ON THE POWER-COMMUTATIVE KERNEL OF LOCALLY NILPOTENT GROUPS

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We define the power-commutative kernel of a group. In particular, we describe the power-commutative kernel of locally nilpotent groups, and of finite groups having a nontrivial center.

A group G is called *power commutative*, or a *PC-group*, if $[x^m, y^n] = 1$ implies $[x, y] = 1$ for all $x, y \in G$ such that $x^m \neq 1$, $y^n \neq 1$. So power-commutative groups are those groups in which commutativity of nontrivial powers of two elements implies commutativity of the two elements. Clearly, G is a PC-group if and only if $C_G(x) = C_G(x^n)$ for all $x \in G$ and all integers n such that $x^n \neq 1$. Obvious examples of PC-groups are groups in which commutativity is a transitive relation on the set of nontrivial elements (*CT-groups*) and groups of prime exponent.

Recall that a group G is called an *R-group* if $x^n = y^n$ implies $x = y$ for all $x, y \in G$ and for all positive integers n. In other words, R-groups are groups in which the extraction of roots is unique. A result due to Mal'cev and Cernikov (see, e.g., [3]) states that every nilpotent torsion-free group is an R-group. There is a natural connection between PC-groups and R-groups. For, as pointed out in [3], a torsion-free group is a PC-group if and only if it is an R-group.

In [5], Wu gave the classification of locally finite PC-groups. In particular, she proved that a finite group is a PC-group if and only if the centralizer of each nontrivial element is abelian or of prime exponent. This result implies that a finite group having a nontrivial center is a PC-group if and only if it is abelian or it has prime exponent. Moreover, the class of PC-groups is contained in the class of groups in which the centralizer of each nontrivial element is nilpotent. This class of groups was investigated by many authors (see, e.g., [1, 4]).

In analogy to what is done in [2] to define the commutative-transitive kernel of a group, we introduce an ascending series

$$\{1\} = P_0(G) \leq P_1(G) \leq \cdots \leq P_t(G) \leq \cdots \tag{1}$$

of characteristic subgroups of G contained in the derived subgroup G'. We define $P_1(G)$ as

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the subgroup of G' generated by those commutators $[x, y]$ such that there exist positive integers n, m with $x^n \neq 1$, $y^m \neq 1$, and $[x^n, y^m] = 1$. If $t > 1$ then $P_t(G)$ is defined by $P_t(G)/P_{t-1}(G) = P_1(G/P_{t-1}(G))$. Finally, the *PC-kernel* of G is the subgroup $P(G)$ of G' defined by

$$P(G) = \bigcup_{t\in\mathbb{N}} P_t(G). \tag{2}$$

Obviously, for any group G, the PC-kernel $P(G)$ is characteristic in G, $G/P(G)$ is a PC-group, and G is a PC-group if and only if $P(G) = \{1\}$.

Let $\mathscr{X}$ be a class of groups. Then one can ask whether there exists a nonnegative integer n such that $P_n(G) = P(G)$ for all $G \in \mathscr{X}$. Of course $P(G) = P_n(G)$ if and only if $G/P_n(G)$ is a PC-group.

In this paper, we give affirmative answers to the previous question when $\mathscr{X}$ is the class of locally nilpotent groups, or the class of finite groups having a nontrivial center. In both cases, we prove that $P(G) = P_1(G)$ for all $G \in \mathscr{X}$.

Our first results are concerned with the power-commutative kernel of finite nilpotent groups.

Proposition 1. *Let p be a prime and G a finite p-group. Then $G/P_1(G)$ is a PC-group.*

Proof. Notice that $P_1(G) \leq M$ for every maximal subgroup M of G since $P_1(G) \leq G' \leq \Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G. This implies that $M/P_1(G)$ is a maximal subgroup of $G/P_1(G)$ if and only if M is a maximal subgroup of G.

Let G be a counterexample of least order. For any maximal subgroup M of G we obtain $M/P_1(G) \simeq (M/P_1(M))/(P_1(G)/P_1(M))$. Hence $M/P_1(G)$ is a PC-group since it is a quotient of a finite PC-group (see [5]). It follows that a maximal subgroup of $G/P_1(G)$ is abelian or it has exponent p.

Put $\overline{G} = G/P_1(G)$ and $\overline{H} = H/P_1(G)$ for all $P_1(G) \leq H \leq G$. If every maximal subgroup $\overline{M}$ of $\overline{G}$ has exponent p, then G is cyclic or of exponent p. In any case $\overline{G}$ is a PC-group, that is a contradiction. So we may assume that $\overline{G}$ has a maximal subgroup $\overline{M}$ such that $\overline{M}$ is abelian and $\overline{M}^p \neq 1$. Consider $g \in \overline{G} \setminus \overline{M}$, so $\overline{G} = \langle \overline{M}, g\rangle$. Moreover $|\overline{G} : \overline{M}| = p$.

If there exists $a \in \overline{M}$ such that $(ga)^p \neq 1$, then $(ga)^p \in \overline{M} \setminus \{1\}$. So, for all $y \in \overline{M}$ we get $[y, (ga)^p] = 1$, hence $[y, g] = [y, ga] = 1$. It follows that $\overline{G}$ is abelian, a contradiction. Thus $(ga)^p = 1$ for all $a \in \overline{M}$, and in particular $g^p = 1$. It follows that $a^{g^{p-1}+\cdots+g+1} = (ga)^p = 1$ for all $a \in \overline{M}$. This implies $a^p = 1$ for all $a \in C_{\overline{M}}(g)$, so $(C_{\overline{M}}(g))^p = C_{\overline{M}^p}(g) = 1$. But $\overline{M}^p \cap Z(\overline{G}) \neq 1$ since $\overline{M}^p \neq 1$, that is a contradiction. □

Proposition 2. *Let G be a finite nilpotent group of order $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ ($p_1, \dots, p_t$ distinct primes). If $t > 1$ then $G/P_1(G)$ is abelian.*

Proof. Let G_{p_i} be the Sylow p_i-subgroup of G for all $i \in \{1, \dots, t\}$; we will prove that $(G_{p_i})' \leq P_1(G)$ for all $i \in \{1, \dots, t\}$. Let $x, y \in G_{p_i} \setminus \{1\}$, $a \in G_{p_1} \times \cdots \times G_{p_{i-1}} \times G_{p_{i+1}} \times \cdots \times G_{p_t}$. Put $|a| = m$ and $|x| = p_i{}^r$. Now $|ax| = mp_i{}^r$ as $(m, p_i{}^r) = 1$. Since $(ax)^{p_i{}^r} = a^{p_i{}^r}$ has order m we get $[(ax)^{p_i{}^r}, y] = [a^{p_i{}^r}, y] = 1$. Thus $[ax, y] = [x, y] \in P_1(G)$. □

Corollary 3. *Let G be a finite nilpotent group; then $G/P_1(G)$ is abelian or it has exponent p. In both cases $G/P_1(G)$ is a PC-group.*

Proof. The result is an immediate consequence of the previous propositions and [5, Theorem 4]. □

Now we prove that the equality $P(G) = P_1(G)$ holds for every nilpotent group G.

THEOREM 4. *Let G be a nilpotent group. Then $G/P_1(G)$ is a PC-group.*

Proof. If G is torsion-free then G is a PC-group (see [3]), so $P_1(G) = \{1\}$ and the result is true. So we may suppose that the torsion subgroup T of G is nontrivial.

First of all, notice that if for elements $x, y \in G \setminus \{1\}$ there exists a positive integer n such that $x^n \neq 1$ and $[x^n, y] = 1$, then $[x, y] \in T$. This is obvious if $x \in T$ or $y \in T$, so we may assume $x, y \notin T$. Then $\langle x, y\rangle T/T \leq G/T$. So $\langle xT, yT\rangle$ is torsion-free, and $[(xT)^n, yT] = T$ implies $[x, y] \in T$. This means that $P_1(G) \subseteq T$.

If for any $x, y \in G$ the commutator $[x, y]$ is periodic, then it is easy to see that there exists a positive integer m such that $[x, y^m] = 1$. In fact, $\langle x, y\rangle$ is a FC-group since $\langle x, y\rangle/Z(\langle x, y\rangle)$ is finite, and therefore the set $\{x^{y^t} | t \in \mathbb{Z}\}$ is finite.

Now notice that if $x \in T$ then $[x,g] \in P_1(G)$ for all $g \in G \setminus T$. In fact, $[x,g] \in T$ implies that there exists a positive integer m such that $[x,g^m] = 1$. So we get $[x,g] \in P_1(G)$ because $g^m \neq 1$.

Finally, let $x, y \in G \setminus P_1(G)$ such that $x^n \notin P_1(G)$ and $[x^n, y] \in P_1(G)$. If $x, y \in T$ then $\langle x, y\rangle$ is a finite nilpotent group and Corollary 3 implies that $\langle x, y\rangle/P_1(\langle x, y\rangle)$ is a finite PC-group. Hence $\langle x, y\rangle/P_1(G) \cap \langle x, y\rangle$ is a PC-group and $[x, y] \in P_1(G)$. If $x \in T$ or $y \in T$ then $[x, y] \in P_1(G)$, as noticed before. So we may suppose $x, y \in G \setminus T$. Since $[x^n, y] \in P_1(G) \subseteq T$, we get $[x^n, y] \in T$ and so there exists a positive integer m such that $[x^n, y^m] = 1$. Therefore $[x, y] \in P_1(G)$, and the proof is complete. □

THEOREM 5. *Let G be a locally nilpotent group. Then $P(G) = P_1(G)$.*

Proof. Let $x, y \in G \setminus P_1(G)$ such that $x^n \notin P_1(G)$ and $[x^n, y] \in P_1(G)$. Then

$$[x^n, y] = \prod_{i=1}^{r} [a_i, b_i], \tag{3}$$

where $a_i, b_i \in G$ for all $i = 1, 2, \dots, r$, and $[a_i^{\alpha_i}, b_i^{\beta_i}] = 1$ for some positive integers α_i and β_i such that $a_i^{\alpha_i} \neq 1$ and $b_i^{\beta_i} \neq 1$.

Let $H = \langle x, y, a_1, \dots, a_r, b_1, \dots, b_r\rangle$. Then H is nilpotent, so $H/P_1(H)$ is a PC-group by Theorem 4. Since $[a_i, b_i] \in P_1(\langle a_i, b_i\rangle) \leq P_1(H)$ for all $i = 1, 2, \dots, r$, we get $[x^n, y] \in P_1(H)$. Thus $[x, y] \in P_1(H)$, and therefore $[x, y] \in P_1(G)$. □

Now it is possible to prove that $P(G) = P_1(G)$ for any finite group G such that $Z(G) \neq \{1\}$.

PROPOSITION 6. *Let G be a finite group such that $Z(G) \neq \{1\}$. Then $[a, b] \in P_1(G)$ for all $a, b \in G \setminus \{1\}$ such that $(|a|, |b|) = 1$.*

Proof. Put $|a| = n$ and $|b| = m$. Then there exists $z \in Z(G) \setminus \{1\}$ such that $|z|$ does not divide n or m. Suppose $|z|$ does not divide n. Then $[(az)^n, b] = [a^n z^n, b] = [z^n, b] = 1$. Moreover $(az)^n = z^n \neq 1$ and this yields $[az, b] = [a, b] \in P_1(G)$. □

Proposition 7. *Let G be a finite group such that $Z(G) \neq \{1\}$. Then $G/P_1(G)$ is nilpotent.*

Proof. We may assume that the order of $G/P_1(G)$ is not a prime power. Let p be any prime divisor of $|G/P_1(G)|$. Then p divides $|G|$ and $PP_1(G)/P_1(G)$ is a Sylow p-subgroup of $G/P_1(G)$ whenever P is a Sylow p-subgroup of G. We are going to show that $PP_1(G)/P_1(G)$ is normal in $G/P_1(G)$. Let $q \neq p$ be any prime dividing $|G/P_1(G)|$, and let Q be a Sylow q-subgroup of G. Then $QP_1(G)/P_1(G)$ centralizes $PP_1(G)/P_1(G)$, by Proposition 6. Thus the normalizer in $G/P_1(G)$ of $PP_1(G)/P_1(G)$ contains a Sylow q-subgroup of $G/P_1(G)$ for all prime divisors of its order. Therefore this normalizer is actually $G/P_1(G)$, and the result follows. □

Theorem 8. *Let G be a finite group such that $Z(G) \neq \{1\}$. Then $G/P_1(G)$ is abelian or it has exponent p.*

Proof. Since $G/P_1(G)$ is nilpotent by Proposition 7, by [5] it suffices to show that $G/P_1(G)$ is a PC-group. Suppose not, and let G be a counterexample of least order. We may assume G is not nilpotent, hence $P_1(G) \nleq \Phi(G)$. Thus there exists a maximal subgroup M of G such that $P_1(G) \nsubseteq M$. In particular $G' \nsubseteq M$. If $Z(G) \nsubseteq M$, then there exists $z \in Z(G) \setminus M$. Since M is maximal, it follows that $\langle z \rangle M = G$. Hence M is normal in G, and G/M is cyclic. This in turn implies that $G' \subseteq M$, a contradiction. Thus $Z(G) \subseteq M$, and so $Z(M) \neq \{1\}$. Then $M/P_1(M)$ is a PC-group and therefore $G/P_1(G) \simeq (M/P_1(M))/((M \cap P_1(G))/P_1(M))$ is a PC-group, the final contradiction. □

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