A VORONOVSKAYA-TYPE THEOREM FOR A POSITIVE LINEAR OPERATOR

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We consider a sequence of positive linear operators which approximates continuous functions having exponential growth at infinity. For these operators, we give a Voronovskaya-type theorem.

1. Introduction

Sequences of positive linear operators are often used in approximation theory. Let $(L_n)_{n\geq 1}$ be such a sequence, where the operators L_n are defined on a suitable linear subspace E of $C(I)$, $I \subset \mathbb{R}$ an interval. An important problem is the investigation of the limit

$$\lim_{n\to\infty} n(L_n f - f) \tag{1.1}$$

in order to obtain information about the rate of convergence and the saturation properties of the sequence (L_n).

The above formula is called *Voronovskaya's formula* for the sequence $(L_n)_{n\geq 1}$.

This paper is devoted to establishing a Voronovskaya-type formula for the sequence of positive linear operators introduced in [1], which approximate continuous functions of exponential order. To obtain the operators, we consider $g(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(1) \neq 0$, an analytic function in the disk $|z| < R$, $R > 1$, and we define the polynomials p_k by the relation

$$g(u)\cosh(ux) = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1.2}$$

where $\cosh x = \sum_{k=0}^{\infty}(x^{2k}/(2k)!)$ is the hyperbolic cosine of x. Therefore, the polynomials are

$$p_k(x) = \sum_{\nu=0}^{k} a_\nu \frac{x^{k-\nu}}{(k-\nu)!}\cdot\frac{1+(-1)^{k-\nu}}{2}. \tag{1.3}$$

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Let $C[0,\infty)$ be the set of all real-valued functions continuous on $[0,\infty)$ and $w_p(x) = e^{-px}$, $x \geq 0$, $p > 0$, the weight function. We will work in the space of functions $C_p = \{f \in C[0,\infty) : w_p f$ is uniformly continuous and bounded on $[0,\infty)\}$, with the norm $\|f\|_p = \sup_{x\in[0,\infty)} w_p(x)|f(x)|$.

We define the operator $P_n : C_p \to C_r$, $r > p$, by the relation

$$P_n(f;x) = \frac{1}{g(1)\cosh(nx)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{1.4}$$

We consider that $a_n/g(1) \geq 0$, $n = 0,1,\ldots$, which implies that the P_n operator is positive.

We proved in [1] the following theorem.

Theorem 1.1. *If $f \in C_p$, then for each $x \geq 0$, $\lim_{n\to\infty} P_n(f;x) = f(x)$, the convergence being uniform in each interval $[0,a]$.*

Remark 1.2. (1) If in (1.2) we consider $g(u) = \cosh u$, the operator P_n becomes

$$L_n(f;x) = \frac{1}{\cosh 1 \cosh(nx)} \sum_{k=0}^{\infty} p_{2k}(nx) f\left(\frac{2k}{n}\right), \tag{1.5}$$

where

$$p_{2k}(x) = \frac{(1+x)^{2k} + (1-x)^{2k}}{2(2k)!}, \tag{1.6}$$

which was studied in [2].

(2) If instead of (1.2) we consider the relation

$$\cosh(ux) = \sum_{k=0}^{\infty} p_k(x) u^k, \tag{1.7}$$

we obtain

$$p_{2k}(x) = \frac{x^{2k}}{(2k)!}. \tag{1.8}$$

The operator

$$L_n^*(f;x) = \frac{1}{\cosh(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{2k}}{(2k)!} f\left(\frac{2k}{n}\right) \tag{1.9}$$

was studied by Leśniewicz and Rempulska [3].

2. Auxiliary results

In order to prove a Voronovskaya-type theorem, we need some auxiliary results.

Lemma 2.1. *For $x \in [0,\infty)$ and $n \in \mathbb{N}$,*

$$P_n(e_0;x) = 1,$$
$$P_n(e_1;x) = x\tanh(nx) + \frac{1}{n}\cdot\frac{g'(1)}{g(1)}, \tag{2.1}$$
$$P_n(e_2;x) = x^2 + \frac{x}{n}\tanh(nx)\frac{2g'(1)+g(1)}{g(1)} + \frac{1}{n^2}\cdot\frac{g''(1)+g'(1)}{g(1)},$$

where $e_i(x) = x^i$, $i \in \{0,1,2\}$, and $\tanh x$ *is the hyperbolic tangent of x.*

Lemma 2.2. *For $x \in [0,\infty)$ and $n \in \mathbb{N}$, the following hold:*

$$P_n(t-x;x) = -x(1-\tanh(nx)) + \frac{1}{n}\cdot\frac{g'(1)}{g(1)},$$
$$P_n((t-x)^2;x) = (1-\tanh(nx))\left[2x^2 - \frac{x}{n}\left(1+\frac{2g'(1)}{g(1)}\right)\right] + \frac{x}{n} + \frac{1}{n^2}\cdot\frac{g''(1)+g'(1)}{g(1)},$$
$$P_n((t-x)^4;x) = (1-\tanh(nx))\left(a_1x^4 - a_2\frac{x^3}{n} + a_3\frac{x^2}{n^2} - a_4\frac{x}{n^3}\right) + a_5\frac{x^2}{n^2} + a_6\frac{x}{n^3} + a_7\frac{1}{n^4}, \tag{2.2}$$

where a_i, $i = \overline{1,7}$, are positive constants:

$$a_1 = 8, \qquad a_2 = 12 + \frac{16g'(1)}{g(1)}, \qquad a_3 = 4\left(1+\frac{6g'(1)+3g''(1)}{g(1)}\right),$$
$$a_4 = 1 + \frac{14g'(1)+18g''(1)+4g^{(3)}(1)}{g(1)}, \qquad a_5 = 3, \tag{2.3}$$
$$a_6 = 1 + \frac{6g''(1)+10g'(1)}{g(1)}, \qquad a_7 = \frac{g'(1)+7g''(1)+6g^{(3)}(1)+g^{(4)}(1)}{g(1)}.$$

Lemmas 2.1 and 2.2 can be proved by means of successive partial differentiation with respect to u in the generating relation (1.2), and putting then $u = 1$.

Lemma 2.3. *For every fixed point $x_0 \in [0,\infty)$,*

$$\lim_{n\to\infty} nP_n(t-x_0;x_0) = \frac{g'(1)}{g(1)}, \qquad \lim_{n\to\infty} nP_n\left((t-x_0)^2;x_0\right) = x_0. \tag{2.4}$$

Proof. Because $1-\tanh(nx) = 2/(e^{2nx}+1)$, by Lemma 2.2 we have

$$nP_n(t-x_0;x_0) = \frac{-2nx_0}{e^{2nx_0}+1} + \frac{g'(1)}{g(1)},$$
$$nP_n\left((t-x_0)^2;x_0\right) = \frac{2nx_0}{e^{2nx_0}+1}\left[2x_0^2 - \frac{x_0}{n}\left(1+2\frac{g'(1)}{g(1)}\right)\right] + x_0 + \frac{1}{n}\cdot\frac{g''(1)+g'(1)}{g(1)}. \tag{2.5}$$

Therefore Lemma 2.3 holds. ∎

Lemma 2.4. *For each fixed point $x_0 \in [0,\infty)$, there is a positive constant $M_1(x_0)$, depending only on x_0 such that*

$$P_n\left((t-x_0)^4;x_0\right) \le M_1(x_0)\frac{1}{n^2} \tag{2.6}$$

for all $n \in \mathbb{N}$.

Proof. For $x \ge 0$ and $r,n \in \mathbb{N}$, we have

$$x^r(1-\tanh(nx)) \le \frac{2^{1-r}}{n^r}r!. \tag{2.7}$$

By Lemma 2.2, it results that

$$\begin{aligned} P_n\left((t-x_0)^4;x_0\right) &\le a_1\frac{2^{-3}}{n^4}4! - a_2\frac{2^{-2}}{n^4}3! + f_3\frac{2^{-1}}{n^4}2! \\ &\quad - a_4\frac{1}{n^4} + a_5\frac{x_0^2}{n^2} + a_6\frac{x_0}{n^3} + a_7\frac{1}{n^4} \le M_1(x_0)\frac{1}{n^2}. \end{aligned} \tag{2.8}$$

We proved in [1] the following lemma. □

Lemma 2.5. *Let $p>0$, let $r>p$, and let n_0 be a natural number such that $n_0 > p/(\ln r - \ln p)$. Then there exists a positive constant $M_{p,r}$ depending only on p and r such that*

$$e^{-rx}\left(P_n(t-x)^2e^{pt};x\right) \le M_{p,r}\frac{x+1}{n} \tag{2.9}$$

for all $x \ge 0$ and $n \ge n_0$.

Lemma 2.6. *Let $x_0 \in [0,\infty)$ be a fixed point and $\varphi(\cdot;x_0) \in C_p$ a function such that*

$$\lim_{t\to x_0}\varphi(t;x_0) = 0. \tag{2.10}$$

Then

$$\lim_{n\to\infty}P_n(\varphi(t;x_0);x_0) = 0. \tag{2.11}$$

Proof. Let $r>p>0$. For every fixed $x_0 \ge 0$ and $n \in \mathbb{N}$, we have

$$e^{-rx_0}P_n(\varphi(t;x_0);x_0) = \frac{e^{-rx_0}}{g(1)\cosh(nx)}\sum_{k=0}^{\infty}p_k(nx_0)\varphi\left(\frac{k}{n};x_0\right). \tag{2.12}$$

By the properties of function $\varphi(\cdot;x_0)$, it results that for all $\varepsilon>0$ there exists a positive constant $\delta(\varepsilon)$ such that if $|t-x_0|<\delta$, then $|\varphi(t;x_0)|<\varepsilon/2$, $t \ge 0$. Moreover, there exists a positive constant $M_2 \equiv M_2(p)$ such that

$$e^{-pt}\left|\varphi(t;x_0)\right| \le M_2 \quad \forall t \ge 0. \tag{2.13}$$

Now we can write

$$e^{-rx_0}P_n(\varphi(t;x_0);x_0) \le \frac{e^{-rx_0}}{g(1)\cosh(nx_0)} \sum_{|k/n-x_0|<\delta} p_k((nx_0)) \left|\varphi\left(\frac{k}{n};x_0\right)\right| + \frac{e^{-rx_0}}{g(1)\cosh nx_0} \sum_{|k/n-x_0|\ge\delta} p_k(nx_0) \left|\varphi\left(\frac{k}{n};x_0\right)\right| := S_1 + S_2. \tag{2.14}$$

By the above properties of function $\varphi(\cdot;x_0)$, it follows that

$$\begin{aligned} S_1 &< \frac{\varepsilon}{2}\cdot\frac{e^{-rx_0}}{g(1)\cosh(nx_0)} \sum_{|k/n-x_0|<\delta} p_k(nx_0) < \frac{\varepsilon}{2}e^{-rx_0}P_n(1,x_0) < \frac{\varepsilon}{2}, \\ S_2 &= \frac{e^{-rx_0}}{g(1)\cosh(nx_0)} \sum_{|k/n-x_0|\ge\delta} p_k((nx_0)) \left|\varphi\left(\frac{k}{n};x_0\right)\right| e^{-pk/n}e^{pk/n} \\ &\le M_2\frac{e^{-rx_0}}{g(1)\cosh nx_0} \sum_{|k/n-x_0|\ge\delta} p_k(nx_0)e^{pk/n}. \end{aligned} \tag{2.15}$$

But if

$$\left|\frac{k}{n} - x_0\right| \ge \delta, \tag{2.16}$$

then

$$1 \le \frac{1}{\delta^2}\left(\frac{k}{n} - x_0\right)^2, \tag{2.17}$$

and by Lemma 2.5, we can write

$$\begin{aligned} S_2 &\le M_2\frac{1}{\delta^2}\cdot\frac{e^{-rx_0}}{g(1)\cosh(nx_0)} \sum_{|k/n-x_0|\ge\delta} p_k(nx_0)\left(\frac{k}{n}-x_0\right)^2 e^{pk/n} \\ &\le M_2\frac{e^{-rx_0}}{\delta^2}P_n\left((t-x_0)^2e^{pt};x_0\right) \le M_2\frac{1}{\delta^2}M_{p,r}\frac{x_0+1}{n} \end{aligned} \tag{2.18}$$

for $n \ge n_0$, $n_0 > p/(\ln r - \ln p)$. It results that for a fixed x_0, ε, δ there exists a natural number $n_0 = n_0(x_0,\varepsilon,\delta,M_2,p,r)$ such that for all $n > n_0$, we have $S_2 < \varepsilon/2$.

Therefore, for all $n > n_0$, we have

$$e^{-rx_0}P_n(\varphi(t;x_0);x_0) < \varepsilon \quad \left(\text{i.e., } \lim_{n\to\infty} e^{-rx_0}P_n(\varphi(t;x_0);x_0) = 0\right). \tag{2.19}$$

It results that $\lim_{n\to\infty} P_n(\varphi(t;x_0);x_0) = 0$. □

3. A Voronovskaya-type theorem

Now we are in the position to state the main result of this paper.

For a fixed $p > 0$, let

$$C_p^2 = \{f \in C_p \text{ such that } f', f'' \in C_p\}. \tag{3.1}$$

Theorem 3.1. *If $f \in C_p^2$, then*

$$\lim_{n\to\infty} n\{P_n(f;x) - f(x)\} = \frac{x}{2}f''(x) + f'(x)\frac{g'(1)}{g(1)} \tag{3.2}$$

for every fixed $x \in [0,\infty)$.

Proof. We use the Taylor formula for a fixed point $x_0 \in [0,\infty)$. For all $t \in [0,\infty)$, we have

$$f(t) = f(x_0) + (t-x_0)f'(x_0) + \frac{1}{2}(t-x_0)^2 f''(x_0) + g(t;x_0)(t-x_0)^2, \tag{3.3}$$

where $g(t;x_0)$ is the Peano form of the remainder, $g(\cdot;x_0) \in C_p$, and $\lim_{t\to x_0} g(t;x_0) = 0$. Because $P_n(e_0;x) = 1$, we can write

$$\begin{aligned} P_n(f;x_0) - f(x_0) &= f'(x_0)P_n(t-x_0;x_0) + \frac{1}{2}f''(x_0)P_n((t-x_0)^2;x_0) \\ &\quad + P_n(g(t;x_0)(t-x_0)^2;x_0). \end{aligned} \tag{3.4}$$

By Cauchy's inequality, we have

$$P_n(g(t;x_0)(t-x_0)^2;x_0) \le \{P_n(g^2(t;x_0);x_0)\}^{1/2}\{P_n((t-x_0)^4;x_0)\}^{1/2}. \tag{3.5}$$

The function $\varphi(t;x_0) = g^2(t;x_0)$, $t \ge 0$, satisfies the conditions of Lemma 2.6; therefore

$$\lim_{n\to\infty} P_n(g^2(t;x_0);x_0) = 0. \tag{3.6}$$

Moreover, by Lemma 2.4, we have

$$nP_n(g(t;x_0)(t-x_0)^2;x_0) \le \{P_n(g^2(t;x_0);x_0)\}^{1/2}\left(n^2 M_1(x_0)\frac{1}{n^2}\right)^{1/2}. \tag{3.7}$$

It results that $\lim_{n\to\infty} nP_n(g(t;x_0)(t-x_0)^2;x_0) = 0$. By the above results and by Lemma 2.3, we obtain

$$\lim_{n\to\infty} n(P_n(f;x_0) - f(x_0)) = f'(x_0)\frac{g'(1)}{g(1)} + \frac{x_0}{2}f''(x_0). \tag{3.8}$$

□

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