THE DOCK PROBLEM REVISITED

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The well-known semi-infinite dock problem of the theory of scattering of surface water waves is reexamined and known results are recovered by utilizing a Fourier type of analysis, giving rise to Carleman-type singular integral equations over semi-infinite ranges.

1. Introduction

The dock problem (cf. [2], [4, 5]), which is that of understanding the scattering of surface water waves by a thin semi-infinite rigid plate floating on the free surface of water of infinite depth, gives rise to the following mixed boundary value problem, involving Laplace's equation in two dimensions, with (x,y) representing the rectangular Cartesian coordinates, assuming linearised theory of water waves:

$$\nabla^2\phi = 0, \quad -\infty < x < \infty,\ y > 0, \tag{1.1}$$

$$K\phi + \phi_y = 0 \quad \text{on } y = 0,\ x < 0, \tag{1.2}$$

$$\phi_y = 0 \quad \text{on } y = 0,\ x > 0, \tag{1.3}$$

$$r\frac{\partial\phi}{\partial r} = 0 \quad \text{as } r = \left(x^2 + y^2\right)^{1/2} \longrightarrow 0, \tag{1.4}$$

$$\phi \text{ and } \phi_x \text{ are continuous at } x = 0,\ y > 0, \tag{1.5}$$

$$\phi \longrightarrow 0 \quad \text{as } y \longrightarrow \infty, \tag{1.6}$$

$$\phi \longrightarrow \begin{cases} e^{-Ky+iKx} + \mathrm{R}e^{-Ky-iKx} & \text{as } x \longrightarrow -\infty, \\ 0 & \text{as } x \longrightarrow \infty. \end{cases} \tag{1.7}$$

Here $\mathrm{Re}\{(g^2/\sigma^3)\phi(x,y)e^{-i\sigma t}\}$ denotes the velocity potential (actual) describing the fluid motion assumed irrotational, where σ is the circular frequency and g is the acceleration due to gravity, $K = \sigma^2/g$, R is the unknown reflection coefficient due to a progressive wave train described by the complex velocity potential function $e^{-Ky+iKx}$ incident on the semi-infinite rigid plate occupying the position $y = 0$, and $x \geq 0$.

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A direct use of a Fourier type of analysis (cf. [7]) is shown to reduce the above boundary value problem to either of two possible singular integral equations of the Carleman type over a semi-infinite range. It is then shown that the closed form solutions of both of these Carleman equations are possible, giving rise to closed form solution of the problem under consideration. The associated reflection coefficient R is determined and is found to agree with the known result. The free surface profile and the pressure distribution on the dock are depicted graphically at initial time $t = 0$ against the distance. These figures coincide with those given in [2].

The present analysis is believed to be more straightforward and simple, as compared to the existing methods of [2] and [4, 5] to handle this class of problems in the theory of surface water waves.

2. The detailed analysis

Using Havelock's expansion of water wave potential (cf. [7]), we look for the following representations of the function $\phi(x,y)$ in the regions $x < 0$ and $x > 0$ $(y > 0)$, satisfying (1.1), (1.2), (1.3), (1.6), and (1.7):

$$\phi(x,y) = e^{-Ky+iKx} + \mathrm{R}e^{-Ky-iKx} + \frac{2}{\pi}\int_0^\infty \frac{A(\xi)}{\xi^2+K^2}L(\xi,y)e^{\xi x}d\xi, \quad \text{for } x < 0, \tag{2.1}$$

$$\phi(x,y) = \frac{2}{\pi}\int_0^\infty \frac{B(\xi)}{\xi}\cos\xi y e^{-\xi x}d\xi, \quad \text{for } x > 0, \tag{2.2}$$

where

$$L(\xi,y) = \xi\cos\xi y - K\sin\xi y, \tag{2.3}$$

and $A(\xi)$, $B(\xi)$ are two unknown functions to be determined along with the unknown reflection coefficient R.

We emphasize, at this stage itself, that the representation (2.2) demands that we must have

$$B(0) = 0 \tag{2.4}$$

to help the integral in (2.2) converge.

The conditions (1.5) give the following relations:

$$\begin{aligned}(1+R)e^{-Ky} + \frac{2}{\pi}\int_0^\infty \frac{A(\xi)}{\xi^2+K^2}L(\xi,y)d\xi &= \frac{2}{\pi}\int_0^\infty \frac{B(\xi)}{\xi}\cos\xi y\,d\xi, \quad y > 0,\\ iK(1-R)e^{-Ky} + \frac{2}{\pi}\int_0^\infty \frac{\xi A(\xi)}{\xi^2+K^2}L(\xi,y)d\xi &= -\frac{2}{\pi}\int_0^\infty B(\xi)\cos\xi y\,d\xi, \quad y > 0.\end{aligned} \tag{2.5}$$

Now there are two possible ways of handling the two relations (2.5) as far as the application of Fourier analysis is concerned. The first possibility is to use a Fourier cosine inversion formula to both the relations (2.5) to determine $B(\xi)$ in terms of $A(\xi)$, and the second possibility is to use the Havelock's expansion theorem (cf. [7]) to determine $A(\xi)$ in terms of $B(\xi)$.

Then by using the first of the above two approaches, we obtain that

$$\frac{B(\xi)}{\xi} = \frac{(1+R)K}{\xi^2+K^2} + \frac{\xi A(\xi)}{\xi^2+K^2} - \frac{2K}{\pi}\int_0^\infty \frac{uA(u)}{(u^2-\xi^2)(u^2+K^2)}du, \quad \xi > 0, \tag{2.6}$$

$$-B(\xi) = \frac{i(1-R)K}{\xi^2+K^2} + \frac{\xi^2 A(\xi)}{\xi^2+K^2} - \frac{2K}{\pi}\int_0^\infty \frac{u^2 A(u)}{(u^2-\xi^2)(u^2+K^2)}du, \quad \xi > 0, \tag{2.7}$$

which on elimination of $B(\xi)$, give rise to the following singular integral equation of the Carleman type:

$$\xi C(\xi) - \frac{K}{\pi}\int_0^\infty \frac{C(u)}{u-\xi}du = -K\left(\frac{1}{\xi - iK} + \frac{R}{\xi + iK}\right), \quad \xi > 0, \tag{2.8}$$

where

$$C(\xi) = \frac{2\xi A(\xi)}{\xi^2+K^2}. \tag{2.9}$$

We observe that (2.8) contains the unknown constant R (the unknown reflection coefficient), and this can be determined by utilizing the convergence criterion (2.4).

Also, by using the second approach, we obtain that

$$A(\xi) = B(\xi) - \frac{2K\xi}{\pi}\int_0^\infty \frac{B(u)}{u(\xi^2-u^2)}du, \quad \xi > 0, \tag{2.10}$$

provided that

$$1 + R = \frac{4K^2}{\pi}\int_0^\infty \frac{B(u)}{u(u^2+K^2)}du, \tag{2.11}$$

$$\xi A(\xi) = -\xi B(\xi) + \frac{2K\xi}{\pi}\int_0^\infty \frac{B(u)}{\xi^2-u^2}du, \quad \xi > 0, \tag{2.12}$$

provided that

$$1 - R = \frac{4iK}{\pi}\int_0^\infty \frac{B(u)}{u^2+K^2}du. \tag{2.13}$$

The following generalized identities (cf. [1]) have been utilized in deriving the results (2.6), (2.7), (2.10), and (2.12):

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon y}\cos uy\cos\xi y\,dy = \frac{\pi}{2}\{\delta(\xi-u)+\delta(\xi+u)\},\\
&\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon y}\sin uy\sin\xi y\,dy = \frac{\pi}{2}\{\delta(\xi-u)-\delta(\xi+u)\},\\
&\lim_{\epsilon\to 0}\int_0^\infty e^{-\epsilon y}\sin uy\cos\xi y\,dy = \frac{u}{u^2-\xi^2},
\end{aligned} \tag{2.14}$$

where $u, \xi > 0$ and $\delta(x)$ is the Dirac delta function.

Also the singular integral occurring in (2.8) and elsewhere in the present work is to be understood as its Cauchy principal value (cf. [3]).

Eliminating $A(\xi)$ between the relations (2.10) and (2.12), we obtain a second singular integral equation of the Carleman type, as given by

$$\xi B(\xi)+\frac{K}{\pi}\int_0^\infty \frac{B(u)}{u-\xi}du=c, \quad \text{(say) } \xi>0, \tag{2.15}$$

where c can be regarded as an unknown constant to be determined, along with the other unknown constant R (the unknown reflection coefficient), by using the two constraints (2.11) and (2.13).

Now both the integral equations (2.8) and (2.15) are of the same type and each of them can be cast into a Riemann-Hilbert problem involving the complex-plane, with a cut along the positive real axis, which can finally be solved by using standard techniques available in [3] or [6].

The two Riemann-Hilbert problems for the two integral equations (2.8) and (2.15) are given by

$$\begin{aligned} &\Phi^+(\xi)-\frac{\xi+iK}{\xi-iK}\Phi^-(\xi)=-K\left\{\frac{R}{\xi^2+K^2}+\frac{1}{(\xi-iK)^2}\right\}, \quad (\xi>0),\\ &\Lambda^+(\xi)-\frac{\xi-iK}{\xi+iK}\Lambda^-(\xi)=\frac{c}{\xi+iK}, \quad (\xi>0), \end{aligned} \tag{2.16}$$

respectively, involving the two sectionally analytic functions $\Phi(\zeta)$ and $\Lambda(\zeta)$, $[\zeta=\xi+i\eta]$, in the cut ζ plane, where

$$\left.\begin{aligned} \Phi(\zeta)&=\frac{1}{2\pi i}\int_0^\infty\frac{C(u)}{u-\zeta}du,\\ \Lambda(\zeta)&=\frac{1}{2\pi i}\int_0^\infty\frac{B(u)}{u-\zeta}du, \end{aligned}\right\} \quad \eta\neq 0, \tag{2.17}$$

with $\Phi^+(\xi)=\Phi(\xi+i0)$, $\Phi^-(\xi)=\Phi(\xi-i0)$, $\Lambda^+(\xi)=\Lambda(\xi+i0)$, and $\Lambda^-(\xi)=\Lambda(\xi-i0)$.

The solutions of the above two Riemann-Hilbert problems are straightforward (cf. [3]) and we find that

$$\begin{aligned} \Phi(\zeta)&=-\frac{K}{2\pi i}\Phi_0(\zeta)\int_0^\infty\left\{\frac{R}{u^2+K^2}+\frac{1}{(u-iK)^2}\right\}\frac{1}{\Phi_0^+(u)(u-\zeta)}du,\\ \Lambda(\zeta)&=\frac{\Lambda_0(\zeta)}{2\pi i}\int_0^\infty\frac{c}{\Lambda_0^+(u)(u+iK)(u-\zeta)}du, \end{aligned} \tag{2.18}$$

with

$$\Phi_0(\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u+iK)/(u-iK))}{u-\zeta}du\right], \tag{2.19a}$$

$$\Lambda_0(\zeta) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u-iK)/(u+iK))-2\pi i}{u-\zeta}du\right]. \tag{2.19b}$$

The solutions of the integral equations (2.8) and (2.15) can be finally determined by using the Plemelj's formulae as given by

$$C(\xi) = \Phi^+(\xi) - \Phi^-(\xi), \tag{2.20}$$

$$B(\xi) = \Lambda^+(\xi) - \Lambda^-(\xi). \tag{2.21}$$

Evaluating the various integrals appearing in the relations (2.18), by using standard techniques involving contour integration (cf. [1] and Appendix A to the present paper), we find that

$$C(\xi) = -\frac{K}{\xi^2+K^2}\frac{\Phi_0^+(\xi)}{\Phi_0(iK)} - \frac{KR}{(\xi+iK)^2}\frac{\Phi_0^+(\xi)}{\Phi_0(-iK)}, \quad \xi > 0, \tag{2.22}$$

$$B(\xi) = \frac{cD_1}{\pi}\frac{\Lambda_0^+(\xi)}{\xi - iK}, \tag{2.23}$$

where D_1 is an unknown constant.

We next use the relations (2.6) and (2.9) along with the result (2.22) and find that

$$B(\xi) = \frac{K}{2}\Phi_0(-\xi)\left\{\frac{1}{(\xi+iK)\Phi_0(iK)} + \frac{R}{(\xi-iK)\Phi_0(-iK)}\right\} \tag{2.24}$$

is obtained after evaluating several integrals by using appropriate contour integration procedures (see Appendix B).

Then, by using the condition (2.4), we find that we must have (see Appendix C.1)

$$R = \frac{\Phi_0(-iK)}{\Phi_0(iK)} = \exp\left(-\frac{i\pi}{4}\right) \tag{2.25}$$

obtained by using the relation (2.19a).

Again, by using the result (2.23) in the two relations (2.11) and (2.13), and by evaluating the various integrals (see Appendix C.2), after using the relation (2.19b), we find that

$$R = \frac{\Lambda_0(iK)}{\Lambda_0(-iK)} = \exp\left(-\frac{i\pi}{4}\right), \tag{2.26}$$

$$\frac{cD_1}{\pi} = \frac{K}{2\Lambda_0(-iK)}. \tag{2.27}$$

We thus find that the mixed boundary value problem under consideration gets solved completely either by the aid of the relations (2.24), (2.25), (2.22), and (2.9), or by the aid of the relations (2.23), (2.26), (2.27), and (2.12), which determine the unknown functions $A(\xi)$ and $B(\xi)$ and the unknown reflection coefficient R completely so that the complete knowledge of the potential $\phi(x,y)$ can be obtained by using the relations (2.1) and (2.2).

We find that the value of R is $e^{-i\pi/4}$, as obtained by Holford [4, 5], by using a completely different analysis.

It is rather interesting to verify (see Appendix D) that the two representations of $B(\xi)$, as given by the relations (2.23) along with (2.27) and (2.24) along with (2.25), are identical.

3. Discussion

The exact form of $\phi(x,y)$ can be obtained from (2.1) for $x<0$ and from (2.2) for $x>0$ after substituting $A(\xi)$ and $B(\xi)$ in terms of $\Phi(\xi)$. This should coincide with the result for the potential function (except for a multiplying constant) given in [2] ($\mathrm{Re}\chi^R(z)$ given there). However, this is not verified here directly. Instead we obtain here the free surface depression $\eta(x,t)$ and the pressure $p(x,t)$ on the dock by using Bernoulli's equation, and depict them graphically against the nondimensional distance Kx at time $t=0$ (actually $K\eta(x,0)$ and $Kp(x,0)/\rho g$) for $x<0$ and $x>0$, respectively.

Using Bernoulli's equation, the free surface distribution $\eta(x,t)$ $(x<0)$ is obtained as

$$\begin{aligned}\eta(x,t) &= -\frac{1}{K}\,\mathrm{Re}\left\{i\phi(x,0)e^{-i\sigma t}\right\}\\ &= \frac{1}{K}\left[\sin(Kx-\sigma t)-\sin\left(\frac{\pi}{4}+Kx+\sigma t\right)+\frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{8}+\sigma t\right)I(Kx)\right],\end{aligned}\tag{3.1}$$

where

$$I(s)=\int_0^{\pi/2}\cos\theta\exp\left\{s\cot\theta+\frac{\sin\theta}{\pi}\int_0^{\pi/2}\frac{\alpha}{\sin\alpha\sin(\theta-\alpha)}d\alpha\right\}d\theta.\tag{3.2}$$

Similarly, the pressure on the dock $p(x,t)$ $(x>0)$ is obtained as

$$p(x,t)=\frac{\rho g}{K}\,\mathrm{Re}\left\{i\phi(x,0)e^{-i\sigma t}\right\}=\frac{\rho g}{K}\frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{8}+\sigma t\right)J(Kx),\tag{3.3}$$

where

$$J(s)=\int_0^{\pi/2}\exp\left\{-s\cot\theta+\frac{\sin\theta}{\pi}\int_0^{\pi/2}\frac{\alpha}{\sin\alpha\sin(\theta+\alpha)}d\alpha\right\}d\theta.\tag{3.4}$$

Figures 3.1 and 3.2 depict, respectively, the free surface profile $\eta(x,0)$ against $x(x<0)$ and the pressure distribution $p(x,0)$ on the dock also against $x(x>0)$; η, p, x being nondimensionalised as $K\eta$, $Kp/\rho g$, Kx. These curves in Figures 3.1 and 3.2 can be identified with the curves given in [2] obtained from the potential function $\mathrm{Re}\chi^R(z)$ given there. This indirectly verifies that the solution for the potential function obtained here coincides with the solution $\mathrm{Re}\chi^R(z)$ given in [2] (except for a multiplying constant).

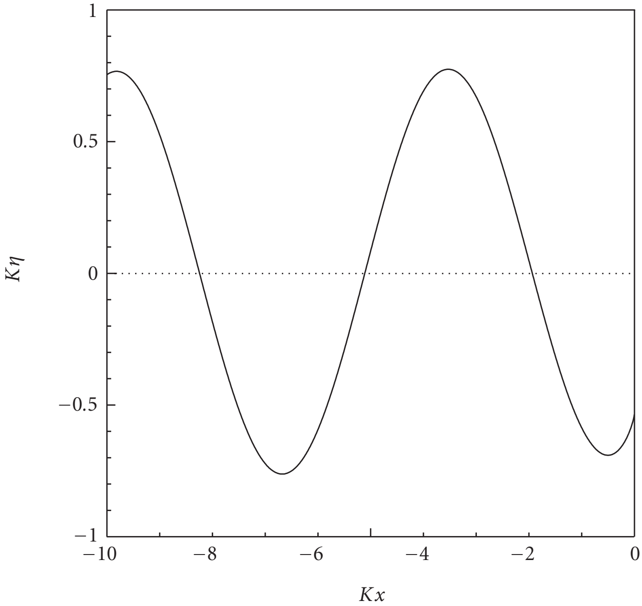

Figure 3.1. Free surface profile at $t = 0$.

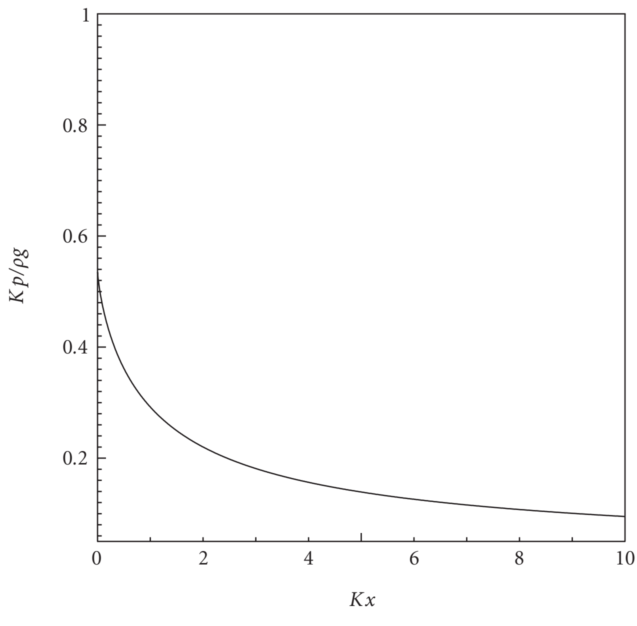

Figure 3.2. Distribution of pressure on the dock at $t = 0$.

Appendices

A. Determination of $C(\xi)$

In this appendix we will describe the contour integration procedure to obtain the value of $C(\xi)$, as given by the relation (2.22).

The relation (2.20) gives

$$C(\xi) = \Phi^+(\xi) - \Phi^-(\xi) = -\frac{K\xi}{\xi + iK}\left\{\frac{R}{\xi^2 + K^2} + \frac{1}{(\xi - iK)^2}\right\}$$
$$-\frac{K^2\Phi_0^+(\xi)}{\pi(\xi + iK)}\int_0^\infty \left\{\frac{R}{u^2 + K^2} + \frac{1}{(u - iK)^2}\right\}\frac{1}{\Phi_0^+(u)(u-\xi)}du, \quad \xi > 0. \tag{A.1}$$

The integrals appearing in (A.1) can be evaluated by considering integrals of the form

$$I(\zeta) = \int_\Gamma \frac{P(\tau)}{Q(\tau)}\frac{1}{\Phi_0(\tau)(\tau - \zeta)}d\tau, \tag{A.2}$$

with Γ a positively oriented closed contour consisting of a loop around the positive real axis and a circle of large radius with centre at the origin, in the complex τ-plane, and $P(\tau)$ and $Q(\tau)$ are polynomials in τ. If these polynomials are such that the contribution to the integral in (A.2) over the circle of large radius vanishes, then

$$I(\zeta) = \int_\Gamma \frac{P(u)}{Q(u)}\left\{\frac{1}{\Phi_0^+(u)} - \frac{1}{\Phi_0^-(u)}\right\}\frac{1}{(u-\zeta)}du$$
$$= -2iK\int_0^\infty \frac{P(u)}{Q(u)}\frac{1}{\Phi_0^+(u)(u - iK)(u-\zeta)}du \tag{A.3}$$

after using the relation

$$\Phi_0^+(\xi) = \frac{\xi + iK}{\xi - iK}\Phi_0^-(\xi). \tag{A.4}$$

Thus by using $P(\tau) = 1$ and $Q(\tau) = -2iK(\tau + iK)$, it is observed that

$$I_1(\zeta) = \int_0^\infty \frac{1}{(u^2 + K^2)\Phi_0^+(u)(u-\zeta)}du$$
$$= -\frac{1}{2iK}\int_\Gamma \frac{d\tau}{(\tau + iK)\Phi_0(\tau)(\tau - \zeta)}$$
$$= -\frac{\pi}{K}\frac{1}{\zeta + iK}\left\{\frac{1}{\Phi_0(\zeta)} - \frac{1}{\Phi_0(-iK)}\right\}. \tag{A.5}$$

Similarly, by choosing $P(\tau) = 1$ and $Q(\tau) = -2iK(\tau - iK)$ in (A.3),

$$\begin{aligned} I_2(\zeta) &= \int_0^\infty \frac{1}{(u-iK)^2\Phi_0^+(u)(u-\zeta)}du \\ &= -\frac{1}{2iK}\int_\Gamma \frac{d\tau}{(\tau-iK)\Phi_0(\tau)(\tau-\zeta)} \\ &= -\frac{\pi}{K}\frac{1}{\zeta-iK}\left\{\frac{1}{\Phi_0(\zeta)} - \frac{1}{\Phi_0(iK)}\right\}. \end{aligned} \tag{A.6}$$

Using Plemelj's formulae,

$$\begin{aligned} \int_0^\infty \frac{1}{(u^2+K^2)\Phi_0^+(u)(u-\xi)}du &= \frac{1}{2}\{I_1^+(\xi)+I_1^-(\xi)\} \\ &= \frac{\pi}{K}\frac{1}{\xi+iK}\left\{\frac{1}{\Phi_0(-iK)} - \frac{\xi}{(\xi-iK)\Phi_0^+(\xi)}\right\}, \end{aligned} \tag{A.7}$$

$$\begin{aligned} \int_0^\infty \frac{1}{(u-iK)^2\Phi_0^+(u)(u-\xi)}du &= \frac{1}{2}\left\{I_2^+(\xi)+I_2^-(\xi)\right\} \\ &= \frac{\pi}{K}\frac{1}{\xi-iK}\left\{\frac{1}{\Phi_0(iK)} - \frac{\xi}{(\xi-iK)\Phi_0^+(\xi)}\right\}. \end{aligned} \tag{A.8}$$

Using (A.7) and (A.8) in (A.1), $C(\xi)$ is obtained as

$$C(\xi) = -\frac{K}{\xi^2+K^2}\frac{\Phi_0^+(\xi)}{\Phi_0(iK)} - \frac{KR}{(\xi+iK)^2}\frac{\Phi_0^+(\xi)}{\Phi_0(-iK)}, \quad \xi > 0. \tag{A.9}$$

B. Proof of (2.24)

Here we will describe the details of the procedure to derive the relation (2.24).

On using (2.21), we obtain

$$\begin{aligned} B(\xi) = {} & \frac{(1+R)K\xi}{\xi^2+K^2} - \frac{K\xi}{2(\xi^2+K^2)}\left\{\frac{\Phi_0^+(\xi)}{\Phi_0(iK)} + R\frac{\Phi_0^-(\xi)}{\Phi_0(-iK)}\right\} \\ & + \frac{K^2\xi}{\pi}\int_0^\infty \frac{1}{(u^2-\xi^2)(\xi^2+K^2)}\left\{\frac{\Phi_0^+(u)}{\Phi_0(iK)} + R\frac{\Phi_0^-(u)}{\Phi_0(-iK)}\right\}du. \end{aligned} \tag{B.1}$$

The integrals in (B.1) can be evaluated by considering the integral

$$J(\zeta) = \int_\Gamma \frac{P(\tau)}{Q(\tau)}\frac{\Phi_0(\tau)}{(\tau^2-\zeta^2)}d\tau, \tag{B.2}$$

where Γ is the same as in (A.2) and $P(\tau), Q(\tau)$ are polynomials such that the contribution to the integral in (B.2) from the circle of large radius vanishes. We obtain

$$\int_0^\infty \frac{\Phi_0^+(u)}{(u^2+K^2)(u^2-\zeta^2)}du = \frac{\pi}{K}\left\{\frac{\Phi_0(\zeta)}{2\zeta(\zeta-iK)} + \frac{\Phi_0(-\zeta)}{2\zeta(\zeta+iK)} - \frac{\Phi_0(iK)}{(\zeta^2+K^2)}\right\}, \tag{B.3}$$

$$\int_0^\infty \frac{\Phi_0^-(u)}{(u^2+K^2)(u^2-\zeta^2)}du = \frac{\pi}{K}\left\{\frac{\Phi_0(\zeta)}{2\zeta(\zeta+iK)} + \frac{\Phi_0(-\zeta)}{2\zeta(\zeta-iK)} - \frac{\Phi_0(-iK)}{(\zeta^2+K^2)}\right\}, \tag{B.4}$$

where $\zeta = \xi + i\eta$ $(\xi > 0)$. Hence the use of Plemelj's formulae produces

$$\int_0^\infty \frac{\Phi_0^+(u)}{(u^2+K^2)(u^2-\xi^2)}du = \frac{\pi}{K}\left\{\frac{\Phi_0^+(\xi)}{2(\xi^2+K^2)} + \frac{\Phi_0(-\xi)}{2\xi(\xi+iK)} - \frac{\Phi_0(iK)}{(\xi^2+K^2)}\right\}, \tag{B.5}$$

$$\int_0^\infty \frac{\Phi_0^-(u)}{(u^2+K^2)(u^2-\xi^2)}du = \frac{\pi}{K}\left\{\frac{\Phi_0^-(\xi)}{2(\xi^2+K^2)} + \frac{\Phi_0(-\xi)}{2\xi(\xi-iK)} - \frac{\Phi_0(-iK)}{(\xi^2+K^2)}\right\}. \tag{B.6}$$

Using the results of (B.6) in (B.1), ultimately $B(\xi)$ is obtained as

$$B(\xi) = \frac{K}{2}\Phi_0(-\xi)\left\{\frac{1}{(\xi+iK)\Phi_0(iK)} + \frac{R}{(\xi-iK)\Phi_0(-iK)}\right\}. \tag{B.7}$$

C. Evaluation of R

C.1. Here we will prove the result (2.25).

From (2.19a), we find that

$$\ln\Phi_0(z) = \frac{1}{\pi}\int_0^{\pi/2} \ln\left(\frac{z-K\tan\theta}{z}\right)d\theta \tag{C.1}$$

so that

$$\ln\left\{\frac{\Phi_0(iK)}{\Phi_0(-iK)}\right\} = \frac{1}{\pi}\int_0^{\pi/2}\ln\left(\frac{i-\tan\theta}{i+\tan\theta}\right)d\theta = \frac{i\pi}{4}. \tag{C.2}$$

Hence we obtain that

$$R = e^{-i\pi/4}. \tag{C.3}$$

C.2. To determine the values of $cD_1/\pi = D$ (say) and R, the relation (2.23) is used in the relations (2.11) and (2.13). This gives rise to the relations

$$\begin{aligned} 1+R &= \frac{4DK^2}{\pi}\int_0^\infty \frac{\Lambda_0^+(\xi)}{\xi(\xi-iK)(\xi^2+K^2)}d\xi, \\ 1-R &= \frac{4iDK}{\pi}\int_0^\infty \frac{\Lambda_0^+(\xi)}{(\xi-iK)(\xi^2+K^2)}d\xi. \end{aligned} \tag{C.4}$$

The integrals in (C.4) can be evaluated by considering integrals of the form (A.2). Thus,

$$\begin{aligned} 1+R &= \frac{2D}{K}\left\{\Lambda_0(iK)+\Lambda_0(-iK)\right\}, \\ 1-R &= \frac{2D}{K}\left\{\Lambda_0(-iK)-\Lambda_0(iK)\right\} \end{aligned} \tag{C.5}$$

so that

$$R=\frac{\Lambda_0(iK)}{\Lambda_0(-iK)}, \qquad D=\frac{cD_1}{\pi}=\frac{K}{2}\frac{1}{\Lambda_0(-iK)}. \tag{C.6}$$

D. Equivalence of (2.23) and (2.24)

Here we prove the following results:

$$\frac{\Lambda_0(-iK)}{\Phi_0(iK)}=\frac{1}{2}, \tag{D.1}$$

$$\frac{\Lambda_0^+(\xi)}{\Phi_0(-\xi)}=\frac{\xi}{\xi+iK} \tag{D.2}$$

so that the expressions as given by the relations (2.23) along with (2.27) and (2.24) along with (2.25) represent the same function $B(\xi)$.

To show (D.1), it may be noted from (2.19a) and (2.19b) that

$$\begin{aligned} \Phi_0(iK) &= \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u+iK)/(u-iK))}{u-iK}du\right], \\ \Lambda_0(-iK) &= \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u-iK)/(u+iK))-2\pi i}{u+iK}du\right]. \end{aligned} \tag{D.3}$$

Using the result

$$\ln\left(\frac{u-iK}{u+iK}\right)+\ln\left(\frac{u+iK}{u-iK}\right)=2\pi i, \tag{D.4}$$

it is found that

$$\begin{aligned} \frac{\Lambda_0(-iK)}{\Phi_0(iK)} &= \exp\left[-\frac{1}{\pi i}\int_0^\infty \ln\left(\frac{u+iK}{u-iK}\right)\frac{u}{u^2+K^2}du\right] \\ &= \exp\left[-\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2}-\theta\right)\tan\theta d\theta\right]=\frac{1}{2}. \end{aligned} \tag{D.5}$$

To show (D.2), the results in (2.19b) and (D.4) are used to obtain

$$\Lambda_0^+(\xi)=\exp\left[\frac{1}{2}\ln\left(\frac{\xi-iK}{\xi+iK}\right)-\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u+iK)/(u-iK))}{u-\xi}du\right], \quad \xi>0. \tag{D.6}$$

Also, from (2.19a), it is seen that

$$\Phi_0(-\xi) = \exp\left[\frac{1}{2\pi i}\int_0^\infty \frac{\ln((u+iK)/(u-iK))}{u+\xi}du\right], \quad \xi > 0. \tag{D.7}$$

Hence

$$\frac{\Lambda_0^+(\xi)}{\Phi_0(-\xi)} = \left(\frac{\xi - iK}{\xi + iK}\right)^{1/2} \exp\left[-\frac{1}{\pi i}\int_0^\infty \ln\left(\frac{u+iK}{u-iK}\right)\frac{u}{u^2-\xi^2}du\right]. \tag{D.8}$$

The integral in (D.8) can be evaluated and its value is $2\pi i \ln\{(\xi^2+K^2)^{1/2}/\xi\}$. Substituting this value in (D.8), the result in (D.2) is obtained.

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