EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A CLASS OF INFINITE-HORIZON SYSTEMS DERIVED FROM OPTIMAL CONTROL

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This paper deals with the existence and uniqueness of solutions for a class of infinite-horizon systems derived from optimal control. An existence and uniqueness theorem is proved for such Hamiltonian systems under some natural assumptions.

1. Introduction

We begin with a simple example to introduce the background of the considered problem. Let U be a bounded closed subset of $\mathbb{R}^m$ and let functions $f : \mathbb{R}^n \times \mathbb{R}^m \times [a,\infty) \to \mathbb{R}^n$, $L : \mathbb{R}^n \times \mathbb{R}^m \times [a,\infty) \to \mathbb{R}$ be differentiable with respect to the first variable. Consider an optimal control system of the form

$$\text{Minimize } J[u(\cdot)] = \int_a^\infty L(x(t),u(t),t)\,dt \tag{1.1}$$

over all admissible controls $u(\cdot) \in L^2([a,\infty);U)$, where the trajectories $x : [a,\infty) \to \mathbb{R}^n$ are differentiable on $[a,\infty)$ and satisfy the dynamic system

$$\dot{x}(t) = f(x(t),u(t),t), \qquad x(a) = x_0. \tag{1.2}$$

From control theory, the well-known Pontryagin maximum principle, an important necessary optimality condition, is usually applied to get optimal controls for this system. By doing this, the following infinite-horizon Hamiltonian system is derived:

$$\begin{gathered} \dot{x}(t) = \frac{\partial H(x(t),p(t),t)}{\partial p}, \\ x(a) = x_0, \\ \dot{p}(t) = \frac{-\partial H(x(t),p(t),t)}{\partial x}, \\ x(\cdot) \in L^2([a,\infty);\mathbb{R}^n), \qquad p(\cdot) \in L^2([a,\infty);\mathbb{R}^n). \end{gathered} \tag{1.3}$$

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Here, $H(x,p,t) = \lambda L(x,\bar{u},t) + \langle p, f(x,\bar{u},t)\rangle$ is the Hamiltonian function for (1.1)-(1.2), $\langle \cdot,\cdot \rangle$ stands for inner product in $\mathbb{R}^n$, $\bar{u}$ is an optimal control, and $x(t)$ is the optimal trajectory corresponding to the optimal control $\bar{u}$.

The existence and uniqueness of solutions for system (1.3) is a very interesting question; if solutions to (1.3) are unique, then the optimal control for system (1.1)-(1.2) can be solved analytically or numerically through (1.3). When we consider the generalization of (1.3) in infinite-dimensional spaces, the following Hamiltonian system is obtained:

$$\begin{gathered} \dot{x}(t) = A(t)x(t) + F(x(t),p(t),t), \\ x(a) = x_0, \\ \dot{p}(t) = -A^*(t)p(t) + G(x(t),p(t),t), \\ x(\cdot) \in L^2([a,\infty);X), \qquad p(\cdot) \in L^2([a,\infty);X), \end{gathered} \tag{1.4}$$

where both $x(t)$ and $p(t)$ take values in a Hilbert space X for $a \le t < \infty$. It is always assumed that $F, G : X \times X \times [a,\infty) \to X$ are nonlinear operators, that $A(t)$ is a closed operator for each $t \in [a,\infty)$, and that $A^*(t)$ is the adjoint operator of $A(t)$.

The following system is called a linear Hamiltonian system, which is a special case of (1.4),

$$\begin{gathered} \dot{x}(t) = A(t)x(t) + B(t)p(t) + \varphi(t), \\ x(a) = x_0, \\ \dot{p}(t) = -A^*(t)p(t) + C(t)x(t) + \psi(t), \\ x(\cdot) \in L^2([a,\infty);X), \qquad p(\cdot) \in L^2([a,\infty);X), \end{gathered} \tag{1.5}$$

where $\varphi(\cdot), \psi(\cdot) \in L^2([a,\infty);X)$, and $B(t)$, $C(t)$ are selfadjoint linear operators from X to X for all $t \in [a,\infty)$.

In [2], Lions has discussed the existence and uniqueness of solutions for system (1.5) and gave an existence and uniqueness result. In [1], Hu and Peng considered the existence and uniqueness of solutions for a class of nonlinear forward-backward stochastic differential equations similar to (1.3) but on finite horizon, they provided an existence and uniqueness theorem for (1.3). Peng and Shi in [3] dealt with the existence and uniqueness of solutions for (1.3) using the techniques developed in [1]. In this paper, we consider the existence and uniqueness of solutions for infinite-dimensional system (1.4).

Throughout the paper, the following basic assumptions hold.

(I) There exists a real number $L > 0$ such that

$$\begin{gathered} ||F(x_1,p_1,t) - F(x_2,p_2,t)|| \le L(||x_1 - x_2|| + ||p_1 - p_2||), \\ ||G(x_1,p_1,t) - G(x_2,p_2,t)|| \le L(||x_1 - x_2|| + ||p_1 - p_2||) \end{gathered} \tag{1.6}$$

for all $x_1, p_1, x_2, p_2 \in X$ and $t \in [a,\infty)$.

(II) There exists a real number $\alpha > 0$ such that

$$\begin{aligned}&\langle F(x_1,p_1,t) - F(x_2,p_2,t), p_1 - p_2\rangle + \langle G(x_1,p_1,t) - G(x_2,p_2,t), x_1 - x_2\rangle \\ &\quad \le -\alpha(\|x_1 - x_2\| + \|p_1 - p_2\|)\end{aligned} \tag{1.7}$$

for all $x_1, p_1, x_2, p_2 \in X$ and $t \in [a, \infty)$.

2. Lemmas

Two lemmas are given in this section. They are essential to prove the main theorem.

Lemma 2.1. *Consider the Hamiltonian system*

$$\begin{gathered}\dot{x}(t) = A(t)x(t) + F_\beta(x,p,t) + \varphi(t), \\ x(a) = x_0, \\ \dot{p}(t) = -A^*(t)p(t) + G_\beta(x,p,t) + \psi(t), \\ x(\cdot) \in L^2([a,\infty);X), \qquad p(\cdot) \in L^2([a,\infty);X),\end{gathered} \tag{2.1}$$

where $\varphi(\cdot), \psi(\cdot) \in L^2([a,\infty);X)$. The functions F_β and G_β are defined as

$$\begin{gathered}F_\beta(x,p,t) := -(1-\beta)\alpha p + \beta F(x,p,t), \\ G_\beta(x,p,t) := -(1-\beta)\alpha x + \beta G(x,p,t).\end{gathered} \tag{2.2}$$

Assume that (2.1) has a unique solution for some real number $\beta = \beta_0 \ge 0$ and any $\varphi(t)$, $\psi(t)$. There exists a real number $\delta > 0$, which is independent of β_0, such that (2.1) has a unique solution for any $\varphi(t)$, $\psi(t)$, and $\beta \in [\beta_0, \beta_0 + \delta]$.

Proof. For any given $\varphi(\cdot), \psi(\cdot), x(\cdot), p(\cdot) \in L^2([a,\infty);X)$ and $\delta > 0$, construct the following Hamiltonian system:

$$\begin{gathered}\dot{X}(t) = A(t)X(t) + F_{\beta_0}(X,P,t) + F_{\beta_0+\delta}(x,p,t) - F_{\beta_0}(x,p,t) + \varphi(t), \\ X(a) = x_0, \\ \dot{P}(t) = -A^*(t)P(t) + G_{\beta_0}(X,P,t) + G_{\beta_0+\delta}(x,p,t) - G_{\beta_0}(x,p,t) + \psi(t), \\ X(\cdot) \in L^2([a,\infty);X), \qquad P(\cdot) \in L^2([a,\infty);X).\end{gathered} \tag{2.3}$$

Note that

$$\begin{aligned}&F_{\beta_0+\delta}(x,p,t) - F_{\beta_0}(x,p,t) \\ &\quad = -(1-\beta_0-\delta)\alpha p + (\beta_0+\delta)F(x,p,t) + (1-\beta_0)\alpha p - \beta_0 F(x,p,t) \\ &\quad = \alpha\delta p + \delta F(x,p,t), \\ &G_{\beta_0+\delta}(x,p,t) - G_{\beta_0}(x,p,t) \\ &\quad = -(1-\beta_0-\delta)\alpha x + (\beta_0+\delta)G(x,p,t) + (1-\beta_0)\alpha x - \beta_0 G(x,p,t) \\ &\quad = \alpha\delta x + \delta G(x,p,t).\end{aligned} \tag{2.4}$$

The assumption of Lemma 2.1 implies that (2.3) has a unique solution for each pair $(x(\cdot),p(\cdot)) \in L^2([a,\infty);X)\times L^2([a,\infty);X)$. Therefore, the mapping J,

$$L^2([a,\infty);X)\times L^2([a,\infty);X) \longrightarrow L^2([a,\infty);X)\times L^2([a,\infty);X), \tag{2.5}$$

given by

$$J(x(\cdot),p(\cdot)) := (X(\cdot),P(\cdot)) \tag{2.6}$$

is well defined.

Let $J(x_1(\cdot),p_1(\cdot)) = (X_1(\cdot),P_1(\cdot))$ and $J(x_2(\cdot),p_2(\cdot)) = (X_2(\cdot),P_2(\cdot))$. Since $X_1(\cdot)-X_2(\cdot) \in L^2([a,\infty);X)$ and $P_1(\cdot)-P_2(\cdot) \in L^2([a,\infty);X)$, there exists a sequence of real numbers $a < t_1 < t_2 < \cdots < t_k < \cdots$ such that $t_k \to \infty$ as $k \to \infty$ and

$$X_1(t_k)-X_2(t_k) \longrightarrow 0, \quad P_1(t_k)-P_2(t_k) \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{2.7}$$

Note that

$$\begin{aligned}
&\frac{d}{dt}\langle X_1(t)-X_2(t),P_1(t)-P_2(t)\rangle \\
&= \langle F_{\beta_0}(X_1,P_1,t)-F_{\beta_0}(X_2,P_2,t)+\alpha\delta(p_1-p_2)+\delta(F(x_1,p_1,t)-F(x_2,p_2,t)),P_1-P_2\rangle \\
&\quad + \langle G_{\beta_0}(X_1,P_1,t)-G_{\beta_0}(X_2,P_2,t)+\alpha\delta(x_1-x_2)+\delta(G(x_1,p_1,t)-G(x_2,p_2,t)),X_1-X_2\rangle \\
&:= I_1+I_2.
\end{aligned} \tag{2.8}$$

Since

$$F_{\beta_0}(X_1,P_1,t)-F_{\beta_0}(X_2,P_2,t) = -\alpha(1-\beta_0)(P_1-P_2)+\beta_0(F(X_1,P_1,t)-F(X_2,P_2,t)) \tag{2.9}$$

implies that

$$\begin{aligned}
I_1 = {}&-\alpha(1-\beta_0)||P_1-P_2||^2+\beta_0\langle F(X_1,P_1,t)-F(X_2,P_2,t),P_1-P_2\rangle \\
&+\alpha\delta\langle p_1-p_2,P_1-P_2\rangle+\delta\langle F(x_1,p_1,t)-F(x_2,p_2,t),P_1-P_2\rangle,
\end{aligned} \tag{2.10}$$

similarly,

$$G_{\beta_0}(X_1,P_1,t)-G_{\beta_0}(X_2,P_2,t) = -\alpha(1-\beta_0)(X_1-X_2)+\beta_0(G(X_1,P_1,t)-G(X_2,P_2,t)) \tag{2.11}$$

implies that

$$\begin{aligned}
I_2 = {}&-\alpha(1-\beta_0)||X_1-X_2||^2+\beta_0\langle G(X_1,P_1,t)-G(X_2,P_2,t),X_1-X_2\rangle \\
&+\alpha\delta\langle x_1-x_2,X_1-X_2\rangle+\delta\langle G(x_1,p_1,t)-G(x_2,p_2,t),X_1-X_2\rangle.
\end{aligned} \tag{2.12}$$

It follows from the estimates for I_1, I_2, and the assumption (I) that

$$\begin{aligned} I_1+I_2 \le & -\alpha(||X_1-X_2||^2+||P_1-P_2||^2) \\ & +\alpha\delta(||p_1-p_2||\,||P_1-P_2||+||x_1-x_2||\,||X_1-X_2||) \\ & +\delta||F(x_1,p_1,t)-F(x_2,p_2,t)||\,||P_1-P_2|| \\ & +\delta||G(x_1,p_1,t)-G(x_2,p_2,t)||\,||X_1-X_2|| \\ \le & -\alpha(||X_1-X_2||^2+||P_1-P_2||^2) \\ & +\delta(2L+\alpha)(||X_1-X_2||^2+||P_1-P_2||^2+||x_1-x_2||^2+||p_1-p_2||^2). \end{aligned} \tag{2.13}$$

Therefore,

$$\begin{aligned} & \frac{d}{dt}\langle X_1(t)-X_2(t),P_1(t)-P_2(t)\rangle \\ & \quad \le -\alpha(||X_1-X_2||^2+||P_1-P_2||^2) \\ & \quad\quad +\delta(2L+\alpha)(||X_1-X_2||^2+||P_1-P_2||^2+||x_1-x_2||^2+||p_1-p_2||^2). \end{aligned} \tag{2.14}$$

Integrating between a and t_k, we have

$$\begin{aligned} & \langle X_1(t_k)-X_2(t_k),P_1(t_k)-P_2(t_k)\rangle - \langle X_1(a)-X_2(a),P_1(a)-P_2(a)\rangle \\ & \quad \le -\alpha\int_a^{t_k}(||X_1-X_2||^2+||P_1-P_2||^2)\,dt+\delta(2L+\alpha) \\ & \quad\quad \times\int_a^{t_k}(||X_1-X_2||^2+||P_1-P_2||^2+||x_1-x_2||^2+||p_1-p_2||^2)\,dt. \end{aligned} \tag{2.15}$$

Letting $k\to\infty$ and noting that (2.7), we obtain

$$\int_a^\infty(||X_1-X_2||^2+||P_1-P_2||^2)\,dt \le \frac{2\delta L+\delta\alpha}{\alpha-2\delta L-\delta\alpha}\int_a^\infty(||x_1-x_2||^2+||p_1-p_2||^2)\,dt. \tag{2.16}$$

Choose a small δ (independent of β_0) such that

$$\frac{2\delta L+\delta\alpha}{\alpha-2\delta L-\delta\alpha}\le\frac{1}{2}, \tag{2.17}$$

then J is a contractive mapping and hence has a unique fixed point. Thus, (2.3) becomes

$$\begin{gathered} \dot{x}(t)=A(t)x(t)+F_{\beta_0+\delta}(x,p,t)+\varphi(t), \\ x(a)=x_0, \\ \dot{p}(t)=-A^*(t)p(t)+G_{\beta_0+\delta}(x,p,t)+\psi(t), \\ x(\cdot)\in L^2([a,\infty);X), \qquad p(\cdot)\in L^2([a,\infty);X). \end{gathered} \tag{2.18}$$

This shows that system (2.1) has a unique solution on $[a,\infty)$ for $\beta\in[\beta_0,\beta_0+\delta]$. The proof is complete. □

Lemma 2.2. *System (2.1) has a unique solution on* $[a,\infty)$ *for* $\beta = 0$, *that is, the system*

$$\begin{gathered}\dot{x}(t) = A(t)x(t) - \alpha p(t) + \varphi(t),\\ x(0) = x_0,\\ \dot{p}(t) = -A^*(t)p(t) - \alpha x(t) + \psi(t),\\ x(\cdot) \in L^2([a,\infty);X), \qquad p(\cdot) \in L^2([a,\infty);X),\end{gathered} \tag{2.19}$$

has a unique solution on $[a,\infty)$.

For the proof, see [2, Section 6.2, Chapter III].

3. Main theorem

Theorem 3.1. *System (1.4) has a unique solution under assumptions (I) and (II).*

Proof. By Lemma 2.2, system (2.1) has a unique solution on $[a,\infty)$ in the case $\beta_0 = 0$. It follows from Lemma 2.1 that there exists a real number $\delta > 0$ such that (2.1) has a unique solution on $[a,\infty)$ for any $\beta \in [0,\delta]$ and $\varphi,\psi \in L^2([a,\infty);X)$. Let $\beta_0 = \delta$ in Lemma 2.1. Repeating this procedure implies that (2.1) has a unique solution on $[a,\infty)$ for any $\beta \in [\delta,2\delta]$ and φ, $\psi \in L^2([a,\infty);X)$. After finitely many steps, one can show that system (2.1) has a unique solution for $\beta = 1$. Therefore, it is proved that system (1.4) has a unique solution on $[a,\infty)$ by letting $\beta = 1$, $\varphi(t) \equiv 0$, and $\psi(t) \equiv 0$. □

Remark 3.2. Consider system (1.5). Note that

$$\begin{aligned}&\langle F(x_1,p_1,t) - F(x_2,p_2,t), p_1 - p_2\rangle + \langle G(x_1,p_1,t) - G(x_2,p_2,t), x_1 - x_2\rangle\\ &\quad = \langle B(t)(p_1 - p_2), p_1 - p_2\rangle + \langle C(t)(x_1 - x_2), x_1 - x_2\rangle.\end{aligned} \tag{3.1}$$

By Theorem 3.1, system (1.5) has a unique solution if it is assumed that both $B(t)$ and $C(t)$ are uniformly negative definite on $[a,\infty)$, that is, there exists a real number $\gamma > 0$ such that $\langle B(t)x,x\rangle \le -\gamma\|x\|^2$ and $\langle C(t)x,x\rangle \le -\gamma\|x\|^2$ for all $x \in X$, $x \ne 0$, and $t \in [a,\infty)$.

Remark 3.3. Consider the control system

$$\dot{x}(t) = A(t)x(t) + Bu(t), \qquad x(a) = x_0, \tag{3.2}$$

with a quadratic cost functional

$$J[u(\cdot)] = \int_a^\infty [\langle Qx(t),x(t)\rangle + \langle Ru(t),u(t)\rangle]\,dt, \tag{3.3}$$

where $u(t)$ and $x(t)$ take values in Hilbert spaces U and X, where $B \in \mathcal{L}[U,X]$, and where $Q \in \mathcal{L}[X,X]$ and $R \in \mathcal{L}[U,U]$ are selfadjoint operators.

From optimal control theory, the following Hamiltonian system is derived:

$$\begin{gathered}\dot{x}(t)=A(t)x(t)-BR^{-1}Bp(t),\\ x(a)=x_0,\\ \dot{p}(t)=-A^*(t)p(t)-Qx(t),\\ x(\cdot)\in L^2([a,\infty);X),\qquad p(\cdot)\in L^2([a,\infty);X).\end{gathered}\tag{3.4}$$

This is a special case of system (1.5). Therefore, system (3.4) has a unique solution if both $BR^{-1}B$ and Q are positive definite.

References

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