]^{n_j}\right). \tag{3.3}$$

Consider the roots λ_i for $1 \leq i \leq n_j$ of the polynomial P. These roots satisfy the equation

$$P(\lambda_i) = 0, \tag{3.4}$$

that is,

$$\phi\left(\left[\lambda_i - x\right]^{n_j}\right) = 0. \tag{3.5}$$

In view of (1), this implies that $[\lambda_i - x]^{n_j}$ is not in Ω. Hence $(\lambda_i - x)$ is also not in Ω. So $\lambda_i \in \mathrm{Sp}^{\Omega}(x)$ for $1 \leq i \leq n_j$ by definition of spectrum.

Also by Lemma 2.5 we get

$$\phi(x)^2 - \phi(x^2) = \frac{\sum_{i=1}^{n_j} \lambda_i^2}{{n_j}^2} - \frac{1}{n_j}\phi(x^2). \tag{3.6}$$

Since $n_j \in \mathbb{N}$ is arbitrary and spectrum is bounded, allowing $n_j \to \infty$ and noting $|\sum_{i=1}^{n_j} \lambda_i^2| \leq n_j M_x{}^2$ gives

$$\phi(x)^2 = \phi(x^2). \tag{3.7}$$

Now ϕ is multiplicative by Theorem 2.9. □

The following example shows that the third condition in the above theorem does not imply any of the first two equivalent conditions.

Example 3.2. Consider $X = \mathbb{C}^2$ with coordinatewise multiplication, then $(1,1)$ is the unit element. Let

$$\Omega = \{(z_1, z_2) : z_1 \neq 0\}, \tag{3.8}$$

then Ω is an open set satisfying the conditions of hypothesis. Define $\phi : X \to \mathbb{C}$ by $\phi(z_1, z_2) = z_2$. Then ϕ is multiplicative. But $(1,0) \in \Omega$ and $\phi(1,0) = 0 \notin \mathrm{Sp}^{\Omega}(1,0) = \{1\}$.

4. Ransford spectrum in a real algebra

In this section, we extend the ideas in Section 3 to the case of a real algebra. For this, first we need to define Ransford's spectrum in this case.

Definition 4.1. Let E be a real algebra with unit 1. Let $\Omega_{\mathbb{R}}$ be a subset of E that satisfies

(1) $0 \notin \Omega_{\mathbb{R}}$,

(2) $1 \in \Omega_{\mathbb{R}}$,

(3) $\mathbb{R}^* \Omega_{\mathbb{R}} \subseteq \Omega_{\mathbb{R}}$ where $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$.

For every $x \in E$, spectrum of x is defined as

$$\mathrm{Sp}^{\Omega_{\mathbb{R}}}(x) := \{s + it \in \mathbb{C} : (x - s)^2 + t^2 \notin \Omega_{\mathbb{R}}\}. \tag{4.1}$$

(4) for a certain increasing sequence $n_1, n_2, n_3, \ldots$ (i.e., $n_1 < n_2 < n_3 \cdots$) where $n_j \in \mathbb{N}$ for $j = 1, 2, 3, \ldots$,

$$x \in \Omega_{\mathbb{R}} \Longrightarrow x^{n_j} \in \Omega_{\mathbb{R}}. \tag{4.2}$$

Example 4.2. In $\mathbb{R}$ with usual multiplication, the set $\mathbb{R}^*$ satisfies all conditions with a sequence $1, 2, 3, \ldots$.

Example 4.3. In $\mathbb{R}^2$ with coordinatewise multiplication, $\mathbb{R}^2 \setminus \{0\}$ satisfies all conditions with a sequence $1, 2, 3, \ldots$.

Consider the complexification $E_{\mathbb{C}}$ of E and a subset $\Omega_{\mathbb{C}}$ of $E_{\mathbb{C}}$ defined by

$$\Omega_{\mathbb{C}} := \{(a, b) \in E \times E : a^2 + b^2 \in \Omega_{\mathbb{R}}\}. \tag{4.3}$$

Then $\Omega_{\mathbb{C}}$ satisfies the following conditions:

(1) $(0,0) \notin \Omega_{\mathbb{C}}$,

(2) $(1,0) \in \Omega_{\mathbb{C}}$,

(3) $\mathbb{C}^*\Omega_{\mathbb{C}} \subseteq \Omega_{\mathbb{C}}$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$,

(4) $x \in \Omega_{\mathbb{C}} \Rightarrow x^{n_j} \in \Omega_{\mathbb{C}}$ for the same increasing sequence $n_1 < n_2 < n_3 \cdots$ where $n_j \in \mathbb{N}$ for $j = 1,2,3,\ldots$.

The following propositions and lemma are general in the sense that $\text{Inv}(A)$ and $\text{Inv}(A_{\mathbb{C}})$ in Propositions 2.1, 2.2, and 2.3, and Lemma 2.4 are replaced by $\Omega_{\mathbb{R}}$, $\Omega_{\mathbb{C}}$. But proofs are similar.

Proposition 4.4. $1 \in \Omega_{\mathbb{R}}$ *if and only if* $(1,0) \in \Omega_{\mathbb{C}}$.

Proposition 4.5. $(a,b) \in \Omega_{\mathbb{C}}$ *if and only if* $(a,-b) \in \Omega_{\mathbb{C}}$.

Proposition 4.6. $\text{Sp}^{\Omega_{\mathbb{C}}}(a,0) = \text{Sp}^{\Omega_{\mathbb{R}}}(a)$.

Lemma 4.7. *Let* $\phi : E \to \mathbb{C}$ *be real linear and unital. Define* $F : E \times E \to \mathbb{C}$ *as*

$$F(a,b) = \phi(a) + i\phi(b). \tag{4.4}$$

Then F *is complex linear. Assume for all* a,b *in* E*, satisfying* $ab = ba$ *and* $a^2 + b^2$ *in* $\Omega_{\mathbb{R}}$,

$$(\phi(a))^2 + (\phi(b))^2 \neq 0. \tag{4.5}$$

If $F(a,b) = 0$*, then* $(a,b) \notin \Omega_{\mathbb{C}}$.

Theorem 4.8. *Let* E *and* $\Omega_{\mathbb{R}}$ *be as defined in Definition 4.1 and* $\text{Sp}^{\Omega_{\mathbb{R}}}(a)$ *be bounded for every* $a \in E$*. Let* $\phi : E \to \mathbb{C}$ *be real linear and unital. Suppose further that*

$$(\phi(a))^2 + (\phi(b))^2 \neq 0 \tag{4.6}$$

for all $a,b \in E$ *such that* $ab = ba$ *and* $a^2 + b^2 \in \Omega_{\mathbb{R}}$*, then* ϕ *is multiplicative.*

Proof. Fix $a \in E$ and $n_j \in \mathbb{N}$. Define $P : \mathbb{C} \to \mathbb{C}$ as follows:

$$P(z) = F\left(\left[z(1,0) - (a,0)\right]^{n_j}\right). \tag{4.7}$$

Consider the roots λ_i for $1 \le i \le n_j$ of the polynomial P. The equation

$$P(\lambda_i) = 0, \tag{4.8}$$

that is,

$$F\left(\left[\lambda_i(1,0) - (a,0)\right]^{n_j}\right) = 0 \tag{4.9}$$

implies that $[\lambda_i(1,0) - (a,0)]^{n_j} \notin \Omega_{\mathbb{C}}$ by Lemma 4.7. Hence $\lambda_i(1,0) - (a,0)$ is also not in $\Omega_{\mathbb{C}}$ by property (5) in definition. That is, $\lambda_i \in \text{Sp}^{\Omega_{\mathbb{C}}}(a,0)$, which is equivalent to $\lambda_i \in \text{Sp}^{\Omega_{\mathbb{R}}}(a)$ for $1 \le i \le n_j$ by Proposition 4.6. Also by Lemma 2.5, we get

$$F((a,0))^2 - F((a,0)^2) = \frac{\sum_{i=1}^{n_j} \lambda_i^2}{{n_j}^2} - \frac{1}{n_j} F((a,0)^2). \tag{4.10}$$

Hence

$$(\phi(a))^2 - \phi(a^2) = \frac{\sum_{i=1}^{n_j} \lambda_i^2}{n_j^2} - \frac{1}{n_j}\phi(a^2). \tag{4.11}$$

Since $n_j \in \mathbb{N}$ is arbitrary and $\mathrm{Sp}^{\Omega_{\mathbb{R}}}(a)$ is bounded for every a in E, letting $n_j \to \infty$ and noting $|\sum_{i=1}^{n_j} \lambda_i^2| \le n_j M_a^2$ gives

$$(\phi(a))^2 = \phi(a^2). \tag{4.12}$$

Now the conclusion follows by Theorem 2.9. □

The following example shows that the condition $(\phi(a))^2 + (\phi(b))^2 \neq 0$ for all $a, b \in E$ such that $ab = ba$ and $a^2 + b^2 \in \Omega_{\mathbb{R}}$, which is a sufficient condition for a function to be multiplicative, is not necessary.

Example 4.9. Consider $E = \mathbb{R}^2$ with coordinatewise multiplication, then $(1,1)$ is the unit element. Let

$$\Omega_{\mathbb{R}} = \{(x_1, x_2) : x_1 \neq 0\}, \tag{4.13}$$

then Ω satisfies all the conditions of hypothesis. Define $\phi : A \to \mathbb{C}$ by $\phi(x_1, x_2) = x_2$. Then ϕ is multiplicative but not satisfying the condition. To see this, take $a = (1,0)$, $b = (0,0)$. Then $ab = ba$ and $a^2 + b^2 = (1,0) \in \Omega_{\mathbb{R}}$ but $(\phi(a))^2 + (\phi(b))^2 = 0$.

5. Operators

In this section, we give sufficient conditions for a linear transformation, between spectrally bounded algebras, to be multiplicative.

Let X and Ω be as in Theorem 3.1. Ransford defined Ω-radical, in [9], as

$$\mathrm{Rad}^{\Omega}(X) := \{a \in X : a + \Omega = \Omega\}. \tag{5.1}$$

If $\mathrm{Rad}^{\Omega}(X) = \{0\}$, then X is said to be Ω semisimple. Zalduendo [12] defined the subsets M_Ω and $\tilde{\Omega}$ as follows:

$$\begin{aligned} M_\Omega &:= \{\phi : X \to \mathbb{C} : \phi \text{ is linear}, \ \phi(1) = 1, \ \phi(\Omega) \subseteq \mathbb{C}^*\}, \\ \tilde{\Omega} &= \bigcap \{(Ker\phi)^c : \phi \in M_\Omega\} = \{a \in X : \phi(a) \neq 0 \ \forall \phi \in M_\Omega\}, \end{aligned} \tag{5.2}$$

and proved

$$\mathrm{Rad}^{\tilde{\Omega}}(X) = \{a \in X : \phi(a) = 0, \ \forall \phi \in M_{\tilde{\Omega}}\}. \tag{5.3}$$

With this notation, Theorem 3.1 implies that every ϕ in M_Ω is multiplicative.

Example 5.1. For X and Ω in Example 3.2,

$$M_\Omega = \{\phi\}, \tag{5.4}$$

a singleton set, where $\phi : A \to \mathbb{C}$ defined as $\phi(z_1, z_2) = z_1$, and $\widetilde{\Omega} = \Omega$, so

$$\text{Rad}^{\widetilde{\Omega}}(X) = \text{Rad}^{\Omega}(X) = \{(z_1, z_2) : z_1 = 0\}. \tag{5.5}$$

Theorem 5.2. *Let A and B be complex algebras with unit 1. Assume Ω_A and Ω_B are subsets of A and B, also each of them is as in Theorem 3.1, and A and B are spectrally bounded with respect to them. Let $T : A \to B$ be a linear map such that $T(1) = 1$. If $T(\Omega_A) \subseteq \Omega_B$, then $T(ab) - (Ta)(Tb) \in \text{Rad}^{\widetilde{\Omega}_B}(B)$. If in addition B is $\widetilde{\Omega}_B$ semisimple, then T is multiplicative.*

Proof. Consider

$$M_{\Omega_B} := \{\phi : B \to \mathbb{C} : \phi \text{ is linear}, \ \phi(1) = 1, \ \phi(\Omega_B) \subseteq \mathbb{C}^*\}. \tag{5.6}$$

For every $\phi \in M_{\Omega_B}$, $\phi \circ T : A \to \mathbb{C}$, is a linear map with $(\phi \circ T)(1) = \phi(T(1)) = 1$ and

$$(\phi \circ T)(\Omega_A) = \phi(T(\Omega_A)) \subseteq \phi(\Omega_B) \subseteq \mathbb{C}^*. \tag{5.7}$$

In other words $\phi \circ T \in M_{\Omega_A}$. Hence by Theorem 3.1, $\phi \circ T$ is multiplicative. Thus,

$$(\phi \circ T)(ab) = (\phi \circ T)(a)(\phi \circ T)(b). \tag{5.8}$$

That is,

$$\phi(T(ab)) = \phi(T(a))\phi(T(b)) = \phi(T(a)T(b)) \tag{5.9}$$

as ϕ is multiplicative. Hence

$$\phi(T(ab) - (Ta)(Tb)) = 0. \tag{5.10}$$

Since ϕ is arbitrary in M_{Ω_B}, we get $T(ab) - T(a)T(b) \in \text{Rad}^{\widetilde{\Omega}_B}(B)$ by (5.3). If B is $\widetilde{\Omega}_B$ semisimple, then $\text{Rad}^{\widetilde{\Omega}_B}(B) = \{0\}$. Hence $T(ab) = T(a)T(b)$. □

Theorem 5.3. *Let A and B be real algebras with unit 1 and B commutative. Let $T : A \to B$ be a linear map such that $T(1) = 1$. Assume Ω_A and Ω_B are subsets of A and B, respectively, also each of them is as in Definition 4.1, and A and B are spectrally bounded with respect to them. Suppose $(T(a))^2 + (T(b))^2 \in \Omega_B$ whenever $ab = ba$ and $a^2 + b^2 \in \Omega_A$. Then,*

$$\phi(T(ab) - T(a)T(b)) = 0 \quad \textit{for every } \phi \in N_{\Omega_B}, \tag{5.11}$$

where

$$N_{\Omega_B} := \{\phi : B \to \mathbb{C} : \phi \textit{ is linear}, \ \phi(1) = 1, (\phi(a))^2 + (\phi(b))^2 \neq 0 \textit{ for } ab = ba, \ a^2 + b^2 \in \Omega_B\}. \tag{5.12}$$

Proof. For $\phi \in N_{\Omega_B}$, $\phi \circ T : A \to \mathbb{C}$ is a linear map with $(\phi \circ T)(1) = 1$. By assumption, whenever $ab = ba$ and $a^2 + b^2 \in \Omega_A$, $T(a)^2 + T(b)^2 \in \Omega_B$, which implies, as B is commutative, $((\phi \circ T)(a))^2 + ((\phi \circ T)(b))^2 \neq 0$. Hence by Theorem 4.8, $\phi \circ T$ is multiplicative. That is,

$$(\phi \circ T)(ab) = (\phi \circ T(a))(\phi \circ T(b)) = \phi(T(a)T(b)) \tag{5.13}$$

as ϕ is multiplicative by Theorem 4.8. Hence we get

$$\phi(Tab - TaTb) = 0 \quad \text{for every } \phi \in N_{\Omega_B}. \tag{5.14}$$

□

Since every real Banach algebra is a spectrally bounded real algebra, we show in the next corollary that [7, Theorem 7] follows from the above theorem using [7, Theorem 2].

Corollary 5.4. *Let A and B be real Banach algebras with units and suppose that B is commutative and semisimple. Let $T : A \to B$ be a linear map such that $T(1) = 1$. Then the following are equivalent:*

(1) *$T(ab) = T(a)T(b)$ for all a, b in A,*
(2) *$\mathrm{Sp}((T(a))^2 + (T(b))^2) \subseteq \mathrm{Sp}(a^2 + b^2)$ for all a, b in A such that $ab = ba$,*
(3) *$(T(a))^2 + (T(b))^2$ is invertible for all a, b in A such that $ab = ba$ and $a^2 + b^2$ is invertible.*

Proof. (1) implies (2) and (2) implies (3) are straight forward. The nontrivial part is (3) implies (1). Replacing Ω_A and Ω_B by the set of all invertible elements of A and B, respectively, in Theorem 5.3, we get

$$\phi(T(ab) - T(a)T(b)) = 0 \quad \text{for every } \phi \in N_{\Omega_B}. \tag{5.15}$$

But from [7, Theorem 2], N_{Ω_B} is the set of all multiplicative functions on B. As B is semisimple, $T(ab) = T(a)T(b)$ for all a, b in A. □

The assumption, commutativity, on B in Theorem 5.3 is necessary by [7, Example 10]. Here we give an example which shows that semisimple condition on B is necessary, in Theorem 5.2, to get T as multiplicative operator.

Example 5.5. Let X and Ω be as in Example 3.2. Then X is semisimple by the explanation in Example 5.1. Now define $T : X \to X$ as $T(z_1, z_2) = (z_1, (z_1 + z_2)/2)$. Clearly T satisfies hypothesis of Theorem 5.2 but is not multiplicative.

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