THE CONVERGENCE OF MEAN VALUE ITERATION FOR A FAMILY OF MAPS

B. E. RHOADES AND ŞTEFAN M. ŞOLTUZ

Received 3 January 2005 and in revised form 6 September 2005

We consider a mean value iteration for a family of functions, which corresponds to the Mann iteration with $\lim_{n\to\infty}\alpha_n \neq 0$. We prove convergence results for this iteration when applied to strongly pseudocontractive or strongly accretive maps.

1. Introduction

Let X be a real Banach space. The map $J : X \to 2^{X^*}$ given by

$$Jx := \{f \in X^* : \langle x, f\rangle = \|x\|^2,\ \|f\| = \|x\|\}, \quad \forall x \in X, \tag{1.1}$$

is called *the normalized duality mapping*. Let $y \in X$ and $j(y) \in J(y)$; note that $\langle \cdot, j(y)\rangle$ is a Lipschitzian map.

Remark 1.1. The above J satisfies

$$\langle x, j(y)\rangle \leq \|x\|\|y\|, \quad \forall x \in X,\ \forall j(y) \in J(y). \tag{1.2}$$

Definition 1.2. Let B be a nonempty subset of X. The map $T : B \to B$ is strongly pseudocontractive if there exist $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty,\ j(x-y)\rangle \leq k\|x-y\|^2, \quad \forall x, y \in B. \tag{1.3}$$

A map $S : B \to B$ is called strongly accretive if there exist $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Sx - Sy,\ j(x-y)\rangle \geq k\|x-y\|^2, \quad \forall x, y \in B. \tag{1.4}$$

In (1.3), take $k = 1$ to obtain a pseudocontractive map. In (1.4), take $k = 0$ to obtain an accretive map.

International Journal of Mathematics and Mathematical Sciences 2005:21 (2005) 3479–3485
DOI: 10.1155/IJMMS.2005.3479

Let B be a nonempty and convex subset of $X, T : B \to B$ and $x_0, u_0 \in B$. The Mann iteration (see [3]) is defined by

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n. \tag{1.5}$$

The Ishikawa iteration is defined (see [2]) by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset [0,1)$.

Let $s \geq 2$ be fixed. Let $T_i : B \to B$, $1 \leq i \leq s$, be a family of functions. We consider the following multistep procedure:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_1 y_n^1, \\ y_n^i &= (1-\beta_n^i)x_n + \beta_n^i T_{i+1} y_n^{i+1}, \quad i = 1,\dots,s-2, \\ y_n^{s-1} &= (1-\beta_n^{s-1})x_n + \beta_n^{s-1} T_s x_n. \end{aligned} \tag{1.7}$$

Let $A, b \in (0,1)$ be fixed. The sequence $\{\alpha_n\} \subset (0,1)$ satisfies

$$0 < A \leq \alpha_n \leq b < 2(1-k), \quad \forall n \in \mathbb{N}, \tag{1.8}$$

$$\{\beta_n^i\} \subset [0,1), \quad i = 1,\dots,s-1. \tag{1.9}$$

Let $F(T_1,\dots,T_s)$ denote the common fixed points set with respect to B for the family $T_1,\dots,T_s$. In this paper, we will prove convergence results for iteration (1.7), for strongly pseudocontractive (accretive) maps when $\{\alpha_n\}$ satisfies (1.8). These results improve the recently obtained results from [6], in which $\{\alpha_n\}$ and $\{\beta_n\}$ converge to zero. We give two numerical examples in which iteration (1.7), when $\{\alpha_n\}$ satisfies (1.8), converges faster as in [6]. Note that, in both cases, iteration (1.7) converges faster than Ishikawa iteration.

Lemma 1.3 [4]. *Let X be a real Banach space, and let $J : X \to 2^{X^*}$ be the duality mapping. Then for any given $x, y \in X$,*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x,y \in X, \ \forall j(x+y) \in J(x+y). \tag{1.10}$$

Lemma 1.4 [7]. *Let $\{a_n\}$ be a nonnegative sequence which satisfies the inequality*

$$a_{n+1} \leq (1-t)a_n + \sigma_n, \tag{1.11}$$

where $t \in (0,1)$ is fixed, $\lim_{n\to\infty} \sigma_n = 0$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main result

Theorem 2.1. *Let $s \geq 2$ be fixed, X a real Banach space, and B a nonempty, closed, convex subset of X. Let $T_1 : B \to B$ be a strongly pseudocontractive operator and $T_2,\dots,T_s : B \to B$,* *with $T_i(B)$ bounded for all $1 \le i \le s$, such that $F(T_1,\dots,T_s) \neq \emptyset$. If $A, b \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ satisfies (1.8), $x_0 \in B$, and the following condition is satisfied:*

$$\lim_{n\to\infty} \|T_1 x_{n+1} - T_1 y_n^1\| = 0, \tag{2.1}$$

then iteration (1.7) converges to the unique common fixed point of $T_1,\dots,T_s$, which is the unique fixed point of T_1.

Proof. Any common fixed point of $T_1,\dots,T_s$, in particular, is a fixed point of T_1. However, T_1 can have at most one fixed point since it is strongly pseudocontractive. Let $x^* = F(T_1,\dots,T_s)$. Denote

$$M = \sup_{n\in\mathbb{N}} \{\|T_1 y_n^1\|, \|x_0\|, \|x^*\|\}. \tag{2.2}$$

Then if we assume $\|x_n\| \le M$, by

$$\|x_{n+1}\| \le (1-\alpha_n)\|x_n\| + \alpha_n\|T_1 y_n^1\| \le M, \tag{2.3}$$

we get $\|x_{n+1}\| \le M$.

From (1.2) and (1.10), with

$$\begin{aligned} x &:= (1-\alpha_n)(x_n - x^*), \\ y &:= \alpha_n(T_1 y_n^1 - T_1 x^*), \\ x + y &= x_{n+1} - x^*, \end{aligned} \tag{2.4}$$

we get

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T_1 y_n^1 - T_1 x^*)\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle T_1 y_n^1 - T_1 x^*, j(x_{n+1} - x^*)\rangle \\ &= (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\langle T_1 x_{n+1} - T_1 x^*, j(x_{n+1} - x^*)\rangle \\ &\quad + 2\alpha_n\langle T_1 y_n^1 - T_1 x_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k\|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n\langle T_1 y_n^1 - T_1 x_{n+1}, j(x_{n+1} - x^*)\rangle \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k\|x_{n+1} - x^*\|^2 \\ &\quad + 2\alpha_n\|T_1 y_n^1 - T_1 x_{n+1}\|\|x_{n+1} - x^*\| \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k\|x_{n+1} - x^*\|^2 \\ &\quad + 4\alpha_n\|T_1 y_n^1 - T_1 x_{n+1}\|M. \end{aligned} \tag{2.5}$$

Using (1.8), we obtain

$$(1-\alpha_n)^2 \le 1 - 2\alpha_n + \alpha_n b < 1 - 2\alpha_n + \alpha_n 2(1-k) = 1 - 2\alpha_n k, \tag{2.6}$$

thus,

$$||x_{n+1}-x^*||^2 \le \frac{(1-\alpha_n)^2}{1-2\alpha_n k}||x_n-x^*||^2+\frac{4\alpha_n M}{1-2\alpha_n k}||T_1y_n^1-T_1x_{n+1}||. \tag{2.7}$$

The following inequality is satisfied:

$$\begin{aligned}\frac{(1-\alpha_n)^2}{1-2\alpha_n k} &= \frac{(1-\alpha_n)^2(1-2\alpha_n k+2\alpha_n k)}{1-2\alpha_n k} = (1-\alpha_n)^2\left(1+\frac{2\alpha_n k}{1-2\alpha_n k}\right)\\ &= (1-\alpha_n)^2+\frac{2\alpha_n k(1-\alpha_n)^2}{1-2\alpha_n k} \le (1-\alpha_n)^2+2\alpha_n k \le 1-2\alpha_n+\alpha_n b+2\alpha_n k\\ &= 1-(2(1-k)-b)\alpha_n \le 1-(2(1-k)-b)A.\end{aligned} \tag{2.8}$$

Substituting (2.6) and (2.8) into (2.7), we obtain

$$||x_{n+1}-x^*||^2 \le (1-(2(1-k)-b)A)||x_n-x^*||^2+\frac{4bM}{1-2bk}||T_1y_n^1-T_1x_{n+1}||. \tag{2.9}$$

Set

$$\begin{aligned} a_n &:= ||x_n-x^*||^2,\\ t &:= (2(1-k)-b)A \in (0,1),\\ \sigma_n &:= \frac{4bM}{1-2bk}||T_1y_n^1-T_1x_{n+1}||.\end{aligned} \tag{2.10}$$

From (2.1), we know that $\lim_{n\to\infty}\sigma_n=0$; all the assumptions of Lemma 1.4 are fulfilled and consequently we have $\lim_{n\to\infty}\|x_n-x^*\|=0$. □

In Theorem 2.1, $\{\alpha_n\}$ does not converge to zero, while in [6], $\{\alpha_n\}$ converges to zero.

Theorem 2.2 [6]. *Let $s\ge 2$ be fixed, X a real Banach space with a uniformly convex dual, and B a nonempty, closed, convex subset of X. Let $T_1: B\to B$ be a strongly pseudocontractive operator and $T_2,\ldots,T_s: B\to B$, with $T_i(B)$ bounded for all $1\le i\le s$, such that $F(T_1,\ldots,T_s)\neq\emptyset$. If $\{\alpha_n\}\subset(0,1)$ satisfies $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=+\infty$, and $\{\beta_n^i\}\subset[0,1)$, $i=1,\ldots,s-1$, satisfy $\lim_{n\to\infty}\beta_n^1=0$, then iteration (1.7) converges to a fixed point of $T_1,\ldots,T_s$.*

The Banach space in Theorem 2.1 contains no restrictions.

3. Further results

Denote by I the identity map.

Remark 3.1. Let $T,S: X\to X$ and let $f\in X$ be given. Then,

(i) a fixed point for the map $Tx=f+(I-S)x$, for all $x\in X$, is a solution for $Sx=f$;
(ii) a fixed point for $Tx=f-Sx$ is a solution for $x+Sx=f$.

Remark 3.2 [5]. The following are true.

(i) The operator $T: X \to X$ is a (strongly) pseudocontractive map if and only if $(I - T): X \to X$ is (strongly) accretive.

(ii) If $S: X \to X$ is an accretive map, then $T = f - S: X \to X$ is a strongly pseudocontractive map.

We consider iteration (1.7), with $T_i x = f_i + (I - S_i)x$, $1 \le i \le s$ and $s \ge 2$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n^i\} \subset [0,1)$, $i = 1,\dots,s-1$ satisfying (1.8):

$$\begin{gathered} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\big(f_1 + (I - S_1)y_n^1\big), \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i\big(f_{i+1} + (I - S_{i+1})y_n^{i+1}\big), \quad i = 1,\dots,s-2, \\ y_n^{s-1} = (1-\beta_n^{s-1})x_n + \beta_n^{s-1}\big(f_{s-1} + (I - S_s)x_n\big). \end{gathered} \tag{3.1}$$

Theorem 2.1, Remark 3.1(i), and Remark 3.2(i) lead to the following result.

Corollary 3.3. *Let $s \ge 2$ be fixed, X a real Banach space, and $S_1: X \to X$ a strongly accretive operator, $S_2,\dots,S_s: X \to X$, such that the equations $S_i x = f_i$, $1 \le i \le s$, have a common solution and $T_i(X)$, $1 \le i \le s$, are bounded. If $A, b \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ satisfies (1.8), and condition (2.1) is satisfied, then iteration (3.1) converges to a common solution of $S_i x = f_i$, $1 \le i \le s$.*

We consider iteration (1.7), with $T_i x = f_i - S_i x$, $1 \le i \le s$, and $s \ge 2$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n^i\} \subset [0,1)$, $i = 1,\dots,s-1$, satisfying (1.8):

$$\begin{gathered} x_{n+1} = (1-\alpha_n)x_n + \alpha_n\big(f_1 - S_1 y_n^1\big), \\ y_n^i = (1-\beta_n^i)x_n + \beta_n^i\big(f_{i+1} - S_{i+1} y_n^{i+1}\big), \quad i = 1,\dots,s-2, \\ y_n^{s-1} = (1-\beta_n^{s-1})x_n + \beta_n^{s-1}\big(f_{s-1} - S_s x_n\big). \end{gathered} \tag{3.2}$$

Theorem 2.1, Remark 3.1(ii), and Remark 3.2(ii) lead to the following result.

Corollary 3.4. *Let $s \ge 2$ be fixed, X a real Banach space, and $S_1: X \to X$ an accretive operator, $S_2,\dots,S_s: X \to X$, such that the equations $x + S_i x = f_i$, $1 \le i \le s$, have a common solution and $S_i(X)$, $1 \le i \le s$, are bounded. If $A, b \in (0,1)$, $\{\alpha_n\} \subset (0,1)$ satisfies (1.8), and condition (2.1) is satisfied, then iteration (3.2) converges to a common solution of $x + S_i x = f_i$, $1 \le i \le s$.*

4. Numerical examples

Let $X = \mathbb{R}^2$ be the euclidean plane, consider $x = (a,b) \in \mathbb{R}^2$, with $x^\perp = (b,-a) \in \mathbb{R}^2$. We know that $\langle x, x^\perp \rangle = 0$, $\|x\| = \|x^\perp\|$, $\langle x^\perp, y^\perp \rangle = \langle x, y \rangle$, $\|x^\perp - y^\perp\| = \|x - y\|$, and $\langle x^\perp, y \rangle + \langle x, y^\perp \rangle = 0$, for all $x, y \in \mathbb{R}^2$. Denote by B the closed unit ball. In [1], we can get the following example in which Ishikawa iteration (1.6) converges and (1.5) is not convergent.

Table 4.1

\Iteration (1.7)	Case 1	Case 2
Step 10	(0.2217,0.1480)	(0.0151, −0.0023)
Step 15	(0.1837,0.1184)	(0.0017, −0.0006)
Step 20	(0.1603,0.1015)	(0.0002, −0.0001)
Step 21	(0.1566,0.0989)	10^{-3} · (0.1156, −0.0686)
Step 22	(0.1531,0.0965)	10^{-4} · (0.7406, −0.4641)
Step 23	(0.1499,0.0942)	10^{-4} · (0.4743, −0.3129)
Step 24	(0.1468,0.0921)	10^{-4} · (0.3037, −0.2103)
Step 25	(0.1440,0.0902)	10^{-4} · (0.1945, −0.1409)

Example 4.1 [1]. Let $H = \mathbb{R}^2$ and let

$$B_1 = \left\{x \in \mathbb{R}^2 : \|x\| \leq \frac{1}{2}\right\}, \qquad B_2 = \left\{x \in \mathbb{R}^2 : \frac{1}{2} \leq \|x\| \leq 1\right\}. \tag{4.1}$$

The map $T : B \to B$ is given by

$$Tx = \begin{cases} x + x^{\perp}, & x \in B_1 \\ \dfrac{x}{\|x\|} - x + x^{\perp}, & x \in B_2. \end{cases} \tag{4.2}$$

Then the following are true:

(i) T is Lipschitz and pseudocontractive;

(ii) for all $(\alpha_n)_n \subset (0,1)$, the Mann iteration does not converge to the fixed point of T (which is the point $(0,0)$ and it is unique).

The main result from [2] assures the convergence of the Ishikawa iteration (1.6) applied to the map T given by (4.2). The convergence is very slow. In [6], for the same T, it was shown that iteration (1.7) converges faster. Here, we give an example for which (1.7) with $\{\alpha_n\}$ satisfying (1.8) converges even faster as in [6].

Case 1 [6]. Consider now $T_1(x,y) = 0.5 \cdot (x,y)$, for all $(x,y) \in B$, $T_2 = T$, and $s = 2$, where T is given by (4.2), the initial point $x_0 = (0.5,0.7)$, and $\alpha_n = \beta_n = 1/(n+1)$ in (1.7). The main result from [6] assures the convergence of (1.7).

Case 2. Consider $T_1(x,y) = 0.5 \cdot (x,y)$, for all $(x,y) \in B$, $T_2 = T$, and $s = 2$, where T is given by (4.2), the initial point $x_0 = (0.5,0.7)$, $\alpha_n = 0.7$, for all $n \in \mathbb{N}$, and $\beta_n = 1/(n+1)$ in (1.7). The fixed point for both functions is $(0,0)$. Observe that $k = 0.5$, and $\{\alpha_n\}$ satisfies (1.8):

$$A = 0.7 = \alpha_n = b \leq 2(1-k) = 1, \quad \forall n \in \mathbb{N}. \tag{4.3}$$

Note that Mann iteration does not converge for any $\{\alpha_n\} \subset (0,1)$. Using a Matlab program, we obtain Table 4.1.

Case 3. Consider in (1.7) the same T_1, T_2, $s = 2$, and x_0 as in Case 1 and $\alpha_n = \beta_n = 1/\sqrt{n+1}$.

Table 4.2

\Iteration	Case 3 (1.7)	Case 4 (1.7)	Ishikawa iteration
Step 10	$(0.0631, -0.0333)$	$(0.0044, -0.0164)$	$(0.4545, 0.2689)$
Step 15	$(0.0256, -0.0221)$	$(-0.0010, -0.0018)$	$(0.1289, -0.4827)$
Step 20	$(0.0117, -0.0139)$	$10^{-5} \cdot (-22.6516, -11.0267)$	$(-0.4456, -0.1532)$
Step 11	$(0.0101, -0.0126)$	$10^{-5} \cdot (-15.5657, -5.4373)$	$(-0.4651, -0.0274)$
Step 22	$(0.0087, -0.0115)$	$10^{-5} \cdot (-10.5234, -2.3727)$	$(-0.4511, 0.0941)$
Step 23	$(0.0075, -0.0105)$	$10^{-5} \cdot (-7.0134, -0.7743)$	$(-0.4077, 0.2037)$
Step 24	$(0.0066, -0.0096)$	$10^{-5} \cdot (-4.6140, -0.0022)$	$(-0.3407, 0.2954)$
Step 25	$(0.0057, -0.0088)$	$10^{-5} \cdot (-2.9993, 0.3215)$	$(-0.2562, 0.3654)$
Step 1500	—	—	$(0.0790, -0.0311)$

Case 4. Consider in (1.7) T_1, T_2, $s = 2$, and x_0 as above and $\alpha_n = 0.7$, for all $n \in \mathbb{N}$, $\beta_n = 1/\sqrt{n+1}$.

Also, consider the Ishikawa iteration with the same T as in (4.2), $x_0 = (0.5, 0.7)$, $\alpha_n = \beta_n = 1/\sqrt{n+1}$, for all $n \in \mathbb{N}$. The main result from [2] assures the convergence of Ishikawa iteration. Note that in this case the convergence is very slow. Eventually, Example 4.1 assures that for the same map, Mann iteration does not converge. A Matlab program leads to the evaluations illustrated in Table 4.2.

Acknowledgment

The authors are indebted to the referee for carefully reading the paper and for making useful suggestions.

References

[1] C. E. Chidume and S. A. Mutangadura, *An example of the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2359–2363.
[2] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), no. 1, 147–150.
[3] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[4] C. H. Morales and J. S. Jung, *Convergence of paths for pseudocontractive mappings in Banach spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 11, 3411–3419.
[5] B. E. Rhoades and Ş. M. Şoltuz, *The equivalence of Mann iteration and Ishikawa iteration for non-Lipschitzian operators*, Int. J. Math. Math. Sci. **2003** (2003), no. 42, 2645–2651.
[6] ———, *Mean value iteration for a family of functions*, to appear in Nonlinear Funct. Anal. Appl.
[7] Ş. M. Şoltuz, *Some sequences supplied by inequalities and their applications*, Rev. Anal. Numér. Théor. Approx. **29** (2000), no. 2, 207–212.

B. E. Rhoades: Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, USA
E-mail address: rhoades@indiana.edu

Ştefan M. Şoltuz: Institute of Numerical Analysis, P.O. Box 68-1, 400110 Cluj-Napoca, Romania
E-mail address: smsoltuz@gmail.com