A SIMPLIFICATION FUNCTOR FOR COALGEBRAS

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For an arbitrary-type functor F, the notion of *split* coalgebras, that is, coalgebras for which the canonical projections onto the simple factor split, generalizes the well-known notion of simple coalgebras. In case F weakly preserves kernels, the passage from a coalgebra to its simple factor is functorial. This is *the simplification functor*. It is left adjoint to the inclusion of the subcategory of simple coalgebras into the category $\mathbb{S}\mathrm{et}_F$ of F-coalgebras, making it an epireflective one. If a product of split coalgebras exists, then this is split and preserved by the simplification functor. In particular, if a product of simple coalgebras exists, this is simple too.

1. Introduction

Originally introduced by Aczel and Mendler [1] to model various types of transition systems, coalgebras (or systems) offer a very rich field of mathematics. Basic tools and results in the theory of coalgebras can be found in [3, 8].

Coalgebras of a given $\mathbb{S}\mathrm{et}$-endofunctor F, also called the *type functor* with morphisms between them, form a category denoted by $\mathbb{S}\mathrm{et}_F$. The notions of extensionality and simplicity are tightly connected to two of the main tools in the theory of coalgebras: bisimulations and congruence relations, respectively.

The coalgebraic definition of bisimulation as well as the notion of congruence relation made their first appearance in [1]. In the particular case of weak pullbacks preserving functors, Rutten in [8] has built on results and notions from them. In general, congruences need not be bisimulations, and even the largest bisimulation on a coalgebra need not be transitive in the general case. In [4], Gumm and Schröder have shown that under a weaker condition, namely that of weak kernels preservation in which we are interested in this paper, every congruence relation is a bisimulation and in particular the largest bisimulation on a coalgebra is the largest congruence relation thereon.

This paper both gives a slight overview of some existing insights in the theory of coalgebras in Sections 2 and 3, and in addition, presents some new material such as the notion

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of *split coalgebra* introduced in Section 4. This generalizes the notion of simple coalgebras as does [5] that of extensional coalgebra (see Section 3). The fact that the quotient of a coalgebra with respect to a congruence relation [3] yields again a coalgebra helps to build many coalgebras from a given one. It turns out that amidst these, the quotient with respect to the largest congruence relation is its sole factor which is simple. We call it its *simple factor* (see Section 3). This uniqueness draws our attention to a study in Section 5 of the passage from an F-coalgebra to its simple factor. Fortunately, as is shown here, in case the type functor weakly preserves kernels, this passage is functorial. We call it *the simplication functor* and show that it is left adjoint to the inclusion functor of the subcategory $\mathbb{S}\mathrm{imp}(F)$ of simple F-coalgebras making it an epireflective one. We also show that if a product of split coalgebras exists, then it is split and that the simplification functor is well-behaved with respect to split coalgebras, as far as the preservation of products is concerned. This implies that if a product of simple coalgebras exists, then it is simple too.

2. Preliminaries

2.1. Some categorical notions

2.1.1. Epireflective subcategory. If a functor $I : \mathbb{C} \to \mathbb{D}$ is an inclusion, then a left adjoint $S : \mathbb{D} \to \mathbb{C}$ is called a *reflection*. If $\mathbb{C}$ is a full subcategory and there exists a reflection S such that the unit $\eta_X : X \to (I \circ S)(X)$ is an epimorphism for all objects X in $\mathbb{D}$, the subcategory [9] $\mathbb{C}$ is called *epireflective*. If there exists a reflection S for a subcategory $\mathbb{C}$, then (see [9]) there exists one which is the identity on $\mathbb{C}$, that is, such that $S \circ I = \mathrm{id}_{\mathbb{C}}$. If such a reflection is chosen, then it is called an *epireflector*.

2.1.2. Weak limits preservation.

Definition 2.1. Let $F : \mathbb{C} \to \mathbb{C}$ be a functor and $D : \mathbb{I} \to \mathbb{C}$ a diagram for which D-limits exist in $\mathbb{C}$. Then,

(i) *F weakly preserves D-limits*, if F transforms every limit cone over D into a weak limit cone over $F \circ D$, that is, for every limit $(L, (v_k)_{k\in\kappa})$ of the diagram D, $(F(L), F(v_k)_{k\in\kappa})$ is a weak limit of the diagram $F \circ D$;

(ii) *F preserves weak D-limits* if it transforms every weak limit cone over D into a weak limit cone over $F \circ D$.

It is quite clear that there is a fine linguistic difference between "F preserves weak D-limits" and "F weakly preserves D-limits." But fortunately, it has been proven [3, 4] that the difference disappears in every category in which D-limits exist.

2.2. Some basic facts about coalgebras. Let F be a type functor $\mathbb{C} \to \mathbb{C}$. An F-coalgebra is just a pair $(A,\ \alpha_A : A \to F(A))$ where A is an object called the *carrier* and α_A a morphism called the *structure* of the coalgebra. This will often be denoted by (A, α_A) or by the single letter $\mathcal{A}$. A *homomorphism* from a coalgebra $\mathcal{A} = (A, \alpha_A)$ to a coalgebra $\mathcal{B} = (B, \alpha_B)$ is a map $f : A \to B$ such that $F(f) \circ \alpha_A = \alpha_B \circ f$.

F-coalgebras with their homomorphisms form a category $\mathbb{C}_F$ which is cocomplete provided that $\mathbb{C}$ is cocomplete. More precisely, the forgetful functor $U : \mathbb{C}_F \to \mathbb{C}$ which associates to each F-coalgebra $\mathcal{A}$ its underlying object A creates colimits. This can be found in [2] in the setting of category $\mathbb{S}\mathrm{et}$ which serves as the basic category from now on. As for limits, the forgetful functor [7] creates those which are preserved by the functor F. Epis in $\mathbb{S}\mathrm{et}_F$ are just surjective homomorphisms and every injective homomorphism is a mono [8].

2.2.1. Generation of homomorphisms. The following theorem can be used to prove that a map is a homomorphism.

Theorem 2.2 [7]. *Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be coalgebras, $\varphi : \mathcal{A} \to \mathcal{C}$ a homomorphism, $f : A \to B$ and $g : B \to C$ maps with $\varphi = g \circ f$.*

(1) *If f is a surjective homomorphism, then g is a homomorphism.*
(2) *If g is an injective homomorphism, then f is a homomorphism.*

2.2.2. Bisimulations. The following definition of bisimulation was introduced by Aczel and Mendler [1]: a *bisimulation* between two coalgebras $\mathcal{A}$ and $\mathcal{B}$ is a relation $R \subseteq A \times B$ for which there exists a structure $\alpha_R : R \to F(R)$ such that the projections $\pi_1 : R \to A$ and $\pi_2 : R \to B$ are homomorphisms of coalgebras. In particular, $\varnothing$ is always a bisimulation between $\mathcal{A}$ and $\mathcal{B}$. On the other hand, bisimulations are closed under arbitrary unions so that there is always a largest bisimulation $\sim_{\mathcal{A},\mathcal{B}}$ between two coalgebras $\mathcal{A}$ and $\mathcal{B}$. When $\mathcal{A} = \mathcal{B}$, one talks of bisimulation *on* $\mathcal{A}$ and $\sim_{\mathcal{A}}$ denotes the largest bisimulation on $\mathcal{A}$. For every coalgebra $\mathcal{A}$, the diagonal Δ_A is [3, 7, 8] a bisimulation on $\mathcal{A}$.

We have the following facts about bisimulations.

Theorem 2.3 [7]. *Let $\mathcal{A}$ and $\mathcal{B}$ be coalgebras.*

(1) *Given two homomorphisms $\varphi : \mathcal{P} \to \mathcal{A}$ and $\psi : \mathcal{P} \to \mathcal{B}$, then $(\varphi,\psi)(P) := \{(\varphi(p), \psi(p)) \mid p \in P\}$ is a bisimulation between $\mathcal{A}$ and $\mathcal{B}$.*
(2) *Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. For all bisimulation R on $\mathcal{A}$, $\varphi(R) := \{(\varphi(a), \varphi(a')) \mid (a,a') \in \mathbb{R}\}$ is a bisimulation on $\mathcal{B}$.*

2.2.3. Congruence relations. A *congruence relation on* a coalgebra $\mathcal{A}$ is a relation on A which is the kernel (in $\mathbb{S}\mathrm{et}$) of a homomorphism with domain $\mathcal{A}$. For any congruence relation θ on $\mathcal{A}$ there is [3] a unique coalgebra structure on A/θ for which the canonical projection $\pi_\theta : A \to A/\theta$ is a homomorphism. The corresponding coalgebra is denoted by $\mathcal{A}/\theta$ and called a *factor coalgebra* for $\mathcal{A}$. In [1], it has been shown that there is a largest congruence relation contained in every reflexive relation R denoted by $\mathrm{Con}[R]$. In particular, $\nabla_{\mathcal{A}} := \mathrm{Con}[A \times A]$ denotes the largest congruence relation on $\mathcal{A}$. This is in general a proper subset of $A \times A$.

Lemma 2.4 [4]. *If F weakly preserves kernels, then the largest bisimulation $\sim_{\mathcal{A}}$ on a coalgebra $\mathcal{A}$ is transitive, in fact it is the largest congruence relation $\nabla_{\mathcal{A}}$ on $\mathcal{A}$.*

3. Extensionality and simplicity

3.1. Definitions and some basic results

Definition 3.1 [5]. Let $\mathcal{A}$ be a coalgebra for an arbitrary-type functor F.

(i) $\mathcal{A}$ is called *simple* if it does not have any nontrivial congruence relation.
(ii) $\mathcal{A}$ is called *extensional* if its diagonal is its largest bisimulation.

The following theorem is a characterization of extensional coalgebras. This can be found in [3], and, in [8] (in case F preserves weak pullbacks), as a characterization of "simple" coalgebras.

Theorem 3.2. *For a coalgebra $\mathcal{A}$, the following are equivalent.*

(1) *$\mathcal{A}$ is extensional.*
(2) *Every homomorphism with domain $\mathcal{A}$ is a monomorphism.*
(3) *For every coalgebra $\mathcal{B}$, there is at most one homomorphism $\psi : \mathcal{B} \to \mathcal{A}$.*

The following result shows that extensionality generalizes simplicity.

Corollary 3.3 [5]. *Every simple coalgebra is extensional.*

For functors weakly preserving kernels, both notions agree. The following gives an easy description of simple coalgebras when the terminal coalgebra exists.

Lemma 3.4 [3]. *If the terminal coalgebra exists, then simple coalgebras are precisely isomorphic copies of its subcoalgebras.*

For every $\mathbb{S}\mathrm{et}$-endofunctor F, we denote by $\mathbb{E}xt(F)$ ($\mathbb{S}\mathrm{imp}(F)$, resp.) the fully faithful subcategory of $\mathbb{S}\mathrm{et}_F$ whose objects are extensional coalgebras (simple coalgebras, resp.). Despite the fact that $\mathbb{S}\mathrm{et}_F$ is cocomplete [2] for any type functor F, the subcategories $\mathbb{E}xt(F)$ and $\mathbb{S}\mathrm{imp}(F)$ are not cocomplete in general. More precisely, we have the following important observation.

Remark 3.5. For every type functor different from the empty constant functor, neither the class of simple coalgebras nor that of extensional ones is closed under sums. However, the former is closed under homomorphic images and domains of injective homomorphisms and the latter under domains of monomorphisms but not under homomorphic images.

3.2. Some examples

3.2.1. Nondeterministic labelled transition systems with output. Let Σ be a set with $|\Sigma| \geq 2$ and a and b two different elements in Σ.

Example 3.6. Let $\mathcal{P}$ be the *power set functor* defined by $\mathcal{P}(X) := \{A \mid A \subseteq X\}$, and for all map $f : X \to Y$, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ where $\mathcal{P}(f)(A) := f(A)$, $\Sigma \times \mathcal{I}d$ the functor which associates every set X with $\Sigma \times X$ and every map $f : X \to Y$ with $\mathrm{id}_\Sigma \times f : \Sigma \times X \to \Sigma \times Y$ defined by $(\mathrm{id}_\Sigma \times f)(e,x) = (e, f(x))$. For the arrow representation of coalgebras and the characterizations of homomorphisms and bisimulations between coalgebras in $\mathbb{S}\mathrm{et}_{\mathcal{P}\circ(\Sigma\times\mathcal{I}d)}$, see [8].

The $\mathcal{P} \circ (\Sigma \times \mathcal{I}d)$-coalgebras $\mathcal{S} := (S, \alpha_S)$ defined by $\alpha_S(0) = \{(a,1)\}$ and $\alpha_S(1) = \{(b,1)\}$ and $\mathcal{T} := (T, \alpha_T)$ defined by $T = \{0,1,2\}$ with $\alpha_T(0) = \{(a,1),(b,2)\}$, $\alpha_T(1) = \{(b,1)\}$ and $\alpha_T(2) = \{(a,2)\}$ represented below are extensional.

$$\begin{tikzcd} \mathcal{S} = 0 \arrow[r, "a"] & 1 \arrow[loop right, "b"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{T} = 2 \arrow[loop left, "a"] & 0 \arrow[l, "b"'] \arrow[r, "a"] & 1 \arrow[loop right, "b"] \end{tikzcd}. \tag{3.1}$$

It is well known [6] that $\mathcal{P}$ preserves weak pullbacks. Similarly does the functor $\Sigma \times \mathcal{I}d$. Thus the type functor $\mathcal{P} \circ (\Sigma \times \mathcal{I}d)$ weakly preserves pullbacks. Consequently, it does weakly preserve kernels so that the coalgebras $\mathcal{S}$ and $\mathcal{T}$ are also simple.

3.2.2. $(-)^3_2$-coalgebras. The functor $(-)^3_2$ given on sets as $(X)^3_2 = \{(x_1,x_2,x_3) \in X^3 \mid x_1 = x_2 \text{ or } x_1 = x_3 \text{ or } x_2 = x_3\}$ and on maps $f : X \to Y$ as $(f)^3_2(x_1,x_2,x_3) = (f(x_1), f(x_2), f(x_3))$ constitutes a great source of counterexamples in the theory of coalgebras. It has been introduced in [1] and is an example of functor which does not weakly preserve arbitrary pullbacks [3].

Example 3.7. Consider the $(-)^3_2$-coalgebras $\mathcal{A} := (A, \alpha_A)$, where $A = \{a, b\}$ and $\alpha_A : A \to (A)^3_2$ is defined by $\alpha_A(a) = (a,a,b)$ and $\alpha_A(b) = (a,b,a)$ and $\mathbb{1} := (\mathbf{1}, \alpha_\mathbf{1})$ with $\mathbf{1} = \{*\}$. $\mathbb{1}$ is the terminal coalgebra and hence the only nonempty simple coalgebra (up to isomorphism) and $\mathcal{A}$ is extensional.

3.3. Simple factor coalgebras. Every coalgebra "can be made" simple by taking the quotient with respect to its largest congruence relation. It is well known [3] that when ordered by the set inclusion, congruences on a coalgebra form a complete lattice. The following which extends a result from Rutten [8] for weak pullbacks preserving functors is immediately checked using the fact that [5] the congruence lattice of $\mathcal{A}/\theta$ is isomorphic to the interval above θ of the congruence lattice of $\mathcal{A}$.

Proposition 3.8. *For any functor F, every coalgebra $\mathcal{A}$, and every congruence relation θ on $\mathcal{A}$, the quotient $\mathcal{A}/\theta$ is simple if and only if $\theta = \nabla_{\mathcal{A}}$.*

We call $\mathcal{A}/\nabla_{\mathcal{A}}$ *the simple factor* of the coalgebra $\mathcal{A}$.

4. Split coalgebras

In this section, we introduce the concept of split coalgebra. As the concept of extensional coalgebra, this generalizes that of simple coalgebra.

4.1. Definitions and some basic results

Definition 4.1. Let F be any type functor. A coalgebra $\mathcal{A}$ is called *split* if the canonical projection $\pi_{\nabla_{\mathcal{A}}}$ of $\mathcal{A}$ onto $\mathcal{A}/\nabla_{\mathcal{A}}$ splits.

We have the following characterization of simple coalgebras.

Proposition 4.2. *For every type functor F, a coalgebra is simple if and only if it is split and extensional.*

From Propositions 3.8 and 4.2, it follows that for any coalgebra if there exists a homomorphism $\chi : \mathcal{A}/\nabla_{\mathcal{A}} \to \mathcal{A}$, then automatically $\pi_{\nabla_{\mathcal{A}}} \circ \chi = \mathrm{id}_{\mathcal{A}/\nabla_{\mathcal{A}}}$, that is, $\mathcal{A}$ is a split coalgebra.

This constitutes a key factor in tackling the proof of the following result which permits obtaining split coalgebras when the terminal coalgebra exists.

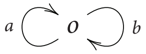
Proposition 4.3. *Let F be any type functor. If there is a terminal coalgebra $\mathbb{1}$, then every coalgebra $\mathcal{A}$, such that there exists a homomorphism $\varphi : \mathbb{1} \to \mathcal{A}$, is split.*

4.2. Some examples

4.2.1. $(-)^3_2$-coalgebras (continued).

Example 4.4. Nonempty split coalgebras are just coalgebras $\mathcal{A}$ for which there exists a homomorphism $\varphi : \mathbb{1} \to \mathcal{A}$, that is, there exists an element $a_0 \in A$ such that $\alpha_A(a_0) = (a_0, a_0, a_0)$.

4.2.2. $\mathcal{P} \circ (\Sigma \times \mathcal{I}d)$-coalgebras (continued).

Example 4.5. The coalgebra $\mathbb{O} =$ 

is obviously split and $\mathcal{M} = \mathbb{O} + \mathbb{O}$ is split but not simple.

5. The simplification functor

It follows from Proposition 3.8 that for every type functor F and all coalgebras $\mathcal{A}$, $\mathcal{A}/\nabla_{\mathcal{A}}$ is the only simple factor coalgebra for $\mathcal{A}$. Can the passage from $\mathcal{A}$ to $\mathcal{A}/\nabla_{\mathcal{A}}$ be made into a functor from $\mathbb{S}\mathrm{et}_F$ to $\mathbb{S}\mathrm{imp}(F)$?

5.1. The simplification functor. The following theorem provides a positive answer to the aforementioned question and shows the adjunction of the thus obtained functor with the inclusion functor $\mathbf{I} : \mathbb{S}\mathrm{imp}(F) \to \mathbb{S}\mathrm{et}_F$.

Theorem 5.1. *Assume that the type functor F weakly preserves kernels. Consider $\mathbf{S} : \mathbb{S}\mathrm{et}_F \to \mathbb{S}\mathrm{imp}(F)$ defined by $\mathbf{S}(\mathcal{A}) = \mathcal{A}/\nabla_{\mathcal{A}}$ and for all homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ and all a in A, $\mathbf{S}(\varphi) : \mathcal{A}/\nabla_{\mathcal{A}} \to \mathcal{B}/\nabla_{\mathcal{B}}$ with $\mathbf{S}(\varphi)(a/\nabla_{\mathcal{A}}) := \varphi(a)/\nabla_{\mathcal{B}}$. Then the following holds.*

(1) *$\mathbf{S}$ is a covariant functor.*

(2) *$\mathbf{S}$ is left adjoint to the inclusion functor $\mathbf{I}$ with unit $\eta : \mathrm{id}_{\mathbb{S}\mathrm{et}_F} \to \mathbf{I} \circ \mathbf{S}$ and counit $\varepsilon : \mathbf{S} \circ \mathbf{I} \to \mathrm{id}_{\mathbb{S}\mathrm{imp}(F)}$ defined by $\eta_{\mathcal{A}} := \pi_{\nabla_{\mathcal{A}}}$, the canonical projection of $\mathcal{A}$ onto $\mathcal{A}/\nabla_{\mathcal{A}}$ for all $\mathcal{A}$ in $\mathbb{S}\mathrm{et}_F$ and $\varepsilon_{\mathcal{A}} : \mathcal{A}/\nabla_{\mathcal{A}} \to \mathcal{A}$ with $\varepsilon_{\mathcal{A}}(a/\nabla_{\mathcal{A}}) := a$ for all $\mathcal{A}$ in $\mathbb{S}\mathrm{imp}(F)$ and all a in A.*

Proof. (1) $\mathbf{S}$ is well defined on objects since $\mathcal{A}/\nabla_{\mathcal{A}}$ is, as seen in Proposition 3.8, always a simple coalgebra. Let now $\varphi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. We want to show that $\mathbf{S}(\varphi)$ is a homomorphism too. Let a and a' be elements of A with $(a, a') \in \nabla_{\mathcal{A}}$. Then $(a, a') \in \sim_{\mathcal{A}}$ for $\nabla_{\mathcal{A}} = \sim_{\mathcal{A}}$ by Lemma 2.4. Now $\varphi(\sim_{\mathcal{A}})$ is a bisimulation on $\mathcal{B}$ by Theorem 2.3. Therefore $(\varphi(a), \varphi(a')) \in \sim_{\mathcal{B}}$ and consequently $(\varphi(a), \varphi(a')) \in \nabla_{\mathcal{B}}$ for $\nabla_{\mathcal{B}} = \sim_{\mathcal{B}}$, that is, $\mathbf{S}(\varphi)$ is a map. We now show that it is even a homomorphism. By the definition of $\mathbf{S}(\varphi)$, we have $\mathbf{S}(\varphi) \circ \pi_{\nabla_{\mathcal{A}}} = \pi_{\nabla_{\mathcal{B}}} \circ f$, where $\pi_{\nabla_{\mathcal{A}}}$ and $\pi_{\nabla_{\mathcal{B}}}$ are the canonical projections of $\mathcal{A}$ and $\mathcal{B}$ onto $\mathcal{A}/\nabla_{\mathcal{A}}$ and $\mathcal{B}/\nabla_{\mathcal{B}}$, respectively. Now $\pi_{\nabla_{\mathcal{B}}} \circ \varphi$ is a homomorphism and $\pi_{\nabla_{\mathcal{A}}}$ is a surjective homomorphism. Therefore by Theorem 2.2, $\mathbf{S}(\varphi)$ is a homomorphism. One can easily check that $\mathbf{S}$ preserves composition of homomorphisms and identity homomorphisms.

(2) We want to show that we have the following identities:

$$\mathbf{I} * \varepsilon \circ \eta * \mathbf{I} = \mathrm{id}_{\mathbf{I}}, \qquad \varepsilon * \mathbf{S} \circ \mathbf{S} * \eta = \mathrm{id}_{\mathbf{S}}. \tag{5.1}$$

We first show that $\mathbf{I} * \varepsilon \circ \eta * \mathbf{I} = \mathrm{id}_{\mathbf{I}}$. Let $\mathscr{A}$ be a coalgebra in $\mathbb{S}\mathrm{imp}(F)$. We have

$$\begin{aligned}(\mathbf{I} * \varepsilon \circ \eta * \mathbf{I})_{\mathscr{A}} &= \mathbf{I}(\varepsilon_{\mathscr{A}}) \circ \eta_{\mathbf{I}(\mathscr{A})} \\ &= \varepsilon_{\mathscr{A}} \circ \eta_{\mathscr{A}}.\end{aligned} \tag{5.2}$$

Now $\mathscr{A}$ is simple, thus it is extensional by Corollary 3.3. Therefore, it follows from Theorem 3.2 that $\varepsilon_{\mathscr{A}} \circ \eta_{\mathscr{A}} = \mathrm{id}_{\mathscr{A}}$.

For the second identity, let $\mathscr{A}$ be an arbitrary coalgebra in $\mathbb{S}\mathrm{et}_F$. We have

$$\begin{aligned}(\varepsilon * \mathbf{S} \circ \mathbf{S} * \eta)_{\mathscr{A}} &= (\varepsilon * \mathbf{S})_{\mathscr{A}} \circ (\mathbf{S} * \eta)_{\mathscr{A}} \\ &= \varepsilon_{\mathbf{S}(\mathscr{A})} \circ \mathbf{S}(\eta_{\mathscr{A}}).\end{aligned} \tag{5.3}$$

Now $\mathbf{S}(\eta_{\mathscr{A}}) : \mathbf{S}(\mathscr{A}) \to \mathbf{S}(\mathbf{S}(\mathscr{A}))$ and $\varepsilon_{\mathbf{S}(\mathscr{A})} : \mathbf{S}(\mathbf{S}(\mathscr{A})) \to \mathbf{S}(\mathscr{A})$ and $\mathbf{S}(\mathscr{A})$ is simple and hence extensional. Thus, automatically, we have $\varepsilon_{\mathbf{S}(\mathscr{A})} \circ \mathbf{S}(\eta_{\mathscr{A}}) = \mathrm{id}_{\mathbf{S}(\mathscr{A})}$. □

The functor $\mathbf{S}$ will be called *the simplification functor* for the functor F.

From Theorem 5.1, we immediately deduce the following.

Corollary 5.2. *If the type functor F weakly preserves kernels, then $\mathbb{S}\mathrm{imp}(F)$ is an epireflective subcategory of $\mathbb{S}\mathrm{et}_F$ and the simplification functor $\mathbf{S}$ is an epireflector (up to isomorphism).*

5.2. Some categorical properties. It is well known [2] that $\mathbb{S}\mathrm{et}_F$ is cocomplete. On the other hand, every left adjoint [9] preserves colimits but does not necessarily preserve limits. The following shows amongst other things that the simplification functor preserves some types of limits, as far as they exist.

Proposition 5.3. *Assume that F is a type functor which weakly preserves kernels. The simplification functor $\mathbf{S}$ preserves all colimits, the terminal coalgebra (as far as it exists) transforms every homomorphism into an injective homomorphism, and, in particular, epimorphisms into isomorphisms.*

The following result shows that the simplification functor preserves some products whenever they exist.

Theorem 5.4. *If F weakly preserves kernels and the product $\Pi_{i\in I}\mathscr{A}_i$ of a family $(\mathscr{A}_i)_{i\in I}$ of split coalgebras exists, then this is split and $(\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i} \cong \Pi_{i\in I}(\mathscr{A}_i/\nabla_{\mathscr{A}_i})$.*

Proof. Let $(\mathscr{A}_i)_{i\in I}$ be a family of split coalgebras such that the product $\Pi_{i\in I}\mathscr{A}_i$ exists in $\mathbb{S}\mathrm{et}_F$. We want to show that it is a split coalgebra too. Thus we need to find a homomorphism $\chi : (\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i} \to \Pi_{i\in I}\mathscr{A}_i$ such that $\pi_{\nabla_{\Pi_{i\in I}\mathscr{A}_i}} \circ \chi = \mathrm{id}_{(\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i}}$. Let $p_i : \Pi_{i\in I}\mathscr{A}_i \to \mathscr{A}_i$ be the canonical projection of $\Pi_{i\in I}\mathscr{A}_i$ to $\mathscr{A}_i$, $i \in I$. Since $\mathscr{A}_i$ is split, there exists a homomorphism $\chi_i : \mathscr{A}_i/\nabla_{\mathscr{A}_i} \to \mathscr{A}_i$ such that $\pi_{\nabla_{\mathscr{A}_i}} \circ \chi_i = \mathrm{id}_{\mathscr{A}_i/\nabla_{\mathscr{A}_i}}$ for all $i \in I$.

But then we have $\chi_i \circ \mathbf{S}(p_i) : (\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i} \to \mathscr{A}_i$ for each $i \in I$. Thus there exists a unique homomorphism $\chi : (\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i} \to \Pi_{i\in I}\mathscr{A}_i$ such that, for all $i \in I$, $p_i \circ \chi = \chi_i \circ \mathbf{S}(p_i)$. Now $(\Pi_{i\in I}\mathscr{A}_i)/\nabla_{\Pi_{i\in I}\mathscr{A}_i}$ is a simple coalgebra, so it is extensional by Corollary 3.3.

Thus by Theorem 3.2, $\pi_{\nabla_{\prod_{i\in I}\mathcal{A}_i}} \circ \chi = \mathrm{id}_{(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i}}$.

$$\begin{tikzcd}
\prod_{i\in I}\mathcal{A}_i \arrow[rr, "p_i"] \arrow[d, "\pi_{\nabla_{\prod_{i\in I}\mathcal{A}_i}}"] & & \mathcal{A}_i \arrow[d, "\pi_{\nabla_{\mathcal{A}_i}}"] \\
(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i} \arrow[rr, "S(p_i)"'] \arrow[u, dotted, bend left, "\chi"] & & \mathcal{A}_i/\nabla_{\mathcal{A}_i} \arrow[u, dotted, bend left, "\chi_i"]
\end{tikzcd} \tag{5.4}$$

To end the proof, we show that $(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i} \cong \prod_{i\in I}(\mathcal{A}_i/\nabla_{\mathcal{A}_i})$. Let $q_i : \mathcal{C} \to A_i/\nabla_{A_i}$ be homomorphisms in $\mathbb{S}\mathrm{et}_F$. We have $u_i := \chi_i \circ q_i : \mathcal{C} \to \mathcal{A}_i$ in $\mathbb{S}\mathrm{et}_F$. Thus, there is a unique $u : \mathcal{C} \to \prod_{i\in I}\mathcal{A}_i$ such that $p_i \circ u = u_i$ for all $i \in I$. Now $(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i}$ is a simple coalgebra. Thus it is extensional and consequently $\pi_{\nabla_{\prod_{i\in I}\mathcal{A}_i}} \circ u$ is the unique homomorphism from $\mathcal{C}$ to $(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i}$. On the other hand, the coalgebras $\mathcal{A}_i/\nabla_{\mathcal{A}_i}$'s are simple. Thus they are extensional so that, automatically, $q_i = \mathbf{S}(p_i) \circ \pi_{\nabla_{\prod_{i\in I}\mathcal{A}_i}} \circ u$ for each i. □

If $(\mathcal{A}_i)_{i\in I}$ is a family of simple coalgebras then by Proposition 4.2, for each i, $\mathcal{A}_i$ is split with $\mathcal{A}_i/\nabla_{\mathcal{A}_i} \cong \mathcal{A}_i$. Thus, if their product exists, then by Theorem 5.4, $(\prod_{i\in I}\mathcal{A}_i)/\nabla_{\prod_{i\in I}\mathcal{A}_i} \cong \prod_{i\in I}\mathcal{A}_i$. Hence we have the following.

Corollary 5.5. *If F weakly preserves kernels and a product of simple coalgebras exists, then this is simple.*

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