COMPATIBLE ELEMENTS IN PARTLY ORDERED GROUPS

JIŘÍ MOČKOŘ AND ANGELIKI KONTOLATOU

Received 16 September 2004 and in revised form 28 September 2005

Some conditions equivalent to a strong quasi-divisor property (SQDP) for a partly ordered group G are derived. It is proved that if G is defined by a family of t-valuations of finite character, then G admits an SQDP if and only if it admits a quasi-divisor property and any finitely generated t-ideal is generated by two elements. A topological density condition in topological group of finitely generated t-ideals and/or compatible elements are proved to be equivalent to SQDP.

1. Introduction

Let G be a partly ordered commutative group (*po*-group). Then G is said to have a *quasi-divisor property* if there exist commutative lattice-ordered group (l-group) $(\Gamma,\cdot,\wedge)$ and an order isomorphism h (the so-called quasi-divisor morphism) from G *into* Γ such that for any $\alpha \in \Gamma$, there exist $g_1,\dots,g_n \in G$ such that $\alpha = h(g_1) \wedge \cdots \wedge h(g_n)$. Moreover, if this embedding h satisfies the condition

$$(\forall \alpha,\beta \in \Gamma_+)\ (\exists \gamma \in \Gamma_+) \quad \alpha\cdot\gamma \in h(G), \quad \beta \wedge \gamma = 1, \tag{1.1}$$

then G is said to have *a strong quasi-divisor property*. Many papers have dealt with *po*-groups with (strong) quasi-divisor property (e.g., see [1, 3, 4, 5, 6, 7, 8]). It is well known that there are some generic examples of such l-group Γ. Namely, if $h: G \to \Gamma$ is a quasi-divisor morphism, then Γ is o-isomorphic to the group $(\mathcal{I}_t^f(G),\times_t)$ of finitely generated t-ideals of G. Recall that a *t-ideal* X_t of G generated by a lower bounded subset $X \subseteq G$ is a set $X_t = \{g \in G : (\forall s \in G)\ s \le X \Rightarrow g \ge s\}$. Then the set $\mathcal{I}_t^f(G)$ of all finitely generated t-ideals of G is a semigroup with operation $\times_t$ defined such that $X_t \times_t Y_t = (X\cdot Y)_t$ (see [2]). It is clear that a map $d: G \to \mathcal{I}_t^f(G)$ defined by $d(g) = \{g\}_t$ is an embedding. Another example of a group Γ is a group $\mathcal{K}(W)$ of compatible elements of a defining family of t-valuations W (see the definitions below). In this note, we want to show that properties of a group $\mathcal{K}(W)$ can be used for deriving new conditions under which quasi-divisor property is also a strong quasi-divisor property.

International Journal of Mathematics and Mathematical Sciences 2005:24 (2005) 4041–4048
DOI: 10.1155/IJMMS.2005.4041

Let $w : G \to G_1$ be an o-homomorphism. Then, w is called t-homomorphism if $w(X_t) \subseteq (w(X))_t$ for any lower bounded subset $X \subseteq G$. Moreover, if G_1 is a totally ordered group (i.e., o-group), then w is called t-valuation. Recall that a family W of t-valuations $w : G \to G_w$ is called a *defining family for* G if

$$(\forall g \in G) \quad g \geq 1 \Longleftrightarrow (\forall w \in W) \quad w(g) \geq 1. \tag{1.2}$$

We say that W is of finite character if

$$(\forall g \in G)\,(\forall' w \in W) \quad w(g) = 1, \tag{1.3}$$

where $\forall'$ means "for all but a finite number." Hence any defining family W of finite character creates an embedding of G into a sum $\sum_{w\in W} G_w$ of o-groups G_w, $w \in W$. Then a quasi-divisors property of G is said to be of *finite character*, if there exists a defining family of t-valuations of finite character for G. If w_1, w_2 are two t-valuations of a po-group G, then w_1 is said to be coarser than w_2 ($w_1 \geq w_2$) if there exists an o-epimorphism $d_{w_1,w_2} : G_{w_1} \to G_{w_2}$ such that $w_2 = d_{w_1,w_2} w_1$. It may be then proved that for any two t-valuations w_1, w_2, there exists a t-valuation $w_1 \wedge w_2$ which is the infimum of w_1, w_2 with respect to this preorder relation. Then, $d_{w_1,w_1\wedge w_2}$ (resp., $d_{w_2,w_1\wedge w_2}$) is an o-epimorphism such that $w_1 \wedge w_2 = d_{w_1,w_1\wedge w_2}$, $w_1 = d_{w_2,w_1\wedge w_2} w_2$. For simplicity, we set $d_{w_1w_2} = d_{w_1,w_1\wedge w_2}$, $d_{w_2w_1} = d_{w_2,w_1\wedge w_2}$ (see the difference between d_{w_1,w_2} and $d_{w_1w_2}$). If W is a system of t-valuations $w : G \to G_w$ of a po-group G and $W' \subseteq W$, then a system $(g_w)_w \in \prod_{w\in W'} G_w$ of elements is called *compatible* provided that $d_{wv}(g_w) = d_{vw}(g_v)$ for all $w, v \in W'$. Finally, $(g_w)_{w\in W'}$ is called W'*-complete* if $\bigcup_{w\in W'} W(g_w) = W'$, where $W(g_w) = \{v \in W : d_{wv}(g_w) \neq 1\}$ for $g_w \neq 1_w$ and $W(1_w) = \{w\}$ for any $w \in W$.

Let W be a defining family of t-valuations of G. Then, we set

$$\mathcal{K}(W) = \left\{ (a_w)_w \in \prod_{w\in W} G_w : (a_w)_w \text{ is compatible} \right\}. \tag{1.4}$$

It can be proved that $\mathcal{K}(W)$ is an l-subgroup in $\prod_{w\in W} G_w$ (see [8]). Now we say that G with a defining family of t-valuations satisfies the positive weak approximation theorem (PWAT) if for any finite subset $F \subseteq W$ and any compatible system $(\alpha_w)_{w\in F} \in \prod_{w\in F} G_w^+$, there exists $g \in G^+$ such that $w(g) = \alpha_w$, $w \in F$. Finally, we say that G with W satisfies the approximation theorem (AT) if for any finite subset $F \subseteq W$ and any compatible and F-complete system $(\alpha_w)_{w\in F} \in \prod_{w\in F} G_w$, there exists $g \in G$ such that

$$\begin{aligned} w(g) &= \alpha_w, \quad w \in F, \\ w(g) &\geq 1, \quad w \in W \setminus F. \end{aligned} \tag{1.5}$$

2. Results

In the theory of quasi-divisors of a po-group, a t-ideal theory has an important position. In the next propositions, we want to show that all t-ideals in a po-group G with a quasi-divisor property of finite character can be derived from the set of compatible elements $\mathcal{K}(W)$ of G, where W is some defining family of t-valuations of G.

Lemma 2.1. *Let* $(\alpha_w)_w \in \mathcal{K}(W)$ *and let* $W_0 = \{w \in W : \alpha_w \neq 1\}$. *Then* $(\alpha_w)_{w\in W'}$ *is* W'*-complete for any* $W_0 \subseteq W' \subseteq W$.

Proof. Let $v \in \bigcup_{w\in W'} W(\alpha_w)$. Then there exists $w \in W_0$ such that $v \in W(\alpha_w)$. Because (α_w, α_v) is compatible, we have $1 \neq d_{wv}(\alpha_w) = d_{vw}(\alpha_v)$ and it follows that $\alpha_v \neq 1$. Hence, $v \in W_0 \subseteq W'$. □

Proposition 2.2. *Let G be a po-group with a quasi-divisor property of finite character and let W be a defining family of t-valuations of G. Let* $(\alpha_w)_w \in \mathcal{K}(W)$. *Then* $X = \{g \in G : (\forall w \in W)\ w(g) \geq \alpha_w\}$ *is a finitely generated t-ideal of G.*

Proof. Because the t-system is defined by a family W of t-valuations, according to [8, Theorem 2.6], the group $\mathcal{K}(W)$ is o-isomorphic to a Lorenzen l-group $\Lambda_t(G)$. It follows that a map $d : G \to \mathcal{K}(W)$ such that $d(g) = (w(g))_w$ is a quasi-divisors morphism. Then for any $(\alpha_w)_w \in \mathcal{K}(W)$, there exist $g_1,\dots,g_n \in G$ such that $d(g_1) \wedge \cdots \wedge d(g_n) = (\alpha_w)_w$. Then $X = (g_1,\dots,g_n)_t$. In fact, for $g \in X$, we have $w(g) \geq \alpha_w$ and it follows that $w(g) \in (w(g_1),\dots,w(g_n))_t$. Because the t-system is defined by W, we have $g \in (g_1,\dots,g_n)_t$, analogously for the other inclusion. □

Corollary 2.3. *Let G be a po-group with a quasi-divisor property of finite character and let W be a defining family of t-valuations of G. Then there exists an o-isomorphism*

$$\sigma : \mathcal{K}(W) \longrightarrow \mathcal{I}_t^f(G) \tag{2.1}$$

such that for $(\alpha_w)_w \in \mathcal{K}(W)$ *and* $J \in \mathcal{I}_t^f(G)$,

$$\begin{gathered}\sigma\big((\alpha_w)_w\big) = \{g \in G : (\forall w \in W)\ w(g) \geq \alpha_w\},\\ \sigma^{-1}(J) = \big(\big(\wedge_{x\in J}\, w(x)\big)_w\big).\end{gathered} \tag{2.2}$$

It is well known that the existence of quasi-divisor property is equivalent to the existence of a defining family of *essential* t-valuations (see [3, Theorem 2.1]). Recall that a t-valuation w of G is essential if $\ker w$ is a directed subgroup of G and w is an o-epimorphism.

Lemma 2.4. *Let w, v be essential t-valuations of G and let* $\alpha \in G_v$ *be such that* $d_{vw}(\alpha) = 1$. *Then there exists* $g \in G$ *such that* $w(g) = 1$, $v(g) \geq \alpha$.

Proof. We may assume that $\alpha > 1$. Let $J = \{x \in G : v(x) \geq \alpha\}$. Let us suppose on contrary that the statement of the lemma is not true. Then for any $x \in J$, we have $w(x) > 1$. Let H be the largest convex subgroup in G_v such that $\alpha \notin H$ and let $w' : G \xrightarrow{v} G_v \to G_v/H$ be the composition of v and canonical morphism. Then $w' \leq w$. In fact, let $x \in G$, $x \geq 1$ be such that $w'(x) > 1$. Because $w'(x) = v(x)H$, we have $v(x) \notin H$, $v(x) > 1$. Then there exists $n \in \mathbb{N}$ such that $v(x)^n \geq \alpha$. In fact, if $v(x)^n < \alpha$ for all $n \in \mathbb{N}$, then the convex subgroup H' generated by $H \cup \{v(x)\}$ does not contain α and $H \subseteq H'$. On the other hand, we have $v(x) \in H' \setminus H$, a contradiction. Then $x^n \in J$ for some $n \in \mathbb{N}$ and according to the assumption, we have $w(x)^n > 1$. Hence $w(x) > 1$ and we proved the implication

$$x \in G, \quad x \geq 1, \quad w'(x) > 1 \Longrightarrow w(x) > 1. \tag{2.3}$$

Let $\rho : G_w \to G_{w'}$ be defined by $\rho(w(g)) = w'(g)$. Then ρ is well defined. In fact, let $w(x) = w(y)$. Since w is essential, there exists $t \in \ker w$ such that $t \geq 1, xy^{-1}$. If $w'(x) \neq w'(y)$, we have, for example, $w'(xy^{-1}) > 1$. Then $w'(t) \geq w'(xy^{-1}) > 1$. According to (2.3), we have $w(t) > 1$, a contradiction with $t \in \ker w$. Thus $w' = \rho \cdot w$ and $w' \leq w$. Then, we have also $w' \leq w \wedge v$. For any $b \in G$ such that $\alpha = v(b)$, we obtain $w'(b) = v(b)H = \alpha H \neq 1$ and $v \wedge w(b) = d_{vw} > v(b) = d_{vw}(\alpha) = 1$, a contradiction, because $v \wedge w \geq w'$. □

Lemma 2.5. *Let $w_1,\dots,w_n$ be essential t-valuations of G and let $(\alpha_1,\dots,\alpha_n) \in \prod_{i=1}^n G_{w_i}^+$ be compatible elements. Then there exists $a_1 \in G$, $a_1 \geq 1$, such that*

$$\forall j \neq 1, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) > \alpha_j. \tag{2.4}$$

Proof. The proof will be done by the induction with respect to n. For $n = 1$, the proof is trivial. Let us assume that the statement is true for any compatible set of $n-1$ elements. Let us assume firstly that $w_1 < w_k$ for some $k \neq 1$. According to the induction assumption, there exists $a \in G_+$ such that

$$\forall j \neq k, 1, \quad w_k(a) = \alpha_k, \quad w_j(a) > \alpha_j. \tag{2.5}$$

Because $w_1 < w_k$, there exists an o-epimorphism $\sigma : G_{w_k} \to G_{w_1}$ such that $w_1 = \sigma \cdot w_k$. Since (α_1, α_k) is compatible, we have $\sigma(\alpha_k) = \alpha_1$. Since $\ker\sigma \neq \{1\}$, there exists $\delta \in \ker\sigma$, $\delta > 1$. From the fact that w_k is essential, it follows that there exists $g \in G$, $g > 1$, such that $w_k(g) = \delta$. We set $a_1 = ga$. Then, we have

$$\begin{gathered} w_1(a_1) = \sigma \cdot w_k(ga) = \sigma(\delta) \cdot \sigma(\alpha_k) = \alpha_1, \\ w_k(a_1) = \delta \cdot \alpha_k > \alpha_k, \\ \forall i \neq k,\ i \geq 2, \quad w_i(a_1) \geq w_i(a) > \alpha_i. \end{gathered} \tag{2.6}$$

Let us assume now that $w_1 \| w_j$, $j \geq 2$. Then $w_j \neq w_1 \wedge w_j$ and for any $j \geq 2$, there exists $\delta_j \in \ker d_{j1}$, $\delta_j > 1$. According to Lemma 2.4, for any $j \geq 2$, there exists $g_j \in G_+$ such that $w_1(g_j) = 1$, $w_j(g_j) \geq \delta_j$. We set $g_1 = \prod_{j\geq 2} g_j$. Then

$$\forall j \geq 2, \quad w_1(g_1) = 1, \quad w_j(g_1) \geq w_j(g_j) \geq \delta_j > 1. \tag{2.7}$$

According to the induction assumption, there exists $a_1 \in G_+$ such that

$$\forall 2 \leq j \leq n-1, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) > \alpha_j. \tag{2.8}$$

Without the loss of generality, we may assume that

$$\forall 2 \leq j, \quad w_1(a_1) = \alpha_1, \quad w_j(a_1) \geq \alpha_j. \tag{2.9}$$

In fact, if $w_n(a_1) < \alpha_n$, then $d_{n1}(\alpha_n \cdot w_n^{-1}(a_1)) = d_{1n}(\alpha_1) \cdot d_{1n}(w_1^{-1}(a_1)) = 1$ and according to Lemma 2.4, there exists $a_1' \in G_+$ such that $w_1(a_1') = 1$, $w_n(a_1') \geq \alpha \cdot w_n^{-1}(a_1)$. Then for $a_1'' = a_1 a_1'$, we have

$$\begin{gathered} w_1(a_1'') = w_1(a_1 a_1') = \alpha_1, \\ \forall n > j \geq 2, \quad w_j(a_1'') \geq w_j(a_1) > \alpha_j, \\ w_n(a_1'') \geq \alpha_n. \end{gathered} \tag{2.10}$$

We set $c_1 = a_1 g_1$, where a_1 satisfies the relation (2.9). Then we have

$$\begin{gathered} w_1(c_1) = w_1(a_1) = \alpha_1, \\ w_j(c_1) > w_j(a_1) \geq \alpha_j, \quad j \geq 2. \end{gathered} \tag{2.11}$$

□

If G admits a quasi-divisor property of finite character, the existence of a map

$$\sigma : \mathcal{K}(W) \longrightarrow \mathcal{I}_t^f(G) \tag{2.12}$$

follows immediately from Proposition 2.2. Between the l-group of compatible elements $\mathcal{K}(W)$ and a semigroup $\mathcal{I}_t^f(G)$ of finitely generated t-ideals of *any po*-group G, there exists another naturally defined map, namely,

$$\tau : \mathcal{I}_t^f(G) \longrightarrow \mathcal{K}(W) \tag{2.13}$$

such that $\tau(X_t) = (\wedge w(X))_{w\in W} = (\wedge w(X_t))_{w\in W} \in \mathcal{K}(W)$. τ is well defined and it can be proved easily that τ is a semigroup monomorphism (because t-ideals are defined by W). If G admits a quasi-divisor property of finite character, then σ and τ are mutually inverse o-isomorphisms (see Corollary 2.3). Moreover, if $h : G \to \mathcal{I}_t^f(G)$ and $d : G \to \mathcal{K}(W)$ are natural embedding maps such that $h(g) = (g)_t$ and $d(g) = (w(g))_{w\in W}$, then the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{I}_t^f(G) & \xrightarrow{\tau} & \mathcal{K}(W) & \xrightarrow{\sigma} & \mathcal{I}_t^f(G) \\ \uparrow{\scriptstyle h} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle h} \\ G & = & G & = & G \end{array} \tag{2.14}$$

In the group $\mathcal{K}(W)$, a group topology $\mathcal{T}_W$ can be defined such that $\ker \hat{w} = \{(\alpha_v)_v \in \mathcal{K}(W) : \alpha_w = 1\}$ is a subbase of neighborhoods of 1 for any $w \in W$ (clearly, $\hat{w} : \mathcal{K}(W) \to G_w$ is the projection map). Then the semigroup monomorphism $\tau : \mathcal{I}_t^f(G) \to \mathcal{K}(W)$ induces a semigroup topology $\mathcal{F}_W$ on $\mathcal{I}_t^f(G)$. If for $w \in W$, we define a map $\widetilde{w} : \mathcal{I}_t^f(G) \to G_w$ such that $\widetilde{w}(X_t) = \wedge w(X) (= \wedge w(X_t))$, then for any finite $F \subseteq W$, we obtain

$$\tau^{-1}\left(\bigcap_{w\in F} \ker \hat{w}\right) = \bigcap_{w\in F} \ker \widetilde{w}. \tag{2.15}$$

Hence, the topology $\mathcal{F}_W$ can be defined by maps $\tilde{w}, w \in W$. Moreover, in the ordered semigroup $(\mathcal{I}_t^f(G), \times_t, \leq_t)$, where $X_t \leq_t Y_t$ if $Y_t \subseteq X_t$, a t-ideals structure can be defined analogously as in any *po*-group. The following lemma shows that the topology $\mathcal{F}_W$ is defined also by t-valuations.

Lemma 2.6. *For any $w \in W$, $\tilde{w}$ is a (t,t)-morphism from $(\mathcal{I}_t^f(G), \times_t, \leq_t)$ to G_w.*

Proof. Let $\mathcal{X}_t$ be a t-ideal in $\mathcal{I}_t^f(G)$ generated by a lower bounded subset $\mathcal{X}$ and let $X_t \in \mathcal{X}_t$. Then there exists a finite set $\mathcal{F} \subseteq \mathcal{X}$ such that $X_t \in \mathcal{F}_t$. We set $S = \bigcup_{F_t \in \mathcal{F}} F$. Then, S is a finite subset in G and $S_t \leq_t F_t$ for any $F_t \in \mathcal{F}$. Hence, $X_t \geq_t S_t$ and we have $\wedge w(X) = \wedge w(X_t) \geq \wedge w(S_t) = \wedge w(S)$. Thus $\tilde{w}(X_t) \in (\tilde{w}(S_t))_t = (\wedge_{F_t \in \mathcal{F}} \tilde{w}(F_t))_t = (\tilde{w}(\mathcal{F}))_t$. □

Theorem 2.7. *Let G be defined by a family of t-valuations of finite character. Then the following statements are equivalent.*

(1) *G admits a strong quasi-divisor property.*

(2) *G admits a quasi-divisor property and for any $(\alpha_w)_w \in \mathcal{K}(W)$ and $a \in G$ such that $\alpha_w \leq w(a)$ for all $w \in W$, there exists $b \in G$ such that $\alpha_w = w(a) \wedge w(b)$ for all $w \in W$.*

(3) *G admits a quasi-divisor property and for any $X_t \in \mathcal{I}_t^f(G)$ and $a \in X_t$, there exists $b \in G$ such that $X_t = (a,b)_t$.*

If W is an infinite set, then these statements are equivalent to the following equivalent statements.

(4) *G admits a quasi-divisor property and $h(G)$ is dense in $(\mathcal{I}_t^f(G), \mathcal{F}_W)$.*

(5) *$d(G)$ is dense in $(\mathcal{K}(W), \mathcal{T}_W)$.*

Proof. (1)⇒(2) Let $(\alpha_w)_w \in \mathcal{K}(W)$, $a \in G$ such that $w(a) \geq \alpha_w$ for all $w \in W$. Let $W_1 = \{w \in W : \alpha_w \neq 1\} \cup \{v \in W : v(a) \neq 1\}$. According to Lemma 2.1, $(\alpha_w)_{w \in W_1}$ is compatible and W_1-complete and according to AT, there exists $b \in G$ such that

$$\begin{aligned} w(b) &= \alpha_w, \quad w \in W_1, \\ w(b) &\geq 1, \quad w \in W \setminus W_1. \end{aligned} \tag{2.16}$$

Then for $w \in W_1$, we have $w(a) \wedge w(b) = w(a) \wedge \alpha_w = \alpha_w$, and for $w \in W \setminus W_1$, $w(a) \wedge w(b) = 1 \wedge w(b) = 1 = \alpha_w$.

(2)⇒(3) Let $a \in X_t \in \mathcal{I}_t^f(G)$. Because t-system is defined by W, we have $X_t = \{g \in G : w(g) \geq \wedge w(X),\ w \in W\}$. According to [3, Lemma 2.9], $(\wedge w(X))_w \in \mathcal{K}(W)$ and there exists $b \in G$ such that $\wedge w(X) = w(a) \wedge w(b)$, for all $w \in W$. Then we have $X_t = \{g \in G : w(g) \in (w(a), w(b))_t,\ w \in W\} = (a,b)_t$.

(3)⇒(1) We show that G satisfies the positive weak approximation theorem (PWAT). Let $(\alpha_1, \dots, \alpha_n) \in \prod_{i=1}^n G_{w_i}^+$ be compatible. According to Lemma 2.5, there exist $a_1, \dots, a_n \in G_+$ such that

$$\forall i,\ \forall j \neq i, \quad w_i(a_i) = \alpha_i, \quad w_j(a_i) > \alpha_j. \tag{2.17}$$

We set $b = a_1 \cdots a_n$. Then $b \in (a_1,\dots,a_n)_t$. Hence, there exists $a \in G_+$ such that $(a_1,\dots,a_n)_t = (a,b)_t$. Then for any i, we have

$$w_i(b) = \alpha_i \cdot \prod_{j \neq i} w_i(a_j) > \alpha_i^n \geq \alpha_i. \tag{2.18}$$

Let us assume that there exists i such that $w_i(b) < w_i(a)$. Since $a_i \in (a,b)_t$, we have $\alpha_i = w_i(a_i) \geq w_i(a) \wedge w_i(b) = w_i(b)$, a contradiction. Then we have $\alpha_i = w_i(a_i) \geq w_i(a) \wedge w_i(b) = w_i(a)$. Since $a \in (a_1,\dots,a_n)_t$, we have $w_i(a) \geq w_i(a_1) \wedge \cdots \wedge w_i(a_n) = \alpha_i \wedge \bigwedge_{j \neq i} w_i(a_j) = \alpha_i$. Thus $w_i(a) = \alpha_i$, $i = 1,\dots,n$ and G satisfies the PWAT. According to [7, Theorem 3.5], G admits a strong quasi-divisor property.

Now let W be an infinite set.

(1)⇒(4) Since G admits a quasi-divisor property, $(\mathcal{I}_t^f(G), \times_t)$ is a group and the subbase of neighborhoods of unity in topology $\mathcal{F}_W$ is $\{\ker \widetilde{w} : w \in W\}$. We show that a map $\sigma : \mathcal{K}(W) \to \mathcal{I}_t^f(G)$ is a homeomorphism. Let $\mathbf{a}, \mathbf{b} \in \mathcal{K}(W)$. Then there exist $a_1,\dots,a_n,b_1,\dots,b_m \in G$ such that $\mathbf{a} = d(a_1) \wedge \cdots \wedge d(a_n)$, $\mathbf{b} = d(b_1) \wedge \cdots \wedge d(b_m)$ and we have $\sigma(\mathbf{a}) = (a_1,\dots,a_n)_t$, $\sigma(\mathbf{b}) = (b_1,\dots,b_m)_t$. Then $\mathbf{a} \cdot \mathbf{b} = d(a_1b_1) \wedge \cdots \wedge d(a_nb_m)$ and $\sigma(\mathbf{a} \cdot \mathbf{b}) = (a_1b_1,\dots,a_nb_m)_t = \sigma(\mathbf{a}) \times_t \sigma(\mathbf{b})$. If $\sigma(\mathbf{a}) = (1)_t$, then $(a_1,\dots,a_n)_t = (1)_t$ and it follows easily that $\mathbf{a} = 1$. It is clear that σ is also homeomorphism. According to [8, Theorem 2.6], there exists an o-isomorphism ψ such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda_t(G) & \xrightarrow{\ \psi\ } & \mathcal{K}(W) \\ {\scriptstyle \overline{w}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{w}} \\ G_w & = & G_w \end{array} \tag{2.19}$$

where $\overline{w}$ is a canonical extension of w. Since $G \to \Lambda_t(G)$ is a strong quasi-divisor morphism, it follows that $d : G \to \mathcal{K}(W)$ is a strong quasi-divisor morphism as well. Then, according to [5, Theorem 2.9], $d(G)$ is dense in $(\mathcal{K}(W), \mathcal{T}_W)$ and it follows that $h(G)$ is also dense in $(\mathcal{I}_t^f(G), \mathcal{F}_W)$.

(4)⇒(5) If G admits a quasi-divisor property, then $\mathcal{I}_t^f(G)$ is o-isomorphic to $\Lambda_t(G)$ and according to [8, Theorem 6], it is also o-isomorphic to $\mathcal{K}(W)$. It can be proved easily that $(\mathcal{I}_t^f(G), \mathcal{F}_W)$ is also homeomorphic to $(\mathcal{K}(W), \mathcal{T}_W)$.

(5)⇒(1) It follows directly from [5, Theorem 2.9]. □

References

[1] A. Geroldinger and J. Močkoř, *Quasi-divisor theories and generalizations of Krull domains*, J. Pure Appl. Algebra **102** (1995), no. 3, 289–311.

[2] P. Jaffard, *Les Systèmes d'Idéaux*, Travaux et Recherches Mathématiques, IV, Dunod, Paris, 1960.

[3] J. Močkoř, *t-valuations and the theory of quasi-divisors*, J. Pure Appl. Algebra **120** (1997), no. 1, 51–65.

[4] ———, *Construction of* po-*groups with quasi-divisors theory*, Czechoslovak Math. J. **50(125)** (2000), no. 1, 197–207.

[5] ———, *Topological characterization of ordered groups with quasi-divisor theory*, Czechoslovak Math. J. **52(127)** (2002), no. 3, 595–607.

[6] J. Močkoř and A. Kontolatou, *Divisor class groups of ordered subgroups*, Acta Math. Inform. Univ. Ostraviensis **1** (1993), 37–46.

[7] ———, *Groups with quasi-divisors theory*, Comment. Math. Univ. St. Paul. **42** (1993), no. 1, 23–36.

[8] ———, *Some remarks on Lorenzen r-group of partly ordered groups*, Czechoslovak Math. J. **46(121)** (1996), no. 3, 537–552.

Jiří Močkoř: Department of Mathematics, University of Ostrava, CZ-702 00 Ostrava, Czech Republic

E-mail address: jiri.mockor@osu.cz

Angeliki Kontolatou: Department of Mathematics, School of Natural Sciences, University of Patras, 26500 Patras, Greece

E-mail address: angelika@math.upatras.gr