THE ARMENDARIZ MODULE AND ITS APPLICATION TO THE IKEDA-NAKAYAMA MODULE

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A ring R is called a *right Ikeda-Nakayama* (for short IN-ring) if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators, that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$ for all right ideals I and J of R. R is called *Armendariz ring* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. In this paper, we show that if $R[x]$ is a right IN-ring, then R is a right IN-ring in case R is an Armendariz ring.

1. Introduction

Throughout this work, all rings will be associative with identity. Let R be a ring. A right (or left) annihilator of a subset U of R is defined by $r_R(U) = \{a \in R : Ua = 0\}$ (or $\ell_R(U) = \{a \in R : aU = 0\}$).

Recall that, a ring R is called a *right Ikeda-Nakayama* ring if the left annihilator of the intersection of any two right ideals is the sum of the left annihilators, that is, if $\ell(I \cap J) = \ell(I) + \ell(J)$ for all right ideals I and J of R (cf. [6]). Let ${}_SM_R$ be an (S,R)-bimodule. Extend the notion of an IN-ring to module such as $\ell_S(A \cap B) = \ell_S(A) + \ell_S(B)$ for any submodules A, B of M_R (cf. [10]).

For a module M_R, let $M[x]$ be the set of all formal polynomials in indeterminate x with coefficients from M. Then $M[x]$ becomes a right $R[x]$-module under usual addition and multiplication of polynomials.

We prove that if ${}_{S[x]}M[x]_{R[x]}$-bimodule and U and V are $R[x]$-submodules of $M[x]_{R[x]}$, then for any $t(x) \in \ell_{S[x]}(U \cap V)$, every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$, where $\lambda : S[x] \to \operatorname{End}(M[x]_{R[x]})$ if and only if $M[x]$ is an IN-module.

Following [1], R is called *Armendariz ring* if whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j.

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A module M is called *α-Armendariz* if

(i) for any $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$;

(ii) for any $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x]$, $m(x)f(x) = 0$ implies $m_i a_j = 0$ for each i, j (cf. [8, 9]).

In [5, Proposition 3.1], Hirano showed that R is Armendariz ring if and only if $r\mathrm{Ann}_R(2^R) \to r\mathrm{Ann}_R(2^{R[x]})$; $A \to AR[x]$ is bijective, where $r\mathrm{Ann}_R(2^R) = \{r_R(U) : U \subseteq R\}$. Using this proposition, in this paper, it is shown that if $R[x]$ is a right IN-ring, then R is a right IN-ring, in case R is an Armendariz ring.

2. Ikeda-Nakayama modules

Let $S[x]$ and $R[x]$ be the polynomial rings over rings S and R and, for a module ${}_SM_R$, let $M[x]$ be the set of all formal polynomials in indeterminate x with coefficients from M. Then $M[x]$ becomes an $(S[x],R[x])$-bimodule under usual addition and multiplication of polynomials. Extend the notion of an IN-ring to module such as the following.

Definition 2.1. Recall that $M[x]$ is called an *Ikeda-Nakayama module (IN-module)* if

$$\ell_{S[x]}(U \cap V) = \ell_{S[x]}(U) + \ell_{S[x]}(V) \tag{2.1}$$

for any $R[x]$-submodules U and V of $M[x]_{R[x]}$. Such modules and rings were studied by many authors (cf. [4, 6, 10]). Professor Harmanci asked (private communication) for a description modules M (rings R) such that $M[x]$ ($R[x]$) are Ikeda-Nakayama modules (right Ikeda-Nakayama rings), respectively.

Note that there is a canonical ring homomorphism $\lambda : S[x] \to \mathrm{End}(M[x]_{R[x]})$ given by $\lambda(s(x))(m(x)) = s(x)m(x)$ for $m(x) \in M[x]$ and $s(x) \in S[x]$.

Let U and V be $R[x]$-submodules of $M[x]$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, $\alpha_{t(x)} : U + V \to M[x]$, $u + v \to t(x)u$ is well defined, where $u \in U$ and $v \in V$.

LEMMA 2.2. *Let ${}_{S[x]}M[x]_{R[x]}$-bimodule and U and V be $R[x]$-submodules of $M[x]_{R[x]}$. Then, for any $t(x) \in \ell_{S[x]}(U \cap V)$, every $U + V \xrightarrow{\alpha_{t(x)}} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $M[x]$ is an IN-module.*

In particular, if $U \cap V = 0$, then every $U + V \xrightarrow{\alpha_1} M[x]$ extends commutatively to $M[x]$ by $\lambda(s(x))$ for some $s(x) \in S[x]$ if and only if $S[x] = \ell_{S[x]}(U) + \ell_{S[x]}(V)$.

Proof. Let $t(x) \in \ell_{S[x]}(U \cap V)$. Then $\alpha_{t(x)} : U + V \to M[x]$, $u + v \to t(x)u$ is a well defined $R[x]$-module homomorphism, where $u \in U$ and $v \in V$. By assumption, there exists $s(x) \in S[x]$ such that $\lambda(s(x))$ extends to $\alpha_{t(x)}$. Thus, for all $u \in U$ and $v \in V$, $t(x)u = \alpha_{t(x)}(u+v) = \lambda(s(x))(u+v) = s(x)(u+v)$ and so $(t(x) - s(x))u + (-s(x))v = 0$. It follows that $t(x) - s(x) \in \ell_{S[x]}(U)$ and $-s(x) \in \ell_{S[x]}(V)$. Hence $t(x) = (t(x) - s(x)) + (-s(x)) \in \ell_{S[x]}(U) + \ell_{S[x]}(V)$. The other inclusion is clear.

For converse, assume that $M[x]$ is an IN-module and, for any $t(x) \in \ell_{S[x]}(U \cap V)$, $\alpha_{t(x)} : U + V \to M[x]$ defined as above. For $a(x) \in \ell_{S[x]}(U)$ and $b(x) \in \ell_{S[x]}(V)$, write $t(x) = a(x) + b(x)$. Then, for all $u \in U$ and $v \in V$, $\alpha_{t(x)}(u+v) = t(x)u = (a(x)+b(x))u = a(x)u + b(x)u = 0 + b(x)u = b(x)u = b(x)u + b(x)v = b(x)(u+v) = \lambda(b(x)(u+v))$. $\square$

As a result of Lemma 2.2, we have the following proposition.

Proposition 2.3. *Let $R[x]$ be the set of all polynomials in indeterminate x with coefficients from R. If I and J are right ideals of $R[x]$ such that every $R[x]$-linear map $I+J \to R[x]$ extends to $R[x]$, then*

$$\ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) + \ell_{R[x]}(J). \tag{2.2}$$

In particular, this holds if $I+J = R[x]$, in which case $\ell_{R[x]}(I \cap J) = \ell_{R[x]}(I) \oplus \ell_{R[x]}(J)$.

Let N be an $R[x]$-submodule of $M[x]$ and $N_C = \{m_i \in M : \exists n \in N \text{ with } n = m_0 + m_1x + \cdots + m_tx^t\}$.

Theorem 2.4. *Let M be an Ikeda-Nakayama module and let N and K be $R[x]$-submodules of $M[x]$ such that $\ell_S((N \cap K)_C) = \ell_S(N_C \cap K_C)$. Then $M[x]$ is an IN-module.*

Proof. Let U and V be $R[x]$-submodules of $M[x]$. Let $t(x) \in \ell_{S[x]}(U \cap V)$. Then $\alpha_{t(x)} : U+V \to M[x], u+v \to t(x)u$ is a well defined $R[x]$-homomorphism, where $u \in U$ and $v \in V$. Similarly, for all $t \in \ell_S(U_C \cap V_C)$, the $\alpha_t : U_C + V_C \to M, u' + v' \to tu'$ is a well defined R-homomorphism, where $u' \in U_C$ and $v' \in V_C$. Since M is an IN-module, we have $\ell_S(U \cap V)_C) = \ell_S(N_C \cap K_C) = \ell_S(U_C) + \ell_S(V_C)$ by assumption and definition. Hence there exists a homomorphism $h_t : M \to M$ such that $h_t i = \alpha_t$, where $i : U_C + V_C \to M$ is the inclusion map by [10, Lemma 1]. We define $h' : M[x] \to M[x]$ such that $h'_t(k_0 + k_1x + \cdots + k_nx^n) = h_t(k_0) + h_t(k_1)x + \cdots + h_t(k_n)x^n$. It is clear that h'_t is well defined. Let $t(x) = t_0 + t_1x + t_2x^2 + \cdots + t_nx^n \in \ell_{S[x]}(U \cap V)$. Then $t_0, t_1, \ldots, t_n \in \ell_S((U \cap V)_C) = \ell_S(U_C) + \ell_S(V_C)$. For each $t_j, \alpha_{t_j} : U_C + V_C \to M$, $u' + v' \to tu'$ is a well defined R-homo-morphism, and then we define a map $h_{t_j} : M \to M$ such that $h_{t_j} i = \alpha_{t_j}$, where $i : U_C + V_C \to M$ is the inclusion map. We extend it by defining $h'_{t_j} : M[x] \to M[x]$ such that, for $j = 0, 1, 2, \ldots, n$, $h'_j(k_0 + k_1x + \cdots + k_nx^n) = (h_{t_j}(k_0) + h_{t_j}(k_1)x + \cdots + h_{t_j}(k_n)x^n)x^j$.

To complete the proof, we show that $hi = \alpha_{t(x)}$, where $i' : U+V \to M[x]$ is the inclusion map. Let $h = \sum_{j=0}^n h'_j$ and $u = u_0 + u_1x + \cdots + u_rx^r \in U$ and $v(x) = v_0 + v_1x + \cdots + v_sx^s \in V$. Then $u_0, u_1, \ldots, u_r \in U_C$ and $v_0, v_1, \ldots, v_s \in V_C$. So $h'_j(u+v) = (h_{t_j}(u_0) + h_{t_j}(u_1)x + \cdots + h_{t_j}(u_r)x^r)x^j = t_jx^j(u_0 + u_1x + \cdots + u_rx^r)$ and $h(u+v) = \sum_{j=0}^n h'_j(u+v) = t(x)(u+v)$. Hence $M[x]$ is an IN-module by Lemma 2.2. □

Let α be an endomorphism of R, that is, α is a ring homomorphism from R to R with $\alpha(1) = 1$. Following [9], a module M is called *α-Armendariz* if

(1) for any $m \in M$ and $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$;
(2) for any $m(x) = \sum_{i=0}^n m_ix^i \in M[x]$ and $f(x) = \sum_{j=0}^s a_jx^j \in R[x]$, $m(x)f(x) = 0$ implies $m_ia_j = 0$ for each i, j.

Note that 1-Armendariz module is called Armendariz module.

We denote $r\text{Ann}_R(2^M) = \{r_R(U) \mid U \subseteq M\}$ and $\ell\text{Ann}_R(2^M) = \{\ell_R(U) \mid U \subseteq M\}$. If U is a subset of M, then $\ell_{R[x]}(U) = \ell_R(U)[x]$ and $r_{R[x]}(U) = r_R(U)[x]$. Hence we have the maps

$$\Phi : r\text{Ann}_R(2^M) \longrightarrow r\text{Ann}_{R[x]}(2^{M[x]}) \tag{2.3}$$

defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$ for every $r_R(U) \in r\mathrm{Ann}_R(2^M)$ and

$$\Phi' : \ell\mathrm{Ann}_R(2^M) \longrightarrow \ell\mathrm{Ann}_{R[x]}(2^{M[x]}) \tag{2.4}$$

defined by $\Phi'(\ell_R(U)) = \ell_{R[x]}(U) = \ell_R(U)[x]$ for every $\ell_R(U) \in \ell\mathrm{Ann}_R(2^M)$.

For a polynomial $m(x) \in M[x]$, $C_{m(x)}$ denotes the set of coefficients of $m(x)$ and for a subset V of $M[x]$, C_V denotes the set $\bigcup_{m(x)\in V} C_{m(x)}$. Then

$$r_{R[x]}(V) \cap R = r_R(V) = r_R(C_V), \qquad \ell_{R[x]}(V) \cap R = \ell_R(V) = \ell_R(C_V). \tag{2.5}$$

Hence we also have the maps

$$\Psi : r\mathrm{Ann}_{R[x]}(2^{M[x]}) \longrightarrow r\mathrm{Ann}_R(2^M) \tag{2.6}$$

defined by $\Psi(r_{R[x]}(V)) = r_{R[x]}(V) \cap R$ for every $r_{R[x]}(V) \in r\mathrm{Ann}_{R[x]}(2^{M[x]})$ and

$$\Psi' : \ell\mathrm{Ann}_{R[x]}(2^{M[x]}) \longrightarrow \ell\mathrm{Ann}_R(2^M) \tag{2.7}$$

defined by $\Psi'(\ell_{R[x]}(V)) = \ell_{R[x]}(V) \cap R$ for every $\ell_{R[x]}(V) \in \ell\mathrm{Ann}_{R[x]}(2^{M[x]})$.

Obviously Φ (or Φ') is injective and Ψ (or Ψ') is surjective. Also, Φ (or Φ') is surjective if and only if Ψ (or Ψ') is injective and in this case Φ and Ψ (or Φ' and Ψ') are the inverses of each other.

Proposition 2.5. *Let M_R be a module. Then the following are equivalent.*

(1) *M_R is an Armendariz module.*

(2) *The map $\Phi : r\mathrm{Ann}_R(2^M) \to r\mathrm{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$, for every $r_R(U) \in r\mathrm{Ann}_R(2^M)$, is bijective.*

(3) *The map $\Phi' : \ell\mathrm{Ann}_R(2^M) \to \ell\mathrm{Ann}_{R[x]}(2^{M[x]})$ defined by $\Phi'(\ell_R(U)) = \ell_{R[x]}(U) = \ell_R(U)[x]$, for every $\ell_R(U) \in \ell\mathrm{Ann}_R(2^M)$, is bijective.*

Proof. (1)$\Leftrightarrow$(2). This is [3, Theorem 2.1].

(1)$\Rightarrow$(3). Assume M is an Armendariz module. Obviously Φ' is injective. So it is enough to show Φ' is surjective. Let $\ell_{R[x]}(V) \in \ell\mathrm{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$. Then for $\ell_R(C_V) \in \ell\mathrm{Ann}_R(2^M)$, $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(C_V) = \ell_{R[x]}(V)$. In fact, let $f(x) \in \ell_{R[x]}(C_V)$, where $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $f(x)C_V = 0$. Thus for all $m \in C_V$, $f(x)m = a_0m + a_1mx + \cdots + a_nmx^n = 0$ and hence $a_jm = 0$ for all j. Let $n(x) = n_0 + n_1x + \cdots + n_tx^t \in V$ be arbitrary. Then $f(x)n(x) = 0$ since $n_i \in C_V$ for all i. Hence $f(x) \in \ell_{R[x]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in \ell_{R[x]}(V)$. Then for all $m(x) \in V$, $g(x)m(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_lx^l \in V$. Since M_R is Armendariz, $b_jm_i = 0$ for all i and j. Hence $g(x)m_i = 0$ for all i. So $g(x) \in \ell_{R[x]}(C_V)$ since $m(x) \in V$ is arbitrary. Consequently for each $\ell_{R[x]}(V) \in \ell\mathrm{Ann}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$ there exists $\ell_R(C_V) \in \ell\mathrm{Ann}_R(2^M)$ such that $\Phi'(\ell_R(C_V)) = \ell_{R[x]}(V)$, and therefore Φ' is surjective.

(3)⇒(1). Conversely, assume $f(x)m(x) = 0$, where $m(x) = m_0 + m_1x + \cdots + m_tx^t \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$. By hypothesis, $\ell_{R[x]}(m(x)) = \ell_R(U)[x]$ for some $U \subseteq M$. Then $f(x) \in \ell_R(U)[x]$ and hence $a_j \in \ell_R(U)$ for all j. So $a_j \in \ell_R(U) \subseteq \ell_R(U)[x] = \ell_{R[x]}(m(x))$ then $a_jm(x) = 0$. Consequently, $a_jm_i = 0$ for all i and j. Therefore M_R is an Armendariz module. □

By Proposition 2.5, we can obtain [5, Proposition 3.1].

Proposition 2.6. *Let R be a ring. The following statements are equivalent.*

(1) *R is Armendariz ring.*

(2) *$rAnn_R(2^R) \to rAnn_R(2^{R[x]})$; $A \to AR[x]$ is bijective, where $rAnn_R(2^R) = \{r_R(U) : U \subseteq R\}$.*

(3) *$\ell Ann_R(2^R) \to \ell Ann_R(2^{R[x]})$; $B \to R[x]B$ is bijective, where $\ell Ann_R(2^R) = \{\ell_R(U) : U \subseteq R\}$.*

Now, we give the main result of this work.

Theorem 2.7. *Let R be an Armendariz ring. If $R[x]$ is a right IN-ring, then R is a right IN-ring.*

Proof. Let I and J be right ideals of R. Since R is an Armendariz ring, we have $\ell_{R[x]}(I) = \ell_R(I)[x]$ by Proposition 2.6, for every right ideal I of R. Note that $\ell_{R[x]}(I) = \ell_{R[x]}(I[x])$. By assumption, $\ell_{R[x]}(I) + \ell_{R[x]}(J) = \ell_{R[x]}(I[x]) + \ell_{R[x]}(J[x]) = \ell_{R[x]}(I[x] \cap J[x]) = \ell_{R[x]}((I \cap J)[x]) = \ell_{R[x]}(I \cap J)$. Then $\ell_R((I \cap J)[x]) = \ell_R(I[x]) + \ell_R(J[x]) = (\ell_R(I) + \ell_R(J))[x]$ implies that $\ell_R(I \cap J) = \ell_R(I) + \ell_R(J)$. So R is a right IN-ring. □

Example 2.8. (i) Since $\mathbb{Z}$ is an Armendariz ring, $\mathbb{Z}$ is a right IN-ring if and only if $\mathbb{Z}[x]$ is an IN-ring.

(ii) Let R be a trivial extension of $\mathbb{Z}$ and the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$, that is, $R = \mathbb{Z} \oplus \mathbb{Z}_{2^\infty}$ with the following addition and multiplication:

$$\begin{aligned} (n,a) + (m,b) &= (n+m, a+b), \\ (n,a)(m,b) &= (nm, nb + ma). \end{aligned} \tag{2.8}$$

Also R is the subring $\{\left(\begin{smallmatrix} a & n \\ 0 & a \end{smallmatrix}\right) : a \in \mathbb{Z},\ n \in \mathbb{Z}_{2^\infty}\}$. R is an IN-ring by [10]. As Lee and Zhou pointed out [8, Corollary 2.7], R is an Armendariz ring. We consider the right ideals I and J of $R[x]$:

$$\begin{aligned} I &= \left\{ \begin{pmatrix} px^2 & u(x) \\ 0 & px^2 \end{pmatrix} : u(x) \in \mathbb{Z}_{2^\infty},\ p \text{ is prime} \right\}, \\ J &= \left\{ \begin{pmatrix} qx + qx^2 & 0 \\ 0 & qx + qx^2 \end{pmatrix} : q \text{ is prime and } (p,q) = 1 \right\}. \end{aligned} \tag{2.9}$$

Clearly, $\ell_{R[x]}(I \cap J) = R[x]$ since p and q are primes with $(p,q) = 1$ and so $I \cap J = 0$. But $\ell_{R[x]}(I)$ and $\ell_{R[x]}(J)$ do not contain constant. Therefore, $\ell_{R[x]}(I) + \ell_{R[x]}(J) \neq \ell_{R[x]}(I \cap J)$. So $R[x]$ is not a right IN-ring by Proposition 2.3.

Recall that, a ring R is called *reduced ring* if it has no nonzero nilpotent elements, a ring R is called right *p.p.-ring* for all $a \in R$, $r_R(a) = eR$, where $e^2 = e \in R$ and R is called *Baer ring*, for all $I \leq_R R$, $r_R(I) = eR$, where $e^2 = e \in R$.

As a result of Theorem 2.7, we can say the following corollary.

Corollary 2.9. *Let $R[x]$ be a right IN-ring. Then R is a right IN-ring in each of the following cases.*
(1) $R^2 = 0$.
(2) *R is a reduced ring.*
(3) *R is an Abelian (if every idempotent of R is central) and von Neumann regular ring.*
(4) *R is an Abelian right (left) p.p.-ring.*
(5) *R is an Abelian Baer ring.*

Proof. Assume $R[x]$ is a right IN-ring.
(1) By [1], if $R^2 = 0$, then R is an Armendariz ring.
(2) By [2], reduced rings are Armendariz.
(3) Every Abelian von Neumann regular ring is a reduced ring.
(4) By [1, Theorem 6] or [7, Lemma 7], if R is an Abelian right (left) p.p.-ring, then R is an Armendariz (a Reduced and so Armendariz) ring.
(5) Every Abelian Baer ring is a reduced ring.

Hence R is a right IN-ring by Theorem 2.7. □

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References

[1] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Communications in Algebra **26** (1998), no. 7, 2265–2272.
[2] E. P. Armendariz, *A note on extensions of Baer and P.P.-rings*, Journal of the Australian Mathematical Society, Series A **18** (1974), 470–473.
[3] M. Baser and A. Harmanci, *Reduced and p.q.-bear modules*, http://www.math.nthu.edu.tw/~tjm/myweb/FrameToAppear.htm.
[4] V. Camillo, W. K. Nicholson, and M. F. Yousif, *Ikeda-Nakayama rings*, Journal of Algebra **226** (2000), no. 2, 1001–1010.
[5] Y. Hirano, *On annihilator ideals of a polynomial ring over a noncommutative ring*, Journal of Pure and Applied Algebra **168** (2002), no. 1, 45–52.
[6] M. Ikeda and T. Nakayama, *On some characteristic properties of quasi-Frobenius and regular rings*, Proceedings of the American Mathematical Society **5** (1954), no. 1, 15–19.
[7] N. K. Kim and Y. Lee, *Armendariz rings and reduced rings*, Journal of Algebra **223** (2000), no. 2, 477–488.
[8] T.-K. Lee and Y. Zhou, *Armendariz and reduced rings*, Communications in Algebra **32** (2004), no. 6, 2287–2299.

[9] ______, *Reduced modules*, Rings, Modules, Algebras, and Abelian Groups, Lecture Notes in Pure and Appl. Math., vol. 236, Marcel Dekker, New York, 2004, pp. 365–377.

[10] R. Wisbauer, M. F. Yousif, and Y. Zhou, *Ikeda-Nakayama modules*, Beiträge zur Algebra und Geometrie **43** (2002), no. 1, 111–119.

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