

POSITIVE PERIODIC SOLUTIONS OF A DISCRETE MUTUALISM MODEL WITH TIME DELAYS

YONGKUN LI

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A discrete periodic mutualism model with time delays is investigated. By using Gaines and Mawhin's continuation theorem of coincidence degree theory, the existence of positive periodic solutions of the model is established.

1. Introduction

Two species cohabit a common habitat and each species enhances the average growth rate of the other, this type of ecological interaction is known as facultative mutualism [8]. In [6], the author has studied the existence of positive periodic solutions of the periodic mutualism model

$$\begin{aligned}\frac{dN_1(t)}{dt} &= r_1(t)N_1(t) \left[\frac{K_1(t) + \alpha_1(t)N_2(t - \tau_2(t))}{1 + N_2(t - \tau_2(t))} - N_1(t - \sigma_1(t)) \right], \\ \frac{dN_2(t)}{dt} &= r_2(t)N_2(t) \left[\frac{K_2(t) + \alpha_2(t)N_1(t - \tau_1(t))}{1 + N_1(t - \tau_1(t))} - N_2(t - \sigma_2(t)) \right],\end{aligned}\tag{1.1}$$

where $r_i, K_i, \alpha_i \in C(\mathbb{R}, \mathbb{R}^+)$, $\alpha_i > K_i$, $i = 1, 2$, $\tau_i, \sigma_i \in C(\mathbb{R}, \mathbb{R}^+)$, $i = 1, 2$, $r_i, K_i, \alpha_i, \tau_i, \sigma_i$ ($i = 1, 2$) are functions of period $\omega > 0$. Since the study on periodic solutions of a population model is of great interest in mathematical biology [5] and many authors [1, 7] have argued that the discrete-time models governed by difference equations are more appropriate than the continuous ones when the populations have nonoverlapping generations, then, discrete-time models can provide efficient computational types of continuous models for numerical simulations. It is reasonable to study the discrete-time mutualism model governed by difference equations.

One of the ways of deriving difference equations modeling the dynamics of populations with nonoverlapping generations is based on appropriate modifications of the corresponding models with overlapping generations [2, 4]. In this approach, differential equations with piecewise constant arguments have been proved to be useful. Following the same idea and the same method in [2, 4], one can easily derive the following discrete

analog of (1.1), which takes the form of

$$\begin{aligned} x_1(k+1) &= x_1(k) \exp \left[\frac{r_1(k)K_1(k) + r_1(k)\alpha_1(k)x_2(k - \tau_2(k))}{1 + x_2(k - \tau_2(k))} - r_1(k)x_1(k - \sigma_1(k)) \right], \\ x_2(k+1) &= x_2(k) \exp \left[\frac{r_2(k)K_2(k) + r_2(k)\alpha_2(k)x_1(k - \tau_1(k))}{1 + x_1(k - \tau_1(k))} - r_2(k)x_2(k - \sigma_2(k)) \right]. \end{aligned} \tag{1.2}$$

The exponential form of (1.2) is more biologically reasonable than that directly derived by replacing the differential by difference in (1.1). Our purpose in this paper is to use Mawhin’s continuous theorem [3] to study the existence of positive periodic solutions of (1.2).

Let \mathbb{Z} , \mathbb{R} , and \mathbb{R}^2 denote the sets of all integers and two-dimensional Euclidean vector space, respectively. Throughout this paper, we always assume that the following hold.

(H₁) For $i = 1, 2$, $r_i, K_i, \alpha_i : \mathbb{Z} \rightarrow (0, \infty)$ and $\tau_i, \sigma_i : \mathbb{Z} \rightarrow [0, \infty)$ are all ω -periodic, that is,

$$\begin{aligned} r_i(k + \omega) &= r_i(k), & K_i(k + \omega) &= K_i(k), & \alpha_i(k + \omega) &= \alpha_i(k), \\ \tau_i(k + \omega) &= \tau_i(k), & \sigma_i(k + \omega) &= \sigma_i(k), & k \in \mathbb{Z}. \end{aligned} \tag{1.3}$$

(H₂) For $i = 1, 2$, $\alpha_i(k) > K_i(k)$, $k \in \mathbb{Z}$.

2. Existence of a positive periodic solution

In order to use Mawhin’s continuous theorem to establish the existence of at least one positive periodic solution of (1.2), we need to make some preparations.

Let X, Y be normed vector spaces, let $L : \text{Dom } L \subset X \rightarrow Y$ be a linear mapping, and let $N : X \rightarrow Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim } \text{Im } L < +\infty$ and $\text{Im } L$ is closed in Y . If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$. It follows that the mapping $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of the mapping by K_P . If Ω is an open bounded subset of X , the mapping N will be called L -compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

For convenience, we introduce Mawhin’s continuous theorem [3, page 40] as follows.

LEMMA 2.1. *Let L be a Fredholm mapping of index zero and let N be L -compact on $\bar{\Omega}$. Assume that*

- (i) *for each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
- (ii) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \text{Ker } L$ and*

$$\deg(JNQ, \Omega \cap \text{Ker } L, 0) \neq 0. \tag{2.1}$$

Then the operator equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$.

In what follows, we will use the notations

$$I_\omega = \{0, 1, \dots, \omega - 1\}, \quad \bar{u} = \frac{1}{\omega} \sum_{k=0}^{\omega-1} u(k), \tag{2.2}$$

where $\{u(k)\}$ is an ω -periodic sequence of real numbers defined for $k \in \mathbb{Z}$.

The following result was given by [2, Lemma 3.2].

LEMMA 2.2. *Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be ω -periodic, that is, $f(k + \omega) = f(k)$. Then for any fixed $k_1, k_2 \in I_\omega$, and any $k \in \mathbb{Z}$,*

$$\begin{aligned} f(k) &\leq f(k_1) + \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|, \\ f(k) &\geq f(k_2) - \sum_{s=0}^{\omega-1} |f(s+1) - f(s)|. \end{aligned} \tag{2.3}$$

The following result was given by [6, Lemma 2.2].

LEMMA 2.3. *Let*

$$f(x, y) = \left(a_1 - \frac{a_1 - b_1}{1 + e^y} - c_1 e^x, a_2 - \frac{a_2 - b_2}{1 + e^x} - c_2 e^y \right) \tag{2.4}$$

and $\Omega = \{(x, y)^T \in \mathbb{R}^2 : |x| + |y| < M\}$, where $M, a_i, b_i, c_i \in \mathbb{R}^+$ are constants, $a_i > b_i$, $i = 1, 2$, and $M > \max\{|\ln(a_i/c_i)|, |\ln(b_i/c_i)|, i = 1, 2\}$. Then

$$\deg \{f, \Omega, (0, 0)\} \neq 0. \tag{2.5}$$

Now we state our fundamental theorem about the existence of a positive ω -periodic solution of (1.2).

THEOREM 2.4. *Assume that (H_1) and (H_2) hold, then (1.2) has at least one positive ω -periodic solution.*

Proof. Consider the following system of difference equations with delays:

$$\begin{aligned} y_1(k+1) - y_1(k) &= r_1(k) \left[\frac{K_1(k) + \alpha_1(k) \exp \{y_2(k - \tau_2(k))\}}{1 + \exp \{y_2(k - \tau_2(k))\}} - \exp \{y_1(k - \sigma_1(k))\} \right], \\ y_2(k+1) - y_2(k) &= r_2(k) \left[\frac{K_2(k) + \alpha_2(k) \exp \{y_1(k - \tau_1(k))\}}{1 + \exp \{y_1(k - \tau_1(k))\}} - \exp \{y_2(k - \sigma_2(k))\} \right], \end{aligned} \tag{2.6}$$

where $r_i, K_i, \alpha_i, \tau_i, \sigma_i (i = 1, 2)$ are the same as those in (1.2). It is easy to see that if (2.6) has an ω -periodic solution $\{(y_1^*(k), y_2^*(k))^T\}$, then $\{(x_1^*(k), x_2^*(k))^T\} = \{(\exp \{y_1^*(k)\}, \exp \{y_2^*(k)\})^T\}$ is a positive ω -periodic solution of (1.2). Therefore, to complete the proof, it suffices to show that system (2.6) has at least one ω -periodic solution.

Define

$$I_2 = \{y = \{y(k)\} : y(k) \in \mathbb{R}^2, k \in \mathbb{Z}\}. \tag{2.7}$$

Let $l^\omega \subset l_2$ denote the subspace of all ω -periodic sequences equipped with the norm $\| \cdot \|$, that is,

$$\|y\| = \left\| (y_1, y_2)^T \right\| = \max_{k \in I_\omega} |y_1(k)| + \max_{k \in I_\omega} |y_2(k)|, \tag{2.8}$$

for any $y = \{(y_1(k), y_2(k)), k \in \mathbb{Z}\} \in l^\omega$.

It is not difficult to show that l^ω is a finite-dimensional Banach space.

Let

$$l_0^\omega = \left\{ y = \{y(k)\} \in l^\omega : \sum_{k=0}^{\omega-1} y(k) = 0 \right\}, \tag{2.9}$$

$$l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) = h \in \mathbb{R}^2, k \in \mathbb{Z}\},$$

then it is easy to check that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \quad \dim l_c^\omega = 2. \tag{2.10}$$

Take $X = Y = l^\omega$ and let

$$(Ny)(k) = \begin{bmatrix} r_1(k) \left[\frac{K_1(k) + \alpha_1(k) \exp\{y_2(k - \tau_2(k))\}}{1 + \exp\{y_2(k - \tau_2(k))\}} - \exp\{y_1(k - \sigma_1(k))\} \right] \\ r_2(k) \left[\frac{K_2(k) + \alpha_2(k) \exp\{y_1(k - \tau_1(k))\}}{1 + \exp\{y_1(k - \tau_1(k))\}} - \exp\{y_2(k - \sigma_2(k))\} \right] \end{bmatrix}, \quad y \in X, k \in \mathbb{Z},$$

$$(Ly)(k) = y(k+1) - y(k), \quad y \in X, k \in \mathbb{Z}, \tag{2.11}$$

then it is easy to see that L is a bounded linear operator with

$$\text{Ker } L = l_c^\omega, \quad \text{Im } L = l_0^\omega, \quad \dim \text{Ker } L = 2 = \text{codim Im } L, \tag{2.12}$$

then it follows that L is a Fredholm mapping of index zero.

Define

$$Py = \frac{1}{\omega} \sum_{s=0}^{\omega-1} y(s), \quad y \in X, \quad Qz = \frac{1}{\omega} \sum_{s=0}^{\omega-1} z(s), \quad z \in Y. \tag{2.13}$$

It is not difficult to show that P and Q are continuous projectors such that

$$\text{Im } P = \text{Ker } L, \quad \text{Im } L = \text{Ker } Q = \text{Im}(I - Q). \tag{2.14}$$

Furthermore, the generalized inverse (to L) $K_P : \text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$ exists, which is given by

$$(K_P z)(n) = \sum_{i=0}^{n-1} z(i) - \frac{1}{\omega} \sum_{i=1}^{\omega} \sum_{s=0}^{i-1} z(s), \quad n \in \mathbb{Z}. \tag{2.15}$$

Obviously, QN and $K_p(I - Q)N$ are continuous. Since X is a finite-dimensional Banach space, one can easily show that $\overline{K_p(I - Q)N(\bar{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded, and hence N is L -compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we are in a position to search for an appropriate open bounded subset $\Omega \subset X$ for the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0, 1)$, we have

$$\begin{aligned} y_1(k + 1) - y_1(k) &= \lambda r_1(k) \left[\frac{K_1(k) + \alpha_1(k) \exp \{y_2(k - \tau_2(k))\}}{1 + \exp \{y_2(k - \tau_2(k))\}} - \exp \{y_1(k - \sigma_1(k))\} \right], \\ y_2(k + 1) - y_2(k) &= \lambda r_2(k) \left[\frac{K_2(k) + \alpha_2(k) \exp \{y_1(k - \tau_1(k))\}}{1 + \exp \{y_1(k - \tau_1(k))\}} - \exp \{y_2(k - \sigma_2(k))\} \right]. \end{aligned} \tag{2.16}$$

Assume that $\{(y_1(k), y_2(k))^T\} \in X$ is a solution of system (2.16) for a certain $\lambda \in (0, 1)$. Summing on both sides of (2.16) from 0 to $\omega - 1$ with respect to k , we obtain

$$\sum_{k=0}^{\omega-1} r_1(k) \left[\frac{K_1(k) + \alpha_1(k) \exp \{y_2(k - \tau_2(k))\}}{1 + \exp \{y_2(k - \tau_2(k))\}} - \exp \{y_1(k - \sigma_1(k))\} \right] = 0, \tag{2.17}$$

$$\sum_{k=0}^{\omega-1} r_2(k) \left[\frac{K_2(k) + \alpha_2(k) \exp \{y_1(k - \tau_1(k))\}}{1 + \exp \{y_1(k - \tau_1(k))\}} - \exp \{y_2(k - \sigma_2(k))\} \right] = 0. \tag{2.18}$$

It is easy to see that we can rewrite (2.17) and (2.18), respectively, as

$$\sum_{k=0}^{\omega-1} \frac{r_1(k)(\alpha_1(k) - K_1(k))}{1 + \exp \{y_2(k - \tau_2(k))\}} + \sum_{k=0}^{\omega-1} r_1(k) \exp \{y_1(k - \sigma_1(k))\} = \sum_{k=0}^{\omega-1} r_1(k) \alpha_1(k), \tag{2.19}$$

$$\sum_{k=0}^{\omega-1} \frac{r_2(k)(\alpha_2(k) - K_2(k))}{1 + \exp \{y_1(k - \tau_1(k))\}} + \sum_{k=0}^{\omega-1} r_2(k) \exp \{y_2(k - \sigma_2(k))\} = \sum_{k=0}^{\omega-1} r_2(k) \alpha_2(k). \tag{2.20}$$

Thus, from (2.16) and (2.19), it follows that

$$\begin{aligned} &\sum_{k=0}^{\omega-1} |y_1(k + 1) - y_1(k)| \\ &< \lambda \sum_{k=0}^{\omega-1} r_1(k) \left[\frac{K_1(k) + \alpha_1(k) \exp \{y_2(k - \tau_2(k))\}}{1 + \exp \{y_2(k - \tau_2(k))\}} + \exp \{y_1(k - \sigma_1(k))\} \right] \\ &< \sum_{k=0}^{\omega-1} r_1(k) \alpha_1(k) + \sum_{k=0}^{\omega-1} \frac{r_1(k)(\alpha_1(k) - K_1(k))}{1 + \exp \{y_2(k - \tau_2(k))\}} \\ &\quad + \sum_{k=0}^{\omega-1} r_1(k) \exp \{y_1(k - \sigma_1(k))\} \\ &= 2 \sum_{k=0}^{\omega-1} r_1(k) \alpha_1(k) := M_1, \end{aligned} \tag{2.21}$$

that is,

$$\sum_{k=0}^{\omega-1} |y_1(k+1) - y_1(k)| < M_1. \tag{2.22}$$

In a similar way, by (2.16) and (2.20), we have

$$\sum_{k=0}^{\omega-1} |y_2(k+1) - y_2(k)| < 2 \sum_{k=0}^{\omega-1} r_2(k)\alpha_2(k) := M_2. \tag{2.23}$$

Moreover, from (2.19), it follows that

$$\sum_{k=0}^{\omega-1} r_1(k)\alpha_1(k) \geq \sum_{k=0}^{\omega-1} r_1(k) \exp\{y_1(k - \sigma_1(k))\} \geq \sum_{k=0}^{\omega-1} r_1(k)K_1(k), \tag{2.24}$$

hence

$$\frac{r_1^* \alpha_1^*}{r_1} \omega \geq \sum_{k=0}^{\omega-1} \exp\{y_1(k - \sigma_1(k))\} \geq \frac{r_1 \underline{K}_1}{r_1^*} \omega, \tag{2.25}$$

which implies that there exist points $k'_1, k'_2 \in I_\omega$ such that

$$\begin{aligned} y_1(k'_1 - \sigma_1(k'_1)) &\leq \ln \left[\frac{r_1^* \alpha_1^*}{r_1} \right] := C_1, \\ y_1(k'_2 - \sigma_1(k'_2)) &\geq \ln \left[\frac{r_1 \underline{K}_1}{r_1^*} \right] := C_2, \end{aligned} \tag{2.26}$$

where $r_1^* = \max\{r_1(k), k \in I_\omega\}$, $\alpha_1^* = \max\{\alpha_1(k), k \in I_\omega\}$, $r_1 = \min\{r_1(k), k \in I_\omega\}$, $\underline{K}_1 = \min\{K_1(k), k \in I_\omega\}$. Denote $k'_i - \sigma_1(k'_i) = k_i + n_i\omega$, $k_i \in I_\omega$ and n_i is an integer, $i = 1, 2$. Then

$$y_1(k_1) \leq C_1, \quad y_1(k_2) \geq C_2. \tag{2.27}$$

Similarly, by (2.20), we can obtain that there exist points $k_3, k_4 \in I_\omega$ such that

$$y_2(k_3) \leq \ln \left[\frac{r_2^* \alpha_2^*}{r_2} \right] := C_3, \quad y_2(k_4) \geq \ln \left[\frac{r_2 \underline{K}_2}{r_2^*} \right] := C_4, \tag{2.28}$$

where $r_2^* = \max\{r_2(k), k \in I_\omega\}$, $\alpha_2^* = \max\{\alpha_2(k), k \in I_\omega\}$, $r_2 = \min\{r_2(k), k \in I_\omega\}$, and $\underline{K}_2 = \min\{K_2(k), k \in I_\omega\}$.

Therefore, in view of (2.22)–(2.27) and Lemma 2.2, we have

$$\begin{aligned} y_1(k) &\leq y_1(k_1) + \sum_{s=0}^{\omega-1} |y_1(s+1) - y_1(s)| < C_1 + M_1, \\ y_1(k) &\geq y_1(k_2) - \sum_{s=0}^{\omega-1} |y_1(s+1) - y_1(s)| > C_2 - M_1. \end{aligned} \tag{2.29}$$

Hence,

$$\max_{k \in I_\omega} |y_1(k)| < \max \{ |C_1 + M_1|, |C_2 - M_1| \} := A_1. \quad (2.30)$$

Similarly, it follows from (2.23)–(2.28) and Lemma 2.2 that

$$\max_{k \in I_\omega} |y_2(k)| < \max \{ |C_3 + M_2|, |C_4 - M_2| \} := A_2. \quad (2.31)$$

Clearly, A_1 and A_2 are independent of λ . Denote $M = A_1 + A_2 + D$, where $D > 0$ is taken sufficiently large such that $M > \max \{ |\ln(a_i/c_i)|, |\ln(b_i/c_i)|, i = 1, 2 \}$. Now we take $\Omega = \{y = \{y(k)\} \in X : \|y\| < M\}$. This satisfies condition (i) in Lemma 2.1. When $y = \{(y_1, y_2)^T\} \in \partial\Omega \cap \text{Ker } L = \partial\Omega \cap \mathbb{R}^2, (y_1, y_2)^T$ is a constant vector in \mathbb{R}^2 with $|y_1| + |y_2| = M$. Then

$$QN \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{\bar{r}_1 \alpha_1 - \bar{r}_1 \bar{K}_1}{1 + e^{y_2}} - \bar{r}_1 e^{y_1} \\ \frac{\bar{r}_2 \alpha_2 - \bar{r}_2 \bar{K}_2}{1 + e^{y_1}} - \bar{r}_2 e^{y_2} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (2.32)$$

Furthermore, by Lemma 2.3, we have

$$\deg \{ JQN(y_1, y_2)^T, \Omega, (0, 0) \} \neq 0, \quad (2.33)$$

where the degree is Brouwer degree and the isomorphism J can be chosen to be the identity mapping, since $\text{Im } Q = \text{Ker } L$. By now we know that Ω verifies all the requirements in Lemma 2.1 and then (2.6) has at least one ω -periodic solution. Therefore, (1.2) has at least one positive ω -periodic solution. The proof is complete. \square

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Yongkun Li: Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China
E-mail address: yklie@ynu.edu.cn