SMARANDACHE BCC-ALGEBRAS

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The notions of Smarandache BCC-algebras and Smarandache BCC-ideals are introduced. Conditions for a (special) subset to be a Smarandache BCC-ideal are given.

1. Introduction

Generally, in any human field, a *Smarandache Structure* on a set A means a weak structure **W** on A such that there exists a proper subset B of A which is embedded with a strong structure **S**. In [9], Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideal of groupoids, seminormal subgroupoids, Smarandache Bol groupoids, and strong Bol groupoids and obtained many interesting results about them. Smarandache semigroups are very important for the study of congruences, and they were studied by Padilla [13]. In this paper, we discuss a Smarandache structure on BCC-algebras, and introduce the notion of Smarandache ideals, and investigate its properties. We give conditions for a (special) subset to be a Smarandache BCC-ideal.

2. Preliminaries

BCC-algebras were introduced by Komori [11] in a connection with some problems on BCK-algebras solved in [14], and Dudek [4, 5] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Komori.

An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCC-algebra* if it satisfies the following conditions:

(a1) $(\forall x,y,z\in X)\ (((x*y)*(z*y))*(x*z)=0)$,
(a2) $(\forall x\in X)\ (0*x=0)$,
(a3) $(\forall x\in X)\ (x*0=x)$,
(a4) $(\forall x,y\in X)\ (x*y=0,\ y*x=0\Rightarrow x=y)$.

Note that every BCK-algebra is a BCC-algebra, but the converse is not true. A BCC-algebra which is not a BCK-algebra is called a *proper BCC-algebra.* The smallest proper BCC-algebra has four elements, and, for every $n\geq 4$, there exists at least one proper BCC-algebra [4].

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A nonempty subset I of a BCC-algebra X is called a *BCC-ideal* of X if it satisfies the following assertions:

(a5) $0 \in I$,

(a6) $(\forall x, z \in X)\,(\forall y \in I)\,((x * y) * z \in I \Rightarrow x * z \in I)$.

Note that every BCC-algebra X satisfies the following assertions:

(b1) $(\forall x \in X)\,(x * x = 0)$,

(b2) $(\forall x, y \in X)\,(x * y \leq x)$,

(b3) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow x * z \leq y * z,\ z * y \leq z * x)$,

where $x \leq y$ if and only if $x * y = 0$.

3. Smarandache BCC-algebras

We know that every proper BCC-algebra has at least four elements (see [4]), and that if X is a BCC-algebra, then $\{0, a\}$, $a \in X$, is a BCK-algebra with respect to the same operation on X. Now let us consider a proper BCC-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	1	2
3	3	3	1	0	3
4	4	0	0	0	0

(3.1)

Then $\{0, 1\}$, $\{0, 2\}$, $\{0, 3\}$, $\{0, 4\}$, $\{0, 1, 2\}$, and $\{0, 1, 3\}$ are BCK-algebras with respect to the operation $*$ on X, and note that X does not contain BCK-algebras of order 4. Based on this result, we give the following definition.

Definition 3.1. A *Smarandache BCC-algebra* (briefly, *S-BCC-algebra*) is defined to be a BCC-algebra X in which there exists a proper subset Q of X such that

(i) $0 \in Q$ and $|Q| \geq 4$,

(ii) Q is a BCK-algebra with respect to the same operation on X.

Note that any proper BCC-algebra X with four elements cannot be an S-BCC-algebra. Hence, if X is an S-BCC-algebra, then $|X| \geq 5$. Notice that the BCC-algebra $X = \{0, 1, 2, 3, 4\}$ with Table 3.1 is not an S-BCC-algebra.

Example 3.2. (1) Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0

(3.2)

Then, $(X;*,0)$ is an S-BCC-algebra. Note that $Q=\{0,a,b,c\}$ is a BCK-algebra which is properly contained in X.

(2) Let $(X;*,0)$ be a finite BCK-chain containing at least four elements, and let c be its maximal element. Let $Y=X\cup\{d\}$, where $d\notin X$, and define a binary operation $\odot$ on Y as follows:

$$x\odot y=\begin{cases} x*y & \text{if } x,y\in X,\\ 0 & \text{if } x\in Y,\ y=d,\\ d & \text{if } x=d,\ y=0,\\ c & \text{if } x=d,\ y\in X.\end{cases} \tag{3.3}$$

Then, $(Y;\odot,0)$ is an S-BCC-algebra.

(3) Let $(X;*,0)$ be a BCK-algebra containing at least four elements in which a is the small atom. Let $Y=X\cup\{w\}$, where $w\notin X$, and define a binary operation $\odot$ on Y as follows:

$$x\odot y=\begin{cases} x*y & \text{if } x,y\in X,\\ w & \text{if } y\in X,\ x=w,\\ 0 & \text{if } x=0,\ y=w,\\ 0 & \text{if } x=w=y,\\ a & \text{if } x\in X\setminus\{0\},\ y=w.\end{cases} \tag{3.4}$$

Then, $(Y;\odot,0)$ is an S-BCC-algebra.

In what follows, let X and Q denote an S-BCC-algebra and a nontrivial BCK-algebra which is properly contained in X, respectively, unless otherwise specified.

Definition 3.3. A nonempty subset I of X is called a *Smarandache BCC-ideal* (briefly, *S-BCC-ideal*) of X related to Q if it satisfies the following:

(c1) $0\in I$,

(c2) $(\forall x,z\in Q)\ (\forall y\in I)\ ((x*y)*z\in I\Rightarrow x*z\in I)$.

If I is an S-BCC-ideal of X related to every nontrivial BCK-algebra Q contained in X, we simply say that I is an *S-BCC-ideal* of X.

Example 3.4. Let $X=\{0,a,b,c,d,e\}$ be the S-BCC-algebra described in Example 3.2(1). Then, $I=\{0,a\}$ and $J=\{0,a,b,c,d\}$ are S-BCC-ideals of X related to $Q=\{0,a,b,c\}$.

Proposition 3.5. *Every S-BCC-ideal I of X related to Q satisfies the following:*

(c3) $(\forall x\in Q)\ (\forall a\in I)\ (x*a\in I\ \Rightarrow\ x\in I)$.

(c4) $(\forall x\in Q)\ (\forall a\in I)\ (a*x\in I)$.

(c5) $(\forall x\in Q)\ (\forall a,b\in I)\ (x*((x*a)*b)\in I)$.

Proof. (c3) Taking $z=0$ and $y=a$ in (c2) and using (a3) induce the desired implication.

(c4) For every $x\in Q$ and $a\in I$, we have $(a*a)*x=0*x=0\in I$, and so $a*x\in I$ by (c2).

(c5) Let $x \in Q$ and $a, b \in I$. Then, $(x * a) * (x * a) = 0 \in I$, and so $x * (x * a) \in I$ by (c2). Since

$$((x * b) * ((x * a) * b)) * (x * (x * a)) = 0 \in I, \tag{3.5}$$

it follows from (c3) that $(x * b) * ((x * a) * b) \in I$, so from (c2) that $x * ((x * a) * b) \in I$. □

Corollary 3.6. *For every S-BCC-ideal I of X related to Q, the following implication is valid:*

$$(\forall x \in Q)\ (\forall a \in I) \quad (x \leq a \Longrightarrow x \in I). \tag{3.6}$$

Corollary 3.7. *Let I be an S-BCC-ideal of X relative to Q. Then,*

$$(\forall x \in Q)\ (\forall a, b \in I) \quad (x * a \leq b \Longrightarrow x \in I). \tag{3.7}$$

Theorem 3.8. *Let Q_1 and Q_2 be nontrivial BCK-algebras which are properly contained in X such that $Q_1 \subset Q_2$. Then, every S-BCC-ideal of X related to Q_2 is an S-BCC-ideal of X related to Q_1.*

Proof. Straightforward. □

Corollary 3.9. *If Q is the largest BCK-algebra which is properly contained in X, then every S-BCC-ideal of X related to Q is an S-BCC-ideal of X.*

The converse of Theorem 3.8 is not true in general as seen in the following example.

Example 3.10. Consider an S-BCC-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	1	0	0	0	1
3	3	1	1	0	1	1
4	4	1	1	1	0	1
5	5	5	5	5	5	0

(3.8)

Note that $Q_1 := \{0, 1, 2, 3\}$ and $Q_2 := \{0, 1, 2, 3, 4\}$ are BCK-algebras. Then, the set Q_1 is an S-BCC-ideal of X related to Q_1, but not Q_2. In fact, we know that $(4 * 2) * 0 = 1 \in Q_1$ and $4 * 0 = 4 \notin Q_1$.

Remark 3.11. Note that every BCC-ideal of X is an S-BCC-ideal of X, but the converse is not valid. Example 3.10 shows that there exists a BCK-algebra Q of order $n \geq 4$, which is properly contained in an S-BCC-algebra X such that an S-BCC-ideal of X related to Q is not a BCC-ideal of X.

We provide conditions for a subset to be an S-BCC-ideal.

THEOREM 3.12. *If I is a subset of Q that satisfies conditions* (c1) *and* (c3), *then I is an S-BCC-ideal of X related to Q.*

Proof. Let $x, y \in Q$ and $a \in I$ be such that $(x * a) * y \in I$. Since $a \in I \subseteq Q$ and Q is a BCK-algebra, it follows that $(x * y) * a = (x * a) * y \in I$, so from (c3) that $x * y \in I$. Hence, I is an S-BCC-ideal of X related to Q. □

THEOREM 3.13. *If a nonempty subset I of X satisfies conditions* (c1) *and* (c5), *then I is an S-BCC-ideal of X related to Q.*

Proof. Let $x, y \in Q$ and $a \in I$ be such that $(x * a) * y \in I$. Taking $b = 0$ in (c5), and using (a3), we have $x * (x * a) \in I$. It follows from (a3), (a1), and (c5) that

$$x * y = (x * y) * 0 = (x * y) * (((x * y) * ((x * a) * y)) * (x * (x * a))) \in I. \quad (3.9)$$

Thus, I is an S-BCC-ideal of X related to Q. □

THEOREM 3.14. *Let H be a BCC-subalgebra of X. Then, H is an S-BCC-ideal of X related to Q if and only if it satisfies the following:*

$$(\forall x \in H)\,(\forall y, z \in Q) \quad ((y * x) * z \in H \Longrightarrow y * z \in H). \quad (3.10)$$

Proof. Straightforward. □

Given an element $w \in X \setminus \{0\}$, consider the set

$$[0, w] := \{x \in X \mid x \leq w\}, \quad (3.11)$$

which is called the *initial segment* of w [7]. Obviously, $0 \in [0, w]$ for all $w \in X$. Since $x \leq w$ is equivalent to $xw = 0$, the initial segment of w is de facto the left annihilator of w. In general, $[0, w]$ is not an S-BCC-ideal of X, but it is a subalgebra. For example, let X be the S-BCC-algebra in Example 3.2(1). Then, $[0, e] = \{0, e\}$ is not an S-BCC-ideal of X related to $Q = \{0, a, b, c\}$ since $(b * e) * d = 0 \in [0, e]$, but $b * d = a \notin [0, e]$.

THEOREM 3.15. *For every $c \in X \setminus \{0\}$, if the inequality*

$$(\forall x \in Q) \quad (x * ((x * c) * c) \leq c) \quad (3.12)$$

holds, then $[0, c]$ is an S-BCC-ideal of X related to Q.

Proof. Let $x \in Q$. If $b \in [0, c]$, then $b \leq c$ and hence $(x * c) * c \leq (x * c) * b$ by (b3). It follows from (b3) and assumption that

$$x * ((x * c) * b) \leq x * ((x * c) * c) \leq c. \quad (3.13)$$

Now if $a \in [0, c]$, then $x * c \leq x * a$, and so

$$x * ((x * a) * b) \leq x * ((x * c) * b) \leq c. \quad (3.14)$$

This shows that $x * ((x * a) * b) \in [0, c]$. Applying Theorem 3.13, we conclude that $[0, c]$ is an S-BCC-ideal of X related to Q. □

Theorem 3.16. *The initial segment* $[0,c]$, *where* $c \in X \setminus \{0\}$, *is an S-BCC-ideal of X related to Q if and only if the implication*

$$(\forall x,y \in Q) \quad ((x*c)*y \leq c \Longrightarrow \ x*y \leq c) \tag{3.15}$$

is valid.

Proof. Let $x,y \in Q$ and $a \in [0,c]$ be such that $(x*a)*y \in [0,c]$. Then, $a \leq c$ and $(x*a)*y \leq c$. The inequality $a \leq c$ implies that $(x*c)*y \leq (x*a)*y \leq c$, so from hypothesis that $x*y \leq c$, that is, $x*y \in [0,c]$. Therefore, $[0,c]$ is an S-BCC-ideal of X related to Q. Conversely assume that $[0,c]$, $c \in X \setminus \{0\}$, is an S-BCC-ideal of X related to Q, and let $x,y \in Q$ be such that $(x*c)*y \leq c$. Then, $(x*c)*y \in [0,c]$. Since $[0,c]$ is an S-BCC-ideal of X related to Q and $c \in [0,c]$, it follows from (c2) that $x*y \in [0,c]$ so that $x*y \leq c$. This completes the proof. □

Corollary 3.17. *If* $[0,c]$, $c \in X \setminus \{0\}$, *is an S-BCC-ideal of X related to Q, then*

$$(\forall x \in Q) \quad (x*c \leq c \Longrightarrow x \leq c). \tag{3.16}$$

Theorem 3.18. *For every* $c \in X \setminus \{0\}$, *if the equality*

$$(\forall x,y \in Q) \quad (((x*c)*y)*c = (x*y)*c) \tag{3.17}$$

is valid, then $[0,c]$ *is an S-BCC-ideal of X related to Q.*

Proof. Let $x,y \in Q$ and $a \in [0,c]$ be such that $(x*a)*y \in [0,c]$. Then, $a \leq c$ and $(x*a)*y \leq c$. It follows that

$$(x*y)*c = ((x*c)*y)*c \leq ((x*a)*y)*c \leq c*c = 0, \tag{3.18}$$

so that $(x*y)*c = 0$, that is, $x*y \leq c$. Hence, $x*y \in [0,c]$ and therefore $[0,c]$ is an S-BCC-ideal of X related to Q. □

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