ON ABSOLUTE MATRIX SUMMABILITY METHODS

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We have proved a theorem on $|T,p_n|_k$ summability methods. This theorem includes a known theorem.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By (w_n^δ), we denote the nth Cesàro means of order $\delta(\delta > -1)$ of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C,\delta|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \left| w_n^\delta - w_{n-1}^\delta \right|^k < \infty. \tag{1.1}$$

In the special case for $\delta = 1$, $|C,\delta|_k$ summability reduces to $|C,1|_k$ summability.

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \qquad (P_{-i} = p_{-i} = 0,\ i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\vartheta_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.3}$$

defines the sequence (ϑ_n) of the $(\bar{N},p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|\bar{N},p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \vartheta_n - \vartheta_{n-1} \right|^k < \infty. \tag{1.4}$$

If we take $p_n = 1$ for all values of n, then $|\bar{N},p_n|_k$ summability is the same as $|C,1|_k$ summability.

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Given a normal matrix $T = (t_{nk})$, we associate two lower semimatrices $\overline{T} = (\overline{t}_{nk})$ and $\widehat{T} = (\widehat{t}_{nk})$ as follows:

$$\begin{gathered}\overline{t}_{nk} = \sum_{i=k}^{n} t_{ni}, \quad n,k = 0,1,\dots, \\ \widehat{t}_{00} = \overline{t}_{00} = t_{00}, \qquad \widehat{t}_{nk} = \overline{t}_{nk} - \overline{t}_{n-1,k}, \quad n = 1,2,\dots.\end{gathered} \tag{1.5}$$

It may be noted that $\overline{T}$ and $\widehat{T}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$\begin{gathered}T_n(s) = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} \overline{t}_{nv} a_v, \\ \overline{\Delta} T_n(s) = \sum_{v=0}^{n} \widehat{t}_{nv} a_v.\end{gathered} \tag{1.6}$$

The series $\sum a_n$ is said to be summable $|T, p_n|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\overline{\Delta} T_n(s)\right|^k < \infty. \tag{1.7}$$

In the special case, for $t_{nv} = p_v/P_n$, $|T, p_n|_k$ summability is the same as $|\overline{N}, p_n|_k$ summability.

2. The main result

The object of this paper is to prove the following theorem.

Theorem 2.1. *Let $k \geq 1$. Let (s_n) be a bounded sequence and suppose that (λ_n) is a sequence such that*

$$\begin{gathered}\sum_{n=0}^{m} \left(\frac{P_n}{p_n}\right)^{k-1} |\lambda_n|^k |t_{nn}|^k = O(1) \quad \text{as } m \longrightarrow \infty, \\ \sum_{n=0}^{m} |\Delta\lambda_n| = O(1) \quad \text{as } m \longrightarrow \infty.\end{gathered} \tag{2.1}$$

If

$$\frac{1}{|t_{nn}|} \sum_{v=0}^{n-1} |\Delta_v(\widehat{t}_{nv})| = O(1) \quad \text{as } n \longrightarrow \infty, \tag{2.2}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta_v \widehat{t}_{nv}| \, |t_{nn}|^{k-1} = O\left(\left(\frac{P_v}{p_v}\right)^{k-1} |t_{vv}|^k\right) \quad \text{as } m \longrightarrow \infty, \tag{2.3}$$

$$\frac{1}{|t_{nn}|}\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}| = O(1) \quad \text{as } n \longrightarrow \infty, \tag{2.4}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\hat{t}_{n,v+1}||t_{nn}|^{k-1} = O(1) \quad \text{as } m \longrightarrow \infty, \tag{2.5}$$

then the series $\sum a_n\lambda_n$ *is summable* $|T,p_n|_k$.

Proof. Let (y_n) be the T-transform of the series $\sum a_n\lambda_n$. Then we have, by (1.6),

$$Y_n = y_n - y_{n-1} = \sum_{v=0}^{n}\hat{t}_{nv}a_v\lambda_v. \tag{2.6}$$

Since $\hat{t}_{nn} = t_{nn}$, by Abel's transformation, we get that

$$\begin{aligned} Y_n &= \sum_{v=0}^{n-1}\Delta_v(\hat{t}_{nv}\lambda_v)s_v + \hat{t}_{nn}\lambda_n s_n \\ &= \sum_{v=0}^{n-1}\Delta\lambda_v\hat{t}_{n,v+1}s_v + \sum_{v=0}^{n-1}\lambda_v\Delta_v(\hat{t}_{nv})s_v + s_n t_{nn}\lambda_n \\ &= Y_n(1) + Y_n(2) + Y_n(3). \end{aligned} \tag{2.7}$$

Using Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|Y_n(r)|^k < \infty \quad \text{for } r = 1,2,3. \tag{2.8}$$

Since (s_n) is bounded, when $k > 1$, applying Hölder's inequality with indices k and k', where $1/k + 1/k' = 1$, we have that

$$\begin{aligned} \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|Y_n(1)|^k &\le \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}||s_v|\right\}^k \\ &= O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}||t_{nn}|^{k-1} \\ &\quad\times\left\{\frac{1}{|t_{nn}|}\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}|\right\}^{k-1} \\ &= O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}||t_{nn}|^{k-1} \\ &= O(1)\sum_{v=0}^{m}|\Delta\lambda_v|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\hat{t}_{n,v+1}||t_{nn}|^{k-1} \\ &= O(1)\sum_{v=0}^{m}|\Delta\lambda_v| = O(1) \quad \text{as } m \longrightarrow \infty, \end{aligned} \tag{2.9}$$

by virtue of the hypothesis of Theorem 2.1.

Again using Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|Y_n(2)|^k &\le \sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=0}^{n-1}|\lambda_v||\Delta_v\hat{t}_{nv}||s_v|\right\}^k \\
&= O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=0}^{n-1}|\lambda_v|^k|\Delta_v\hat{t}_{nv}||t_{nn}|^{k-1} \\
&\quad\times\left\{\frac{1}{|t_{nn}|}\sum_{v=0}^{n-1}|\Delta_v\hat{t}_{nv}|\right\}^{k-1} \\
&= O(1)\sum_{n=1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=0}^{n-1}|\lambda_v|^k|\Delta_v\hat{t}_{nv}||t_{nn}|^{k-1} \\
&= O(1)\sum_{v=0}^{m}|\lambda_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta_v\hat{t}_{nv}||t_{nn}|^{k-1} \\
&= O(1)\sum_{v=0}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}|\lambda_v|^k|t_{vv}|^k = O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned} \tag{2.10}$$

by virtue of the hypothesis of Theorem 2.1.

Finally, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|Y_n(3)|^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|t_{nn}|^k|\lambda_n|^k = O(1) \quad \text{as } m \longrightarrow \infty, \tag{2.11}$$

by virtue of the hypothesis of Theorem 2.1.

Therefore, we get that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|Y_n(r)|^k = O(1) \quad \text{as } m \longrightarrow \infty, \text{ for } r = 1,2,3. \tag{2.12}$$

This completes the proof of Theorem 2.1. □

3. An application

Now we will prove the following corollary.

Corollary 3.1 (see [2]). *Let $k \ge 1$. If the sequence (s_n) is bounded and (λ_n) is a sequence such that*

$$\begin{aligned}
\sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n|^k &= O(1) \quad \text{as } m \longrightarrow \infty, \\
\sum_{n=1}^{m}|\Delta\lambda_n| &= O(1) \quad \text{as } m \longrightarrow \infty,
\end{aligned} \tag{3.1}$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$.

Proof. In Theorem 2.1, let $t_{nv} = p_v/P_n$. Then to prove the corollary, it is sufficient to show that the conditions of Theorem 2.1 are satisfied.

If $t_{nn} = p_n/P_n$, (2.1) are automatically satisfied.

Since

$$\begin{aligned}
\Delta_v \hat{t}_{nv} &= \hat{t}_{nv} - \hat{t}_{n,v+1} \\
&= \bar{t}_{nv} - \bar{t}_{n-1,v} - \bar{t}_{n,v+1} + \bar{t}_{n-1,v+1} \\
&= \sum_{i=v}^{n} t_{ni} - \sum_{i=v}^{n-1} t_{n-1,i} - \sum_{i=v+1}^{n} t_{ni} + \sum_{i=v+1}^{n-1} t_{n-1,i} \\
&= \frac{1}{P_n}\sum_{i=v}^{n} p_i - \frac{1}{P_{n-1}}\sum_{i=v}^{n-1} p_i - \frac{1}{P_n}\sum_{i=v+1}^{n} p_i + \frac{1}{P_{n-1}}\sum_{i=v+1}^{n-1} p_i \\
&= -\frac{p_n p_v}{P_n P_{n-1}},
\end{aligned} \tag{3.2}$$

we get

$$\frac{1}{|t_{nn}|}\sum_{v=0}^{n-1} |\Delta_v \hat{t}_{nv}| = \frac{P_n}{p_n}\sum_{v=0}^{n-1} \frac{p_n p_v}{P_n P_{n-1}} = O(1) \quad \text{as } n \longrightarrow \infty. \tag{3.3}$$

Thus condition (2.2) is satisfied.

Using $\Delta_v \hat{t}_{nv}$ and t_{nn},

$$\begin{aligned}
\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta_v \hat{t}_{nv}| \, |t_{nn}|^{k-1} &= \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \frac{p_n p_v}{P_n P_{n-1}} \left(\frac{p_n}{P_n}\right)^{k-1} \\
&= p_v \sum_{n=v+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = \frac{p_v}{P_v} \\
&= \left(\frac{P_v}{p_v}\right)^{k-1} |t_{vv}|^k \quad \text{as } m \longrightarrow \infty,
\end{aligned} \tag{3.4}$$

condition (2.3) is satisfied.

Since

$$\begin{aligned}
\hat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v} &= \sum_{i=v}^{n} t_{ni} - \sum_{i=v}^{n-1} t_{n-1,i} \\
&= \frac{1}{P_n}\sum_{i=v}^{n} p_i - \frac{1}{P_{n-1}}\sum_{i=v}^{n-1} p_i \\
&= P_{v-1}\left(-\frac{1}{P_n} + \frac{1}{P_{n-1}}\right) = P_{v-1}\frac{p_n}{P_n P_{n-1}},
\end{aligned}$$

$$\frac{1}{|t_{nn}|}\sum_{v=0}^{n-1}|\Delta\lambda_v||\hat{t}_{n,v+1}| = \frac{P_n}{p_n}\sum_{v=0}^{n-1}|\Delta\lambda_v|P_v\frac{p_n}{P_nP_{n-1}}$$
$$= \frac{1}{P_{n-1}}\sum_{v=0}^{n-1}|\Delta\lambda_v|P_v = O(1)\sum_{v=0}^{n-1}|\Delta\lambda_v| = O(1) \quad \text{as } n \longrightarrow \infty, \tag{3.5}$$

and condition (2.4) is satisfied.

Finally,

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|\hat{t}_{n,v+1}||t_{nn}|^{k-1} = \sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\frac{P_vp_n}{P_nP_{n-1}}\left(\frac{p_n}{P_n}\right)^{k-1}$$
$$= P_v\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1) \quad \text{as } m \longrightarrow \infty, \tag{3.6}$$

so condition (2.5) is satisfied.

This completes the proof of the corollary. □

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