NEW ERROR INEQUALITIES FOR THE LAGRANGE INTERPOLATING POLYNOMIAL

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A new representation of remainder of Lagrange interpolating polynomial is derived. Error inequalities of Ostrowski-Grüss type for the Lagrange interpolating polynomial are established. Some similar inequalities are also obtained.

1. Introduction

Many error inequalities in polynomial interpolation can be found in [1, 7]. These error bounds for interpolating polynomials are usually expressed by means of the norms $\|\cdot\|_p$, $1 \le p \le \infty$. Some new error inequalities (for corrected interpolating polynomials) are given in [10, 11]. The last mentioned inequalities are similar to error inequalities obtained in recent years in numerical integration and they are known in the literature as inequalities of Ostrowski (or Ostrowski-like, Ostrowski-Grüss) type. For example, in [9] we can find inequalities of Ostrowski-Grüss type for the well-known Simpson's quadrature rule,

$$\left| \int_{x_0}^{x_2} f(t)dt - \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] \right| \le C_n(\Gamma_n - \gamma_n)h^{n+1}, \tag{1.1}$$

where $x_i = x_0 + ih$, for $h > 0$, $i = 1,2$, γ_n, Γ_n are real numbers such that $\gamma_n \le f^{(n)}(t) \le \Gamma_n$, for all $t \in [x_0, x_2]$, and C_n are constants, $n \in \{1,2,3\}$.

The inequalities of Ostrowski type can be also found in [2, 3, 4, 5, 6, 12]. In some of the mentioned papers, we can find estimations for errors of quadrature formulas which are expressed by means of the differences $\Gamma_k - \gamma_k$, $S - \gamma_k$, $\Gamma_k - S$, where Γ_k, γ_k are real numbers such that $\gamma_k \le f^{(k)}(t) \le \Gamma_k$, $t \in [a,b]$ (k is a positive integer while $[a,b]$ is an interval of integration) and $S = [f^{(k-1)}(b) - f^{(k-1)}(a)]/(b-a)$. It is shown that the estimations expressed in such a way can be much better than the estimations expressed by means of the norms $\|f^{(k)}\|_p$, $1 \le p \le \infty$.

As we know there is a close relationship between interpolation polynomials and quadrature rules. Thus, it is a natural try to establish similar error inequalities in polynomial interpolation.

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We first establish general error inequalities, expressed by means of $\|f^{(k)} - P_m\|$, where P_m is any polynomial of degree m and then we obtain inequalities of the above mentioned types. For that purpose, we derive a new representation of remainder of the interpolating polynomial. This is done in Section 2. In Section 3, we obtain the error inequalities of the above-mentioned types. In Section 4, we give some results for derivatives.

Finally, we emphasize that the usual error inequalities in polynomial interpolation (for the Lagrange interpolating polynomial $L_n(x)$) are given by means of the $(n+1)$th derivative while in this paper we can find these error inequalities expressed by means of the kth derivative for $k = 1,2,\dots,n$.

2. Representation of remainder

Let $D = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be a given function. The Lagrange interpolation polynomial is given by

$$L_n(x) = \sum_{i=0}^{n} p_{ni}(x) f(x_i), \tag{2.1}$$

where

$$p_{ni}(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}, \tag{2.2}$$

for $i = 0, 1,\dots,n$. We have the Cauchy relations [7, pages 160-161],

$$\sum_{i=0}^{n} p_{ni}(x) = 1, \tag{2.3}$$

$$\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^j = 0, \quad j = 1,2,\dots,n. \tag{2.4}$$

Let $\bar{D} = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given uniform subdivision of the interval $[a,b]$, that is, $x_i = x_0 + ih$, $h = (b-a)/n$, $i = 0,1,2,\dots,n$. Then the Lagrange interpolating polynomial is given by

$$L_n(x) = L_n(x_0 + th) = (-1)^n \frac{t(t-1)\cdots(t-n)}{n!} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{f(x_i)}{t-i}, \tag{2.5}$$

where $t \notin \{0,1,2,\dots,n\}$, $0 < t < n$.

Lemma 2.1. *Let $P_m(t)$ be an arbitrary polynomial of degree $\le m$ and let $p_{ni}(x)$ be defined by (2.2). Then*

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = 0, \tag{2.6}$$

for $0 \le k+m \le n-1$ and $x \in [a,b]$.

Proof. Let x be a given real number. Then we have

$$P_m(t) = \sum_{j=0}^{m} c_j (x-t)^j, \tag{2.7}$$

for some coefficients $c_j = c_j(x)$, $j = 0,1,2,\dots,m$. (This is a consequence of the Taylor formula.) Thus,

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = \sum_{j=0}^{m} c_j \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} (x-t)^j (t-x_i)^k dt. \tag{2.8}$$

Let $\beta(\cdot,\cdot)$ and $\Gamma(\cdot)$ denote the beta and gamma functions, respectively. We now calculate

$$\begin{aligned}
\int_{x_i}^{x} (x-t)^j (t-x_i)^k dt &= \int_{0}^{x-x_i} (x-x_i-u)^j u^k du \\
&= (x-x_i)^j \int_0^{x-x_i} \left(1-\frac{u}{x-x_i}\right)^j u^k du \\
&= (x-x_i)^{j+k+1} \int_0^1 (1-v)^j v^k dv \\
&= \beta(j+1,k+1)(x-x_i)^{j+k+1} \\
&= \frac{\Gamma(k+1)\Gamma(j+1)}{\Gamma(k+j+2)} (x-x_i)^{j+k+1} \\
&= \frac{k!j!}{(k+j+1)!} (x-x_i)^{j+k+1}.
\end{aligned} \tag{2.9}$$

From (2.8) and (2.9) it follows that

$$\sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt = \sum_{j=0}^{m} c_j \frac{k!j!}{(k+j+1)!} \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{j+k+1}. \tag{2.10}$$

From (2.10) and (2.4) we conclude that (2.6) holds. □

THEOREM 2.2. *Let $f \in C^{n+1}(a,b)$ and let the assumptions of Lemma 2.1 hold. Then*

$$f(x) = L_n(x) + R_{k,m}(x), \tag{2.11}$$

where $L_n(x)$ is given by (2.1) and

$$R_{k,m}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m(t)\right](t-x_i)^k dt. \tag{2.12}$$

Proof. We have

$$R_{k,m}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} f^{(k+1)}(t)(t-x_i)^k dt - \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} P_m(t)(t-x_i)^k dt. \tag{2.13}$$

From (2.13) and (2.6) it follows that

$$R_{k,m}(x)=R_k(x)=\frac{(-1)^k}{k!}\sum_{i=0}^{n}p_{ni}(x)\int_{x_i}^{x}f^{(k+1)}(t)(t-x_i)^k dt. \tag{2.14}$$

For $k=0$ we have

$$\begin{aligned}R_0(x)&=\sum_{i=0}^{n}p_{ni}(x)\int_{x_i}^{x}f'(t)dt\\&=\sum_{i=0}^{n}p_{ni}(x)[f(x)-f(x_i)]=f(x)-L_n(x),\end{aligned} \tag{2.15}$$

since (2.3) holds.

We now suppose that $k\ge 1$. Integrating by parts, we obtain

$$\frac{(-1)^k}{k!}\int_{x_i}^{x}f^{(k+1)}(t)(t-x_i)^k dt=\frac{(-1)^k}{k!}f^{(k)}(x)(x-x_i)^k+\frac{(-1)^{k-1}}{(k-1)!}\int_{x_i}^{x}f^{(k)}(t)(t-x_i)^{k-1}dt. \tag{2.16}$$

In a similar way we get

$$\begin{aligned}&\frac{(-1)^{k-1}}{(k-1)!}\int_{x_i}^{x}f^{(k)}(t)(t-x_i)^{k-1}dt\\&\quad=\frac{(-1)^{k-1}}{(k-1)!}f^{(k-1)}(x)(x-x_i)^{k-1}\frac{(-1)^{k-2}}{(k-2)!}\int_{x_i}^{x}f^{(k-1)}(t)(t-x_i)^{k-2}dt.\end{aligned} \tag{2.17}$$

Continuing in this way, we get

$$\begin{aligned}\frac{(-1)^k}{k!}\int_{x_i}^{x}f^{(k+1)}(t)(t-x_i)^k dt&=\sum_{j=1}^{k}\frac{(-1)^j}{j!}f^{(j)}(x)(x-x_i)^j+\int_{x_i}^{x}f'(t)dt\\&=f(x)-f(x_i)+\sum_{j=1}^{k}\frac{(-1)^j}{j!}f^{(j)}(x)(x-x_i)^j.\end{aligned} \tag{2.18}$$

From (2.14) and (2.18) it follows that

$$\begin{aligned}R_k(x)&=\sum_{i=0}^{n}p_{ni}(x)\left[f(x)-f(x_i)+\sum_{j=1}^{k}\frac{(-1)^j}{j!}f^{(j)}(x)(x-x_i)^j\right]\\&=f(x)-L_n(x)+\sum_{j=1}^{k}\frac{(-1)^j}{j!}f^{(j)}(x)\sum_{i=0}^{n}p_{ni}(x)(x-x_i)^j\\&=f(x)-L_n(x),\quad k=1,2,\dots,n,\end{aligned} \tag{2.19}$$

since (2.3) and (2.4) hold. From (2.14), (2.15), and (2.19) we see that (2.11) holds. □

3. Error inequalities

We now introduce the notations

$$\omega_n(x) = (x-x_0)(x-x_1)\cdots(x-x_n), \tag{3.1}$$

$$C_k(x) = \sum_{i=0}^{n} \frac{|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}||x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.2}$$

$$B_k(x) = \sum_{i=0}^{n} \frac{(S_{ki}-\gamma_{k+1})|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}||x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.3}$$

$$D_k(x) = \sum_{i=0}^{n} \frac{(\Gamma_{k+1}-S_{ki})|x-x_i|^k}{|x_i-x_0|\cdots|x_i-x_{i-1}||x_i-x_{i+1}|\cdots|x_i-x_n|}, \tag{3.4}$$

where $S_{ki} = [f^{(k)}(x) - f^{(k)}(x_i)]/(x-x_i)$, $i = 0,1,\dots,n$, and γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \le f^{(k+1)}(t) \le \Gamma_{k+1}$, $t \in [a,b]$, $k = 0,1,\dots,n-1$.

Let $g \in C(a,b)$. As we know among all algebraic polynomials of degree $\le m$ there exists the only polynomial $P_m^*(t)$ having the property that

$$||g-P_m^*||_\infty \le ||g-P_m||_\infty, \tag{3.5}$$

where $P_m \in \Pi_m$ is an arbitrary polynomial of degree $\le m$. We define

$$E_m(g) = ||g-P_m^*|| = \inf_{P_m\in\Pi_m} ||g-P_m||_\infty. \tag{3.6}$$

Theorem 3.1. *Under the assumptions of Theorem 2.2,*

$$|f(x)-L_n(x)| \le \frac{E_m(f^{(k+1)})}{(k+1)!} C_k(x)|\omega_n(x)|, \tag{3.7}$$

where $C_k(\cdot)$ and $E_m(\cdot)$ are defined by (3.2) and (3.6), respectively.

Proof. Let $P_m(t) = P_m^*(t)$, where $P_m^*(t)$ is defined by (3.6) for the function $g(t) = f^{(k+1)}(t)$. We have

$$\begin{aligned} |R_{k,m}(x)| &= \left| \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_m^*(t)\right](t-x_i)^k dt \right| \\ &\le \frac{||f^{(k+1)}-P_m^*||_\infty}{(k+1)!} C_k(x)|\omega_n(x)| \\ &= \frac{E_m(f^{(k+1)})}{(k+1)!} C_k(x)|\omega_n(x)|, \end{aligned} \tag{3.8}$$

since

$$\left| \int_{x_i}^{x} (t-x_i)^k dt \right| = \frac{|x-x_i|^{k+1}}{k+1}. \tag{3.9}$$

□

Remark 3.2. The above estimate has only theoretical importance, since it is difficult to find the polynomial P^*. In fact, we can find P^* only for some special cases of functions. However, we can use the estimate to obtain some practical estimations—see Theorem 3.3.

Theorem 3.3. *Let the assumptions of Theorem 2.2 hold. If γ_{k+1}, Γ_{k+1} are real numbers such that $\gamma_{k+1} \le f^{(k+1)}(t) \le \Gamma_{k+1}$, $t \in [a,b]$, $k = 0,1,\dots,n-1$, then*

$$|f(x) - L_n(x)| \le \frac{\Gamma_{k+1} - \gamma_{k+1}}{2(k+1)!} C_k(x) |\omega_n(x)|, \tag{3.10}$$

where ω_n and $C_k(\cdot)$ are defined by (3.1) and (3.2), respectively. Also

$$\begin{aligned} |f(x) - L_n(x)| &\le \frac{|\omega_n(x)|}{k!} B_k(x), \\ |f(x) - L_n(x)| &\le \frac{|\omega_n(x)|}{k!} D_k(x), \end{aligned} \tag{3.11}$$

where $B_k(\cdot)$ and $D_k(\cdot)$ are defined by (3.3) and (3.4), respectively.

Proof. We set $P_m(t) = (\Gamma_{k+1} + \gamma_{k+1})/2$ in (2.12). Then we have

$$|f(x) - L_n(x)| = |R_k(x)| \le \frac{1}{k!} \sum_{i=0}^{n} |p_{ni}(x)| \left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_\infty \left| \int_{x_i}^{x} (t - x_i)^k dt \right|. \tag{3.12}$$

We also have

$$\begin{aligned} \left\| f^{(k+1)} - \frac{\Gamma_{k+1} + \gamma_{k+1}}{2} \right\|_\infty &\le \frac{\Gamma_{k+1} - \gamma_{k+1}}{2}, \\ \left| \int_{x_i}^{x} (t - x_i)^k dt \right| &= \frac{|x - x_i|^{k+1}}{k+1}. \end{aligned} \tag{3.13}$$

From the above three relations we get

$$\begin{aligned} |f(x) - L_n(x)| &\le \frac{\Gamma_{k+1} - \gamma_{k+1}}{2(k+1)!} \sum_{i=0}^{n} |p_{ni}(x)| |x - x_i|^{k+1} \\ &= \frac{\Gamma_{k+1} - \gamma_{k+1}}{2(k+1)!} C_k(x) |\omega_n(x)|. \end{aligned} \tag{3.14}$$

The first inequality is proved.

We now set $P_m(t) = \gamma_{k+1}$ in (2.12). Then we have

$$|f(x) - L_n(x)| = |R_k(x)| \le \frac{1}{k!} \sum_{i=0}^{n} |p_{ni}(x)| \left| \int_{x_i}^{x} [f^{(k+1)}(t) - \gamma_{k+1}](t - x_i)^k dt \right|. \tag{3.15}$$

We also have

$$\left| \int_{x_i}^{x} [f^{(k+1)}(t) - \gamma_{k+1}](t-x_i)^k dt \right| \le |x-x_i|^k |f^{(k)}(x) - f^{(k)}(x_i) - \gamma_{k+1}(x-x_i)| = |x-x_i|^{k+1}(S_{ki} - \gamma_{k+1}). \tag{3.16}$$

Thus,

$$\begin{aligned} |f(x) - L_n(x)| &\le \frac{1}{k!}\sum_{i=0}^{n} |p_{ni}(x)|\,|x-x_i|^{k+1}(S_{ki} - \gamma_{k+1}) \\ &= \frac{|\omega_n(x)|}{k!} B_k(x). \end{aligned} \tag{3.17}$$

The second inequality is proved. In a similar way we prove that the third inequality holds. □

Lemma 3.4. *Let $D = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given uniform subdivision of the interval $[a,b]$, that is, $x_i = x_0 + ih$, $h = (b-a)/n$, $i = 0,1,2,\ldots,n$. If $x \in (x_{j-1}, x_j)$, for some $j \in \{1,2,\ldots,n\}$, then*

$$|\omega_n(x)| \le j!(n-j+1)!h^{n+1}, \tag{3.18}$$

$$C_k(x) \le \frac{2^n}{n!}\left\{\frac{1}{2}[n+1+|n-2j+1|]\right\}^k h^{k-n}, \tag{3.19}$$

$$C_k(x)|\omega_n(x)| \le \alpha_{jnk}\frac{n-j+1}{n}\frac{2^n(b-a)^{k+1}}{\binom{n}{j}}, \tag{3.20}$$

where

$$\alpha_{jnk} = \left[\frac{1}{2n}(n+1+|2j-n-1|)\right]^k. \tag{3.21}$$

This lemma is proved in [10].

Remark 3.5. Note that

$$\alpha_{jnk} \le 1 \tag{3.22}$$

and $\alpha_{jnk} = 1$ if and only if $j = 1$ or $j = n$. If we choose $x \in [x_j, x_{j+1}]$, $j = 0,1,\ldots,n-1$, then we get the factor $(j+1)/n$ instead of the factor $(n-j+1)/n$ in (3.20).

Theorem 3.6. *Under the assumptions of Lemma 3.4 and Theorem 3.3,*

$$|f(x) - L_n(x)| \le \frac{\Gamma_{k+1} - \gamma_{k+1}}{(k+1)!}\alpha_{jnk}\frac{n-j+1}{n}\frac{2^{n-1}(b-a)^{k+1}}{\binom{n}{j}}. \tag{3.23}$$

Proof. The proof follows immediately from Theorem 3.3 and Lemma 3.4. □

4. Results for derivatives

Lemma 4.1. *Let* $1 \le j \le n-1$ *and* $j+1 \le r \le n$. *Then*

$$\sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_i)^r = 0. \tag{4.1}$$

Proof. We have (see (2.4))

$$A(x) = \sum_{i=0}^{n} p_{ni}(x)(x-x_i)^r = 0, \quad \text{for } 1 \le r \le n. \tag{4.2}$$

Thus,

$$A'(x) = \sum_{i=0}^{n} p'_{ni}(x)(x-x_i)^r + r\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{r-1} = 0, \tag{4.3}$$

if $1 \le r \le n$. If $n \ge r-1 \ge 1$, that is, $n+1 \ge r \ge 2$, then

$$r\sum_{i=0}^{n} p_{ni}(x)(x-x_i)^{r-1} = 0. \tag{4.4}$$

From (4.3) and (4.4) we get

$$\sum_{i=0}^{n} p'_{ni}(x)(x-x_i)^r = 0, \quad \text{for } 2 \le r \le n. \tag{4.5}$$

(Note that $\{r : 1 \le r \le n\} \cap \{r : 2 \le r \le n+1\} = \{r : 2 \le r \le n\}$. Here we use this fact and similar facts without a special mentioning.)

We now suppose that

$$\sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_i)^r = 0, \tag{4.6}$$

for $j = 1,2,\dots,m$, $m < n-1$ and $j+1 \le r \le n$. We wish to prove that

$$\sum_{i=0}^{n} p_{ni}^{(m+1)}(x)(x-x_i)^r = 0, \quad \text{for } m+2 \le r \le n. \tag{4.7}$$

For that purpose, we first calculate

$$\begin{aligned} A^{(m)}(x) &= \sum_{i=0}^{n} \left[p_{ni}(x)(x-x_i)^r\right]^{(m)} \\ &= \sum_{i=0}^{n}\sum_{k=0}^{m} \binom{m}{k} p_{ni}^{(k)}(x)\frac{r!}{(r-m+k)!}(x-x_i)^{r-m+k} \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k)!}\sum_{i=0}^{n} p_{ni}^{(k)}(x)(x-x_i)^{r-m+k}. \end{aligned} \tag{4.8}$$

We have

$$A^{(m)}(x) = 0, \quad \text{for } r \geq m+1, \tag{4.9}$$

by the above assumption. Thus,

$$A^{(m+1)}(x) = 0. \tag{4.10}$$

On the other hand, we have

$$\begin{aligned} A^{(m+1)}(x) &= \frac{d}{dx} A^{(m)}(x) \\ &= \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k)!} \sum_{i=0}^{n} p_{ni}^{(k+1)}(x)(x-x_i)^{r-m+k} \\ &\quad + \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k-1)!} \sum_{i=0}^{n} p_{ni}^{(k)}(x)(x-x_i)^{r-m+k-1} \\ &= 0. \end{aligned} \tag{4.11}$$

We now rewrite the above relation in the form

$$\begin{aligned} \sum_{i=0}^{n} p_{ni}^{(m+1)}(x)(x-x_i)^{r} + \sum_{k=0}^{m-1} \binom{m}{k} \frac{r!}{(r-m+k)!} \sum_{i=0}^{n} p_{ni}^{(k+1)}(x)(x-x_i)^{r-m+k} \\ + \sum_{k=0}^{m} \binom{m}{k} \frac{r!}{(r-m+k-1)!} \sum_{i=0}^{n} p_{ni}^{(k)}(x)(x-x_i)^{r-m+k-1} = 0. \end{aligned} \tag{4.12}$$

For $r-m+k-1 \geq k+1$, that is, $r \geq m+2$, we have

$$\sum_{i=0}^{n} p_{ni}^{(k)}(x)(x-x_i)^{r-m+k-1} = 0 \tag{4.13}$$

by the above assumption. We also have

$$\sum_{i=0}^{n} p_{ni}^{(k+1)}(x)(x-x_i)^{r-m+k} = 0, \tag{4.14}$$

if $r-m+k \geq k+2$, that is, $r \geq m+2$. Thus (4.7) holds. This completes the proof. □

Theorem 4.2. *Let $f \in C^{n+1}(a,b)$ and let $P_r(t)$ be an arbitrary polynomial of degree $\leq r$ and let $0 \leq k \leq n$, $1 \leq m \leq k$. Then*

$$f^{(m)}(x) = L_n^{(m)}(x) + E_{k,r}(x), \tag{4.15}$$

where

$$E_{k,r}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_r(t) \right] (t-x_i)^k \, dt. \tag{4.16}$$

Proof. We define

$$\begin{aligned} v_i(x) &= \int_{x_i}^{x} \left[f^{(k+1)}(t) - P_r(t) \right] (t - x_i)^k dt \\ &= \int_{x_i}^{x} g(t)(t - x_i)^k dt, \end{aligned} \tag{4.17}$$

where, obviously, $g(t) = f^{(k+1)}(t) - P_r(t)$. We denote

$$R_{k,r}(x) = f(x) - L_n(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}(x) v_i(x), \tag{4.18}$$

see Theorem 2.2. Then we have

$$\begin{aligned} R_{k,r}^{(m)}(x) &= \frac{(-1)^k}{k!} \sum_{i=0}^{n} \left[p_{ni}(x) v_i(x) \right]^{(m)} \\ &= \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m} \binom{m}{j} p_{ni}^{(j)}(x) v_i^{(m-j)}(x) \\ &= \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) v_i(x) + \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-1} \binom{m}{j} p_{ni}^{(j)}(x) v_i^{(m-j)}(x). \end{aligned} \tag{4.19}$$

We introduce the notation

$$B(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-1} \binom{m}{j} p_{ni}^{(j)}(x) v_i^{(m-j)}(x) \tag{4.20}$$

such that

$$R_{k,r}^{(m)}(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} p_{ni}^{(m)}(x) v_i(x) + B(x). \tag{4.21}$$

We now rewrite $B(x)$ in the form

$$B(x) = \frac{(-1)^k}{k!} \sum_{i=0}^{n} \sum_{j=0}^{m-2} \binom{m}{j} p_{ni}^{(j)}(x) v_i^{(m-j)}(x) + \frac{(-1)^k}{k!} m \sum_{i=0}^{n} p_{ni}^{(m-1)}(x) v_i'(x). \tag{4.22}$$

We have

$$v_i'(x) = g(x)(x - x_i)^k \tag{4.23}$$

such that

$$\sum_{i=0}^{n} p_{ni}^{(m-1)}(x) v_i'(x) = g(x) \sum_{i=0}^{n} p_{ni}^{(m-1)}(x)(x - x_i)^k = 0, \tag{4.24}$$

for $k \geq m$—see Lemma 4.1.

We also have

$$v_i^{(m-j)}(x) = \sum_{l=0}^{m-j-1} \binom{m-j-1}{l} g^{(l)}(x) \frac{k!}{(k-m+j+l+1)!}(x-x_i)^{k-m+j+l+1}, \tag{4.25}$$

for $m \geq j+2$ such that

$$\begin{aligned} \sum_{i=0}^{n}\sum_{j=0}^{m-2}\binom{m}{j} p_{ni}^{(j)}(x) v_i^{(m-j)}(x) &= \sum_{j=0}^{m-2}\binom{m}{j}\sum_{l=0}^{m-j-1}\binom{m-j-1}{l}\frac{k!}{(k-m+j+l+1)!} \\ &\quad \times \sum_{i=0}^{n} p_{ni}^{(j)}(x)(x-x_i)^{k-m+j+l+1} \\ &= 0, \end{aligned} \tag{4.26}$$

if $k-m+j+l+1 \geq j+1$, that is, $k \geq m$, since $l \geq 0$—see also Lemma 4.1. Hence, $B(x) = 0$ in all cases. Now from (4.21) it follows that

$$\begin{aligned} R_{k,r}^{(m)}(x) &= \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}^{(m)}(x) v_i(x) \\ &= \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}^{(m)}(x)\int_{x_i}^{x}\left[f^{(k+1)}(t) - P_r(t)\right](t-x_i)^k dt. \end{aligned} \tag{4.27}$$

On the other hand, we have

$$\left[f(x) - L_n(x)\right]^{(m)} = f^{(m)}(x) - L_n^{(m)}(x). \tag{4.28}$$

This completes the proof. □

Theorem 4.3. *Under the assumptions of Theorem 4.2,*

$$\left| f^{(m)}(x) - L_n^{(m)}(x)\right| \leq \frac{E_r\left(f^{(k+1)}\right)}{(k+1)!}\sum_{i=0}^{n}\left| p_{ni}^{(m)}(x)\right| \left| x-x_i\right|^{k+1}, \tag{4.29}$$

where $E_r(\cdot)$ is defined by (3.6).

Proof. Let $P_r(t) = P_r^*(t)$, where $P_r^*(t)$ is defined by (3.6) for the function $g(t) = f^{(k+1)}(t)$. We have

$$\begin{aligned} \left| R_{k,r}^{(m)}(x)\right| &= \left| \frac{(-1)^k}{k!}\sum_{i=0}^{n} p_{ni}^{(m)}(x)\int_{x_i}^{x}\left[f^{(k+1)}(t) - P_r^*(t)\right](t-x_i)^k dt\right| \\ &\leq \frac{\left\| f^{(k+1)}(t) - P_r^*(t)\right\|_\infty}{(k+1)!}\sum_{i=0}^{n}\left| p_{ni}^{(m)}(x)\right| \left| x-x_i\right|^{k+1} \\ &= \frac{E_r\left(f^{(k+1)}\right)}{(k+1)!}\sum_{i=0}^{n}\left| p_{ni}^{(m)}(x)\right| \left| x-x_i\right|^{k+1}, \end{aligned} \tag{4.30}$$

since

$$\left|\int_{x_i}^{x}(t-x_i)^k dt\right| = \frac{|x-x_i|^{k+1}}{k+1}. \tag{4.31}$$

□

THEOREM 4.4. *Under the assumptions of Theorem 3.3 and Lemma 4.1,*

$$\begin{aligned}
|f^{(m)}(x)-L_n^{(m)}(x)| &\le \frac{\Gamma_{k+1}-\gamma_{k+1}}{2(k+1)!}\sum_{i=0}^{n}|p_{ni}^{(m)}(x)|\,|x-x_i|^{k+1},\\
|f^{(m)}(x)-L_n^{(m)}(x)| &\le \frac{1}{k!}\sum_{i=0}^{n}(S_{ki}-\gamma_{k+1})\,|p_{ni}^{(m)}(x)|\,|x-x_i|^{k+1},\\
|f^{(m)}(x)-L_n^{(m)}(x)| &\le \frac{1}{k!}\sum_{i=0}^{n}(\Gamma_{k+1}-S_{ki})\,|p_{ni}^{(m)}(x)|\,|x-x_i|^{k+1}.
\end{aligned} \tag{4.32}$$

Proof. We choose $P_r(t)=\Gamma_{k+1}+\gamma_{k+1}/2$ in Theorem 4.2. Then we get

$$\begin{aligned}
|f^{(m)}(x)-L_n^{(m)}(x)| &\le \frac{1}{k!}\sum_{i=0}^{n}|p_{ni}^{(m)}(x)|\left|\int_{x_i}^{x}\left[f^{(k+1)}(t)-\frac{\Gamma_{k+1}+\gamma_{k+1}}{2}\right](t-x_i)^k dt\right|\\
&\le \frac{\Gamma_{k+1}-\gamma_{k+1}}{2(k!)}\sum_{i=0}^{n}|p_{ni}^{(m)}(x)|\left|\int_{x_i}^{x}(t-x_i)^k dt\right|\\
&= \frac{\Gamma_{k+1}-\gamma_{k+1}}{2(k+1)!}\sum_{i=0}^{n}|p_{ni}^{(m)}(x)|\,|x-x_i|^{k+1}.
\end{aligned} \tag{4.33}$$

If we choose $P_r(t)=\gamma_{k+1}$ in Theorem 4.2, then we get

$$\begin{aligned}
|f^{(m)}(x)-L_n^{(m)}(x)| &\le \frac{1}{k!}\sum_{i=0}^{n}|p_{ni}^{(m)}(x)|\left|\int_{x_i}^{x}[f^{(k+1)}(t)-\gamma_{k+1}](t-x_i)^k dt\right|\\
&\le \frac{1}{k!}\sum_{i=0}^{n}(S_{ki}-\gamma_{k+1})\,|p_{ni}^{(m)}(x)|\,|x-x_i|^{k+1},
\end{aligned} \tag{4.34}$$

since $|\int_{x_i}^{x}[f^{(k+1)}(t)-\gamma_{k+1}]dt| = |f^{(k)}(x)-f^{(k)}(x_i)-\gamma_{k+1}(x-x_i)|$.

In a similar way we prove that the third inequality holds. □

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