A RADICAL FOR RIGHT NEAR-RINGS: THE RIGHT JACOBSON RADICAL OF TYPE-0

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The notions of a right quasiregular element and right modular right ideal in a near-ring are initiated. Based on these $J_0^r(R)$, the right Jacobson radical of type-0 of a near-ring R is introduced. It is obtained that J_0^r is a radical map and $N(R) \subseteq J_0^r(R)$, where $N(R)$ is the nil radical of a near-ring R. Some characterizations of $J_0^r(R)$ are given and its relation with some of the radicals is also discussed.

1. Introduction

Throughout this paper, R stands for a right near-ring. The structure of matrix near-rings was studied by Rao in [3, 4]. It is clear from these papers that right Jacobson-type radicals have an important role to play in the study of Meldrum-van der Walt matrix near-rings. This motivated the authors to develop the right Jacobson-type radicals of near-rings. The aim of this paper is to give a good beginning in this direction of investigation. Left quasiregularity was introduced and studied in near-rings. In this paper, right quasiregularity is developed, and right modules of near-rings and the right Jacobson radical of type-0 are studied.

In Section 2, the notions of a right quasiregular element and a right modular right ideal are introduced. Using these right 0-primitive ideal, right 0-primitive near-ring, and $J_0^r(R)$, the right Jacobson radical of R of type-0 are introduced. A nil subset of R is right quasiregular and the constant part of R is also right quasiregular. It is shown that J_0^r is a radical map and $J_0^r(R)$ is the largest right quasiregular ideal of R. Moreover, $N(R)$, the nil radical of R, is contained in $J_0^r(R)$ and that $P(R) \subseteq N(R) \subseteq J_0^r(R)$. If R is a zero-symmetric near-ring with DCC on the left R-subgroups of R, then $J_0(R) \subseteq J_0^r(R)$.

In Section 3, a right R-group of type-0 is introduced. It is shown that G is a right R-group of type-0 if and only if G is right R-isomorphic to R/K, for some maximal right modular right ideal K of R. A right 0-primitive ideal of R is a prime ideal of R. An ideal P of a d.g. near-ring R is a right 0-primitive ideal of R if and only if $P = (0 : G)$, for some right R-group G of type-0.

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In Section 4, the right Jacobson radical of type-0 of a biregular near-ring is studied. For such a near-ring R, it is shown that $J_0^r(R) = J_0(R) = \{0\}$. Moreover, if R is biregular, then an ideal P of R is right 0-primitive if and only if P is left 0-primitive if and only if P is a maximal right and left modular ideal. It is shown by an example that in general $J_0^r(R)$ differs from the existing Jacobson-type radicals of R and the J_0^r-radical class contains almost all the classes of near-rings with trivial multiplication.

2. Right quasiregularity and the right J_0-radical

Throughout this paper, R is a right near-ring.

Definition 2.1. An element $a \in R$ is called *right quasiregular* if and only if the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R.

Remark 2.2. Note that unlike in the left quasiregularity case, if R is a near-ring and $a \in R$, then the right ideal of R generated by the set $\{x - ax \mid x \in R\}$ is R if and only if a is in the right ideal of R generated by the set $\{x - ax \mid x \in R\}$.

Definition 2.3. A right ideal (left ideal, ideal, subset) K of R is called a *right quasiregular right ideal (left ideal, ideal, subset)* of R, if each element of K is right quasiregular.

Lemma 2.4. *A nilpotent element of R is right quasiregular.*

Proof. Let a be a nilpotent element in R, say $a^n = 0$ for some positive integer n. Let K be the right ideal of R generated by $x - ax$, $x \in R$. Observe that $x = x - a^n x = (x - ax) + (ax - a(ax)) + (a^2x - a(a^2x)) + \cdots + (a^{n-1}x - a(a^{n-1}x)) \in K$. Therefore, $K = R$ and hence a is right quasiregular. □

One of the major differences between right and left quasiregular elements of R and hence between its right and left Jacobson-type radicals is the following.

Lemma 2.5. *The constant part of R is right quasiregular.*

Proof. Let R_c be the constant part of R and $a \in R_c$. For $x \in R$, $(x + a) - a(x + a) = (x + a) - a = x$. Therefore, a is right quasiregular. So, R_c is right quasiregular. □

Lemma 2.6. *Let e be a nonzero distributive idempotent in R. Then e is not a right quasiregular element of R.*

Proof. Let e be a nonzero distributive idempotent in R. Now $(0 : e) = \{a \in R \mid ea = 0\}$ is a right ideal of R. Also $x - ex \in (0 : e)$ for all $x \in R$. But e is not in $(0 : e)$. So $(0 : e) \neq R$. Therefore, e is not right quasiregular. □

Definition 2.7. A right ideal K of R is called *right modular* if there is an element $e \in R$ such that $x - ex \in K$ for all $x \in R$. In this case K is said to be *right modular by e*.

Proposition 2.8. *Let $e \in R$. Then e is right quasiregular if and only if no proper right ideal of R is right modular by e.*

The proof follows from the definitions.

Remark 2.9. Let K be a right ideal of R. If K is right modular by e, then $e \in K$ if and only if $K = R$.

Proposition 2.10. *Let K be a proper right ideal of R. If K is, right modular, then K is contained in a maximal right ideal of R which is also right modular.*

The proof of this result is easy and hence omitted.

Proposition 2.11. *Let K and L be right modular right ideals of R and $R = K + L$. Then $K \cap L$ is also a right modular right ideal of R.*

Proof. Suppose that K is right modular by e_1 and L is right modular by e_2. Let $e_1 = b_{11} + b_{12}$ and let $e_2 = b_{21} + b_{22}$, where $b_{11}, b_{21} \in K$ and $b_{12}, b_{22} \in L$. Let $e = b_{21} + b_{12}$. Let $r \in R$; $r - er = r - (b_{21} + b_{12})r = r - b_{12}r - b_{21}r = (r - e_1r) + (b_{11}r - b_{21}r) \in K$; $r - er = r - (b_{21} + b_{12})r = r - b_{12}r - b_{21}r = (r - b_{12}r + b_{22}r - r) + (r - e_2r) \in L$. Therefore, $r - er \in K \cap L$, and hence $K \cap L$ is a right modular right ideal. □

Proposition 2.12. *If $K_1, K_2, \ldots, K_n$ are maximal right modular right ideals of R such that $\bigcap_{i=1}^{n} K_i = \{0\}$, then R has a left identity.*

The proof follows from Proposition 2.11.

Definition 2.13. $J^r_{1/2}(R)$ is the intersection of all maximal right modular right ideals of R and if R has no maximal right modular right ideals, then $J^r_{1/2}(R) = R$.

Theorem 2.14. *$J^r_{1/2}(R)$ is the largest right quasiregular right ideal of R.*

Proof. Let $q \in J^r_{1/2}(R)$. Suppose that q is not right quasiregular. Let K be the right ideal of R generated by the set $\{x - qx \mid x \in R\}$. Now $q \notin K$. By Zorn's lemma, we get a right ideal M, which is maximal for the property that $K \subseteq M$, $q \notin M$. Now M is a maximal right ideal of R. M is right modular right ideal of R as $x - qx \in M$, for all $x \in R$. As $q \in J^r_{1/2}(R)$, $q \in M$, a contradiction. Therefore each element of $J^r_{1/2}(R)$ is right quasiregular. We see now that each right quasiregular right ideal of R is contained in $J^r_{1/2}(R)$. Let K be a right quasiregular right ideal of R. We claim that $K \subseteq J^r_{1/2}(R)$. Suppose that K is not contained in $J^r_{1/2}(R)$. We get a maximal right modular right ideal M such that $R = M + K$. Suppose that M is right modular by e; $e = m + k$, $m \in M$, $k \in K$; $x - kx = x - (-m + e)x = x - (-mx + ex) = x - ex + mx \in M$. Since $M \neq R$, k is not right quasiregular, which is a contradiction. Therefore, $K \subseteq J^r_{1/2}(R)$. Hence, $J^r_{1/2}(R)$ is the largest right quasiregular right ideal of R. □

Remark 2.15. As expected, $J^r_{1/2}(R)$ is not an ideal of R. For this, consider the nonabelian group G of order 6. Then $M_0(G)$ is a simple near-ring. Since G has only one nonzero maximal normal subgroup, $M_0(G)$ has only one nonzero maximal right ideal which is right modular by the identity element of $M_0(G)$. So, $J^r_{1/2}(M_0(G))$ is not an ideal of $M_0(G)$.

Remark 2.16. Let K be a right ideal of R. We show that there is a unique largest ideal of R contained in K. Now $\{0\}$ is an ideal of R and $\{0\} \subseteq K$. Let I and J be ideals of R and $I \subseteq K$, $J \subseteq K$. Now $I + J$ is an ideal of R. Since K is a subgroup of $(R, +)$, $I + J \subseteq K$. Similarly, if $I_1, I_2, \ldots, I_n$ are ideals of R and $I_j \subseteq K$, for all $1 \leq j \leq n$, then we get that $I_1 + I_2 + \cdots + I_n \subseteq K$. Let $\{J_i \mid i \in \Delta\}$ be the collection of all ideals T of R such that $T \subseteq K$. Let $J = \sum_{i \in \Delta} J_i$. Now J is an ideal of R. It is clear that $J \subseteq K$ as any element $a \in J$ can be written as $a = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$, $a_{i_j} \in J_{i_j}$, $i_j \in \Delta$. Obviously, J is the largest ideal of R contained in K.

Definition 2.17. The largest ideal of R contained in $J^r_{1/2}(R)$ is denoted by $J^r_0(R)$ and is called the *right Jacobson radical of R of type*-0.

THEOREM 2.18. *$R \to J^r_0(R)$ is a radical map.*

Proof. (1) First suppose that R has no maximal right modular right ideal. Now $R = J^r_0(R)$ and that $R/J^r_0(R) = \{0\}$. So $J^r_0(R/J^r_0(R)) = \{0\}$. Suppose now that R has a maximal right modular right ideal. Let $\{M_\alpha \mid \alpha \in \Delta\}$ be the collection of all maximal right modular right ideals of R. Since M_α is a maximal right modular right ideal of R and $J^r_0(R) \subseteq M_\alpha$, $M_\alpha/J^r_0(R)$ is a maximal right modular right ideal of $R/J^r_0(R)$ for all $\alpha \in \Delta$. So $J^r_0(R/J^r_0(R)) \subseteq \bigcap_{\alpha\in\Delta}(M_\alpha/J^r_0(R)) = (\bigcap_{\alpha\in\Delta} M_\alpha)/J^r_0(R)$. Since $J^r_0(R)$ is the largest ideal of R contained in $\bigcap_{\alpha\in\Delta} M_\alpha$, we get that the largest ideal of $R/J^r_0(R)$ contained in $\bigcap_{\alpha\in\Delta}(M_\alpha/J^r_0(R))$ is the zero ideal. Therefore, $J^r_0(R/J^r_0(R)) = \{0\}$.

(2) Let h be a homomorphism of the near-ring R onto a near-ring S. If S has no maximal right modular right ideal, then $J^r_0(S) = S$. Then clearly $h(J^r_0(R)) \subseteq S = J^r_0(S)$. Suppose that S has a maximal right modular right ideal. Let $\{N_\alpha \mid \alpha \in \Delta\}$ be the collection of all maximal right modular right ideals of S. Now $h^{-1}(N_\alpha)$ is a maximal right modular right ideal of R for each $\alpha \in \Delta$. Let $M_\alpha = h^{-1}(N_\alpha)$, $\alpha \in \Delta$. We have that $h(h^{-1}(N_\alpha)) = N_\alpha$, for all $\alpha \in \Delta$, and also $J^r_0(R) \subseteq \bigcap_{\alpha\in\Delta} M_\alpha$. So $h(J^r_0(R)) \subseteq h(\bigcap_{\alpha\in\Delta} M_\alpha) \subseteq \bigcap_{\alpha\in\Delta} h(M_\alpha) = \bigcap_{\alpha\in\Delta} N_\alpha$. Since $h(J^r_0(R))$ is an ideal of S and $J^r_0(S)$ is the largest ideal of S contained in $\bigcap_{\alpha\in\Delta} N_\alpha$, $h(J^r_0(R)) \subseteq J^r_0(S)$. Therefore, $R \to J^r_0(R)$ is a radical map. □

We denote the ideal of R generated by an element a of R by (a). The following result is obvious in view of Theorem 2.14.

THEOREM 2.19. $J^r_0(R) = \{a \in R \mid (a)$ *is a right quasiregular ideal*$\}$.

THEOREM 2.20. *$J^r_0(R)$ is the largest right quasiregular ideal of R.*

The proof follows from Theorem 2.14.

THEOREM 2.21. *The nil radical $N(R)$ of R is contained in $J^r_0(R)$.*

The proof follows from Lemma 2.4 and Theorem 2.20.

COROLLARY 2.22. *$P(R) \subseteq N(R) \subseteq J^r_0(R)$, where $P(R)$ is the prime radical of R.*

Proof. We know that $P(R) \subseteq N(R)$. Therefore from Theorem 2.21, $P(R) \subseteq N(R) \subseteq J^r_0(R)$. □

THEOREM 2.23. *Let R be a zero symmetric right near-ring with DCC on left R-subgroups of R. Then $J_0(R) \subseteq J^r_0(R)$.*

Proof. By Pilz [2, Theorem 5.40], $J_0(R)$ is nilpotent. Therefore, $J_0(R) \subseteq N(R)$. By Theorem 2.21, $N(R) \subseteq J^r_0(R)$. Hence, $J_0(R) \subseteq J^r_0(R)$. □

Definition 2.24. The largest ideal contained in a maximal right modular right ideal of R is called a *right* 0*-primitive ideal* of R.

Remark 2.25. If R is a ring, then $J^r_0(R)$ is the (right) Jacobson radical of R and a right 0-primitive ideal of the near-ring R is a (right) primitive ideal of the ring R.

Theorem 2.26. *If $J_0^r(R) \neq R$, then $J_0^r(R)$ is the intersection of all right* 0*-primitive ideals of R.*

Proof. $J_0^r(R)$ is an ideal contained in each maximal right modular right ideal of R. So $J_0^r(R)$ is contained in each right 0-primitive ideal of R. Hence it is contained in the intersection of all right 0-primitive ideals of R. On the other hand, the intersection of all right 0-primitive ideals of R is an ideal contained in each maximal right modular right ideal of R and that it is contained in $J_0^r(R)$. □

Theorem 2.27. *A maximal right modular ideal of a near-ring R is a right* 0*-primitive ideal of R.*

Proof. Let R be a near-ring and let K be a maximal right modular ideal of R. Since K is a proper right modular right ideal of R, K is contained in a maximal right modular right ideal M of R. Since K is a maximal ideal of R, K is the largest ideal contained in M. Hence, K is a right 0-primitive ideal of R. □

Definition 2.28. R is called a *right* 0*-primitive near-ring*, if $\{0\}$ is a right 0-primitive ideal of R.

Definition 2.29. A 0-primitive ideal of R defined in Pilz [2] is called a left 0-primitive ideal of R and similarly a left 0-primitive near-ring.

Theorem 2.30. *Let P be an ideal of R. P is a right* 0*-primitive ideal of R if and only if R/P is a right* 0*-primitive near-ring.*

Proof. Let P be a right 0-primitive ideal of R. So we get a maximal right modular right ideal M of R such that P is the largest ideal of R contained in M. Now M/P is a maximal right modular right ideal of R/P. Since P is the largest ideal of R contained in M, the zero ideal of R/P is the largest ideal of R/P contained in M/P. Therefore R/P is a right 0-primitive near-ring. Suppose now that R/P is a right 0-primitive near-ring. So we get a maximal right modular right ideal M/P of R/P such that the zero ideal of R/P is the largest ideal of R/P contained in M/P. Clearly M is a maximal right modular right ideal of R. Since the zero ideal of R/P is the largest ideal of R/P contained in M/P, P is the largest ideal of R contained in M. Therefore, P is a right 0-primitive ideal of R. □

Theorem 2.31. *A commutative right* 0*-primitive near-ring is a field.*

Proof. Let R be a commutative right 0-primitive near-ring. We get a modular maximal right ideal M of R such that $\{0\}$ is the largest ideal of R contained in M. Suppose that M is right modular by e. $x - ex \in M$, for all $x \in R$. Since R is commutative, M is an ideal of R. Therefore $M = \{0\}$. Since $x - ex \in M = \{0\}$, $x = ex = xe$. So e is the identity element of R. Now R is a commutative ring with identity. Since $M = \{0\}$ is a maximal ideal of R, R is a field. □

3. Right R-groups of type-0

Definition 3.1. A group $(G,+)$ is called a *right R-group* if there is a mapping $(g,r) \to gr$ of $G \times R$ into G such that (1) $(g+h)r = gr + hr$, (2) $g(rs) = (gr)s$, for all $g,h \in G$ and $r,s \in R$.

A subgroup (normal subgroup) H of a right R-group of G is called an R-*subgroup (ideal)* of G, if $hr \in H$ for all $h \in H$ and $r \in R$.

R is a right R-group. If K is a subgroup of $(R,+)$ and $kr \in K$ for all $k \in K$ and $r \in R$, then K is a right R-subgroup of R. Every right ideal of R is an ideal of the right R-group R. Also, if K is a right ideal of R, then R/K is a right R-group, where $(x+K)r = xr+K$, for all $x+K \in R/K$ and $r \in R$.

Definition 3.2. Let G be a right R-group. An element $g \in G$ is called a *generator* of G if $gR = G$ and $g(r+s) = gr+gs$ for all $r,s \in R$. G is said to be *monogenic* if G has a generator.

Definition 3.3. Let G and H be right R-groups. A mapping $f : G \to H$ is called an R-*homomorphism* if $f(x+y) = f(x)+f(y)$ and $f(xr) = f(x)r$ for all $x,y \in G$ and for all $r \in R$. G is said to be R-*isomorphic* to H if there is a one-one R-homomorphism of G onto H.

Proposition 3.4. *Let G be a right R-group. Then G is monogenic if and only if there is a right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proof. Let G be a right R-group. Suppose that G is monogenic. Let g be a generator of G. Define $h : R \to G$ by $h(r) = gr$, for all $r \in R$. h is an R-homomorphism of R onto G. Let K be the kernel of h. $K = \{r \in R \mid h(r) = 0\}$ is a right ideal of R. Therefore R/K is R-isomorphic to G. We get $b \in R$ such that $g = gb$. For each $x \in R$, $gx = gbx$. Now $g(x-bx) = 0$, that is, $x-bx \in K$. So K is modular by b. Conversely, suppose that K is a right ideal of R modular by e and R/K is R-isomorphic to G. Let f be an R-isomorphism of R/K onto G. Let $f(e+K) = g$. We see that $e+K$ is a generator of the right R-group R/K. Let $r,s \in R$. Now $r-er \in K$. So $r+K = er+K = (e+K)r \in (e+K)R$ and hence $(e+K)R = R/K$. Also $(r+s)-e(r+s)$, $r-er$, $s-es \in K$. Let $k = r-er$ and let $t = s-es$. So $r = k+er$, $s = t+es$. Since $(r+s)-e(r+s) \in K$, we get that $(k+er)+(t+es)-e(r+s) = k+(er+t-er)+er+es-e(r+s) \in K$ and that $er+es-e(r+s) \in K$. Therefore $e(r+s)+K = (er+es)+K$. So $(e+K)(r+s) = (er+K)+(es+K) = (e+K)r+(e+K)s$. This shows that $e+K$ is a generator of R/K. So g is a generator of G and hence G is monogenic. □

Proposition 3.5. *Let K be a right ideal of R. Then K is right modular if and only if there is a right R-group G with a generator g such that $K = (0:g)$.*

Proof. Suppose that K is right modular by e. As seen in the above proposition $e+K$ is a generator of the right R-group R/K. Now $r \in (K : e+K) \Leftrightarrow er+K = K \Leftrightarrow er \in K \Leftrightarrow r \in K$. Therefore $K = (K : e+K)$. Conversely suppose that g is a generator of the right R-group G and $(0:g) = K$. Since $gR = G$, we get $e \in R$ such that $ge = g$. Let $r \in R$. Now $g(r-er) = gr-gr = 0$. Therefore $r-er \in (0:g) = K$. Hence, K is right modular by e. □

Definition 3.6. Let G be a right R-group. G is said to be *simple* if $G \neq \{0\}$ and $\{0\}$ and G are the only ideals of G.

Definition 3.7. A monogenic right R-group G is said to be a *right R-group of type*-0 if G is simple.

Proposition 3.8. *Let G be a right R-group. G is a right R-group of type-*0 *if and only if there is a maximal right modular right ideal K of R such that G is R-isomorphic to R/K.*

Proof. G is a right R-group. Suppose that G is of type-0. Let $g \in G$ be a generator. Therefore from the proof of Proposition 3.4, G is R-isomorphic to R/K for some right modular right ideal K of R. Since G is simple, we get that R/K is also simple. Hence, K is a maximal right ideal of R. Conversely, suppose that G is R-isomorphic to R/K, where K is a maximal right modular right ideal of R. Since $R/K \neq \{K\}$ has exactly two ideals, we get that $\{0\}$ and G are the only ideals of G, where $\{0\} \neq G$. Let K be right modular by e. So $e+K$ is a generator of R/K. Therefore, G is also monogenic. Hence, G is a right R-group of type-0. □

Definition 3.9. Let G be a right R-group. The *annihilator* of G denoted by $(0:G)$ is defined as $(0:G) = \{a \in R \mid Ga = \{0\}\}$.

If A and B are nonempty subsets of R, then $(A:B)$ denotes the set $\{r \in R \mid Br \subseteq A\}$.

Corollary 3.10. *Let K be a right modular right ideal of R. Then $(K:R) \subseteq K$.*

Proof. Since K is a right modular right ideal of R, by Proposition 3.5, there ia a right R-group G with a generator g such that $K = (0:g)$. Therefore, $K = (0:g) \supseteq (0:G) = (0:R/K) = (K:R)$. □

Proposition 3.11. *Let R be a zero-symmetric near-ring and let K be a right ideal of R right modular by e. Then $(K:R) = (K:eR)$ and the largest ideal of R contained in K is the largest ideal of R contained in $(K:R)$.*

Proof. Since $eR \subseteq R$, $(K:R) \subseteq (K:eR)$. Let $x \in (K:eR)$. Now $eyx \in K$, for all $y \in R$. But $yx - eyx \in K$, for all $y \in R$. Therefore, $yx \in K$, for all $y \in R$, that is, $x \in (K:R)$. So $(K:eR) \subseteq (K:R)$. Therefore, $(K:R) = (K:eR)$. Let J be the largest ideal of R contained in K. For $x \in J$, $Rx \subseteq J \subseteq K$. Therefore, $J \subseteq (K:R)$. Let I be an ideal of R contained in $(K:R)$. By Corollary 3.10, $(K:R) \subseteq K$. So, I $\subseteq K$. Therefore, $I \subseteq J$. Hence, J is the largest ideal of R contained in $(K:R)$. □

Proposition 3.12. *Let P be an ideal of a zero-symmetric near-ring R. P is right* 0*-primitive if and only if P is the largest ideal of R contained in* $(0:G)$ *for some right R-group G of type-*0.

Proof. Let P be an ideal of a zero-symmetric near-ring R. Suppose that P is a right 0-primitive ideal of R. So we get a maximal right modular right ideal K of R such that P is the largest ideal of R contained in K. Now by Proposition 3.8, R/K is a right R-group of type-0. By Proposition 3.11, P is the largest ideal of R contained in $(K:R) = (0:R/K)$. Conversely, suppose that P is the largest ideal of R contained in $(0:G)$, where G is a right R-group of type-0. Now G is R-isomorphic to R/K for some maximal right modular right ideal K of R. So $(0:G) = (0:R/K) = (K:R)$. Since P is the largest ideal of R contained in $(0:G) = (K:R)$, by Proposition 3.11, P is the largest ideal of R contained in K. Hence, P is a right 0-primitive ideal of R. □

Proposition 3.13. *Let G be a monogenic right R-group. If R is a distributively generated (d.g.) near-ring then there is a subset T of G such that $h(a+b) = ha+hb$, for all $h \in T$ and $a,b \in R$, and T generates $(G,+)$.*

Proof. Let G be a monogenic right R-group. Suppose that R is a d.g. near-ring. Since G is a monogenic right R-group, by Proposition 3.4, we get a right modular right ideal K of R such that G is R-isomorphic to R/K. Let f be a R-isomorphism of G onto R/K. Let S be the set of distributive elements of R, where S generates $(R,+)$. It is clear that $\overline{S} = \{s+K \mid s \in S\}$ generates $(R/K,+)$. Let $s \in S$ and let $a,b \in R$. Since $s(a+b) = sa+sb$, we have that $s(a+b)+K = (sa+sb)+K$, that is, $(s+K)[a+b] = (s+K)a+(s+K)b$. Therefore, $T = \{f^{-1}(s+K) \mid s \in S\}$ is the required subset of G. □

Proposition 3.14. *Let G be a monogenic right R-group. If R is a d.g. near-ring, then $(0:G)$ is an ideal of R.*

Proof. G is a monogenic right R-group and R is d.g. Let S be the set of distributive elements of R. By Proposition 3.13, we get a subset T of G such that $h(x+y) = hx+hy$, for all $h \in T$ and $x,y \in R$, and T generates $(G,+)$. Let $a,b \in (0:G)$, $p,q \in R$, and let $g \in G$. $h(a+b) = ha+hb = 0+0 = 0$, $h(-b) = -hb = 0$ for all $h \in T$. Now $p = \delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n$, where $\delta_i = \pm 1$ and $s_i \in S$, for $1 \le i \le n$:

$$\begin{aligned} p(q+a)-pq &= (\delta_1 s_1+\delta_2 s_2+\cdots+\delta_n s_n)(q+a)-(\delta_1 s_1+\delta_2 s_2+\cdots+\delta_n s_n)q \\ &= \delta_1 s_1(q+a)+\delta_2 s_2(q+a)+\cdots+\delta_n s_n(q+a) \\ &\quad -(\delta_1 s_1 q+\delta_2 s_2 q+\cdots+\delta_n s_n q) \\ &= \delta_1(s_1 q+s_1 a)+\delta_2(s_2 q+s_2 a)+\cdots+\delta_n(s_n q+s_n a) \\ &\quad -\delta_n s_n q-\delta_{n-1}s_{n-1}q-\cdots-\delta_1 s_1 q. \end{aligned} \tag{3.1}$$

Now

$$\begin{aligned} h[p(q+a)-pq] &= h(\delta_1(s_1 q+s_1 a))+h(\delta_2(s_2 q+s_2 a))+\cdots+h(\delta_n(s_n q+s_n a)) \\ &\quad -h(\delta_n s_n q)-h(\delta_{n-1}s_{n-1}q)-\cdots-h(\delta_1 s_1 q) \\ &= \delta_1(h(s_1 q)+(hs_1)a)+\delta_2(h(s_2 q)+(hs_2)a)+\cdots+\delta_n(h(s_n q) \\ &\quad +(hs_n)a)-\delta_n h(s_n q)-\delta_{n-1}h(s_{n-1}q)-\cdots-\delta_1 h(s_1 q) \\ &= \delta_1 h(s_1 q)+\delta_2 h(s_2 q)+\cdots+\delta_n h(s_n q)-\delta_n h(s_n q) \\ &\quad -\delta_{n-1}h(s_{n-1}q)-\cdots-\delta_1 h(s_1 q)=0. \end{aligned} \tag{3.2}$$

Also $h(ap) = (ha)p = 0p = 0$. Since every element of G is a finite sum of elements h, where $h \in T$ or $-h \in T$, we get that

(1) $g(a+b) = 0$;
(2) $g(-a) = 0$;

(3) $g(p+a-p)=0$;
(4) $g(p(q+a)-pq)=0$;
(5) $g(ap)=0$.

Therefore, $(0:G)$ is an ideal of R. □

Corollary 3.15. *Let R be a d.g. near-ring and let P be an ideal of R. Then P is a right 0-primitive ideal of R if and only if $P=(0:G)$ for some right R-group G of type-0.*

Proof. Since a d.g. near-ring is zero-symmetric, the proof follows from Propositions 3.14 and 3.12. □

Theorem 3.16. *A right 0-primitive ideal of R is a prime ideal of R.*

Proof. Let P be a right 0-primitive ideal of R. We get a right R-module G of type-0 with a generator g such that P is the largest ideal of R contained in $M=(0:g)=\{r\in R\mid gr=0\}$, M is a maximal right modular right ideal of R. Let A and B be ideals of R and $AB\subseteq P$. Suppose that $A\nsubseteq P$ and $B\nsubseteq P$. Since $A\nsubseteq P$, $gA\neq\{0\}$. Clearly gA is a subgroup of G. Let $h\in G$ and let $a\in A$. $h=gr$, for some $r\in R$. $h+ga-h=gr+ga-gr=g(r+a-r)\in gA$, as $r+a-r\in A$. So gA is a normal subgroup of G. Also $(gA)R=g(AR)\subseteq gA$. This shows that gA is an ideal of G. Since $gA\neq\{0\}$ and G is a right R-group of type-0, $gA=G$. Similarly, for B, also $gB=G$. Now $G\supseteq gAB=(gA)B=GB\supseteq gB=G$. Therefore, $gAB=G$, a contradiction to the fact that $gAB=\{0\}$. So either $A\subseteq P$ or $B\subseteq P$. Hence, P is a prime ideal of R. □

4. The right J_0-radical of a biregular near-ring

In this section, it will be shown that the right and left Jacobson radicals of type-0 of a biregular near-ring R are equal, and an ideal P of R is right 0-primitive if and only if P is left 0-primitive.

We recall the following definition of Betsch (Pilz [2, Remark 3.49]).

Definition 4.1. A near-ring R is called *biregular* if there exists a set E of central idempotents of R such that

(1) Re is an ideal of R for all $e\in E$;
(2) for each $r\in R$, there exists an $e\in E$ such that $\mathrm{Re}=(r)$;
(3) $e+f=f+e$, for all $e,f\in E$;
(4) $ef, e+f-ef\in E$, if $e,f\in E$.

A biregular near-ring is zero-symmetric. Let R be a biregular near-ring. Now $\{0\}=(0)=\mathrm{Re}$, for some central idempotent e of R. Also $e=ee\in\mathrm{Re}=\{0\}$. So $e=0$ and hence $R0=\{0\}$. Therefore R is zero-symmetric.

Theorem 4.2. *Let R be a biregular near-ring. Then $J_0^r(R)=\{0\}$.*

Proof. Let R be a biregular near-ring. Let $x\in J_0^r(R)$. Let (x) be the ideal of R generated by x. Now $(x)=(e)$, for some central idempotent $e\in R$. Therefore $e\in J_0^r(R)\subseteq J_{1/2}^r(R)$. Since e is central, e is a distributive idempotent. By Lemma 2.6, $J_{1/2}^r(R)$ contains no nonzero distributive idempotent. Therefore, $e=0$. Hence, $J_0^r(R)=\{0\}$. □

Theorem 4.3. *Let R be a biregular near-ring. Then R is right (left)* 0*-primitive if and only if R is a nonzero simple near-ring with identity.*

Proof. Let R be a biregular near-ring. Suppose that R is a right (left) 0-primitive near-ring. Let $0 \neq a \in R$. Now $(a) = \mathrm{Re}$, for some central idempotent $e \neq 0$ in R. Since e is a central idempotent, $(0:e)$ is an ideal of R. Now $R = \mathrm{Re} + (0:e)$ is a direct sum of the ideals Re and $(0:e)$. Since R is right (left) 0-primitive, $\{0\}$ is a prime ideal of R by Theorem 3.16 (Pilz [2, Theorem 4.34]). Since R is zero-symmetric, $\mathrm{Re}(0:e) \subseteq \mathrm{Re} \cap (0:e) = \{0\}$. So either $\mathrm{Re} = \{0\}$ or $(0:e) = \{0\}$. Since $e \neq 0$, $\mathrm{Re} \neq \{0\}$. Therefore, $(0:e) = \{0\}$. Hence, $R = \mathrm{Re} = (a)$. So R is a simple near-ring with identity e. Conversely, suppose that R is a nonzero simple near-ring with identity. Since R is a near-ring with identity, R has a maximal right (left) ideal K and is right (left) modular by the identity. Since R is simple, $\{0\}$ is the largest ideal of R contained in K. Therefore, $\{0\}$ is a right (left) 0-primitive ideal of R and hence R is right (left) 0-primitive. □

Corollary 4.4. *A right (left)* 0*-primitive ideal of a biregular near-ring is a maximal ideal which is both right and left modular.*

Proof. Let R be a biregular near-ring and let P be a right (left) 0-primitive ideal of R. By Theorem 2.30, R/P is a right (left) 0-primitive near-ring. Since R/P is also biregular, by Theorem 4.3, R/P is a simple near-ring with identity. So, P is a maximal right and left modular ideal. □

Corollary 4.5. *Let R be a biregular near-ring and let P be an ideal of R. Then the following statements are equivalent:*

(1) *P is a right* 0*-primitive ideal of R;*
(2) *P is a left* 0*-primitive ideal of R;*
(3) *P is a maximal ideal of R which is both right and left modular.*

Proof. By Corollary 4.4, (1) implies (3) and (2) implies (3). Let P be a maximal right and left modular ideal of R. Since P is a maximal right (left) modular ideal of R, R/P is a simple near-ring with a left (right) identity. So R/P is a simple near-ring with identity. Since R/P is biregular, by Theorems 4.3 and 2.30, we get that (3) implies (1) and (3) implies (2). □

Corollary 4.6. *Let $(G,+)$ be a group with more than one element. Then $M_0(G)$ is a right and left* 0*-primitive near-ring.*

Proof. Since $M_0(G)$ is a nonzero simple biregular near-ring, it is a right and left 0-primitive by Theorem 4.3. □

Now we observe an interesting fact that unlike the left Jacobson-type radical classes, the J_0^r-radical class contains almost all the classes of near-rings with trivial multiplication.

We consider J_0^r-radical of a near-ring with trivial multiplication. Let $(R,+)$ be a group containing more than one element. Let S be a nonempty subset of R not containing 0. The trivial multiplication on $(R,+)$ determined by S is given by $a \cdot b = a$, if $b \in S$ and 0, if $b \in S^c$, where S^c is the complement of S in R. Then $(R,+,\cdot)$ is a near-ring.

Example 4.7. Let R, S, and S^c be as defined above. Then S^c is nilpotent and hence it is right quasiregular. Moreover, R is right quasiregular if and only if S^c is not a normal subgroup of $(R,+)$ of index 2.

Let $a \in R$. Let $A = \{x - ax \mid x \in R\}$, let $B = \{x - ax \mid x \in S\}$, and let $C = \{x - ax \mid x \in S^c\}$. Now $A = B \cup C$, $B = \{x - a \mid x \in S\} = S + (-a)$, and $C = \{x \mid x \in S^c\} = S^c$. So $A = [S + (-a)] \cup S^c$. Let K be the normal subgroup of $(R,+)$ generated by A. Clearly K is a right ideal of R. Hence, K is the right ideal of R generated by A.

If $a, b \in S^c$, then $ab = 0$. Therefore, S^c is nilpotent and right quasiregular. Now suppose that $a \in S$. First assume that S^c is not a normal subgroup of $(R,+)$. Since $R = S \cup S^c$, the normal subgroup of $(R,+)$ generated by S^c contains an element $s \in S$. So $s \in K$. Since $s, s - a \in K$, $a = -(s - a) + s \in K$. Therefore, a is right quasiregular.

Assume now that S^c is a normal subgroup of $(R,+)$. Since $a \in S$, a is not in S^c. So $-a$ is not an element of S^c, as S^c is a subgroup. Since $R = S^c \cup S$, $R = [S^c + (-a)] \cup [S + (-a)]$. Now S^c and $S^c + (-a)$ are two right cosets of S^c in $(R,+)$. Let S^c, $S^c + (-a)$, $S^c + r_\alpha$, and $\alpha \in \triangle$ be the distinct right cosets of S^c in $(R,+)$. So $R = S^c \cup [S^c + (-a)] \cup [\cup S^c + r_\alpha]$. The complement of $S^c + (-a)$ in R is $S + (-a) = S^c \cup [\cup S^c + r_\alpha]$. If the index of S^c in $(R,+)$ is 2, then $S + (-a) = S^c$ and hence $A = S^c = K \neq R$. Thus a is not right quasiregular.

Now assume that the index of S^c in $(R,+)$ is not 2. So, $S + (-a) \neq S^c$. We will show that $S + (-a)$ is not a normal subgroup of $(R,+)$. Suppose that $S + (-a)$ is a normal subgroup of $(R,+)$. Since $S^c \subseteq S + (-a)$, $R = [S^c + (-a)] \cup [S + (-a)]$, and $[S^c + (-a)] \cap [S + (-a)]$ is empty, $S^c + (-a) = S + (-a) + (-a)$. Since S^c is a proper subset of $S + (-a)$, $S^c + (-a)$ is a proper subset of $S + (-a) + (-a) = S^c + (-a)$, a contradiction. Therefore, $S + (-a)$ is not a normal subgroup of $(R,+)$. So, the normal subgroup of $(R,+)$ generated by $S + (-a)$ contains an element $x + (-a)$ of $S^c + (-a)$, $x \in S^c$. Now $x + (-a)$, $x \in K$, and $a = -(x + (-a)) + x \in K$. Therefore, a is right quasiregular.

Remark 4.8. Let R, S, and S^c be as defined above and let S^c be a normal subgroup of $(R,+)$ of index 2. Then, S^c is a nilpotent ideal of R, $S^c = P(R) = J_0^r(R) = J_2(R)$. Moreover, S^c is the set of right (left) quasiregular elements of R.

We see now by an example that there is a near-ring R which has a left 0-primitive ideal, but has no right 0-primitive ideals and that the right and left Jacobson radicals of R of type-0 are different. We know that every prime ideal of a zero-symmetric near-ring R with DCC on left R-subgroups of R is left 0-primitive, but there is a finite zero-symmetric near-ring $R \neq \{0\}$ in which no (proper) prime ideal is right 0-primitive. Even though for a zero-symmetric near-ring R with DCC on left R-subgroups of R, every left quasiregular left R-subgroup of R is nilpotent, we see that there is a finite zero-symmetric right near-ring R in which no nonzero right quasiregular right R-subgroup of R is nil.

Example 4.9. Let $(R,+)$ be a group containing more than two elements. Define a trivial multiplication in R by $rs = r$ if $s \neq 0$ and 0 if $s = 0$ for all $r, s \in R$. R is a zero-symmetric right near-ring. Clearly, R is a left R-group of type-2. Moreover, R is simple. Therefore, R is 2-primitive on the left R-group R, as $RR \neq \{0\}$. So $J_2(R) = \{0\} = J_0(R)$. But each element of R is right quasiregular. Therefore, $J_0^r(R) = R$. If R is finite, then obviously R has DCC on right (left) R-groups of R, but no nonzero right quasiregular right ideal of R is nilpotent. Moreover, the zero ideal of R is prime but not right 0-primitive.

We recall some of the definitions and results of [3] which are required to observe that right Jacobson radicals are relevant for the study of near-rings in terms of matrix near-rings.

Matrix near-rings were introduced in [1].

Definition 4.10. Let R be a zero-symmetric near-ring with identity. A subset $\{e_{ij} \mid 1 \leq i, j \leq n\}$ of distributive elements in R is said to be a set of *matrix units* in R if and only if $e_{11} + e_{22} + \cdots + e_{nn} = 1$ and $e_{rs}e_{pq} = \delta_{sp}e_{rq}$, where

$$\delta_{sp} = \begin{cases} 1 & \text{if } s = p, \\ 0 & \text{if } s \neq p. \end{cases} \tag{4.1}$$

Proposition 4.11. *Let R be a zero-symmetric near-ring with identity. $R = K_1 \oplus K_2 \oplus \cdots \oplus K_n$, a direct sum of n pairwise isomorphic right ideals K_i of R as right R-groups if and only if R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. In this case $K_i = e_{ii}R$, for all $1 \leq i \leq n$.*

As stated soon after [3, Corollary 15], we have the following.

Theorem 4.12. *Let R be a simple and d.g. near-ring with identity. Then R is isomorphic to a matrix near-ring $M_n(S)$ if and only if R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$.*

Theorem 4.13. *Let R be a simple d.g. near-ring with DCC on right ideals of R and $J^r_{1/2} = \{0\}$. Suppose that any two minimal right ideals of R are isomorphic as right R-groups. Then, $R = K_1 \oplus K_2 \oplus \cdots \oplus K_n$, a direct sum of minimal right ideals K_i and is (isomorphic to) a matrix near-ring $M_n(S)$.*

Proof. Since $J^r_{1/2}(R) = \{0\}$ and R has DCC on right ideals of R, we get that the intersection of a finite number of maximal right modular right ideals of R is zero. So, R is a direct sum of a finite number of minimal right ideals $K_1, K_2, \ldots, K_n$ of R. By Proposition 2.12, R has a left identity as the intersection of a finite number of maximal right modular right ideals of R is zero. Since R is a simple near-ring with left identity, it has an identity. Also, since by our assumption any two minimal right ideals of R are isomorphic as right R-groups, by Proposition 4.11, R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. Therefore, by Theorem 4.12, R is (isomorphic to) a matrix near-ring $M_n(S)$. □

Example 4.14. We give an example of a nonring which satisfies the hypothesis of Theorem 4.13. Let G be a finite simple nonabelian additive group. By [3, Corollary 19], $E(G^2)$ is isomorphic to the matrix near-ring $M_2(E(G))$. As mentioned soon after [3, Corollary 19], $E(G^2) = M_0(G^2)$. So, $M_0(G^2)$ is a simple d.g. near-ring with DCC on right ideals. Let $i \in \{1,2\}$. Let $G_1 = G \times \{0\}$ and let $G_2 = \{0\} \times G$. Since G_i is a maximal (minimal) normal subgroups of G^2, $K_i = (G_i : G^2) = \{m \in M_0(G^2) \mid m(a) \in G_i, \text{ for all } a \in G^2\}$ is a maximal right ideal of $M_0(G^2)$. Moreover, $K_1 \cap K_2 = \{0\}$. Thus $J^r_{1/2}(M_0(G^2)) = \{0\}$. This shows that $M_0(G^2) = K_1 \oplus K_2$, where K_i is a minimal right ideal of $M_0(G^2)$. Define $e_i : G^2 \to G_i$ by $e_i((a_1,a_2)) = (b_1,b_2)$, where $b_j = a_i$ if $j = i$ and 0 if $j \neq i$. Now e_i is a group homomorphism and hence it is a distributive idempotent in $M_0(G^2)$ and $e_iM_0(G^2) \subseteq K_i$. Since e_1 and e_2 are orthogonal distributive idempotents in $M_0(G^2)$ and $e_1 + e_2 = 1$, by [3, Proposition 2], we get that $e_iM_0(G^2)$ is a right ideal of $M_0(G^2)$. Thus, $K_i = e_iM_0(G^2)$.

The mapping $e_{12} : G^2 \to G^2$ defined by $e_{12}((a_1, a_2)) = (a_2, 0)$ is an endomorphism of G^2. So, e_{12} is a distributive element in $M_0(G^2)$. It is an easy verification that the mapping $h : e_2M_0(G^2) \to e_1M_0(G^2)$ defined by $h(e_2m) = e_{12}(e_2m)$ is an isomorphism of the right $M_0(G^2)$-groups. So, K_1 and K_2 are isomorphic as right $M_0(G^2)$-groups. Since a minimal right ideal K of $M_0(G^2)$ is isomorphic to K_j for some $j \in \{1,2\}$ as right $M_0(G^2)$-groups, we get that any two minimal right ideals of $M_0(G^2)$ are isomorphic as right $M_0(G^2)$-groups. So, $M_0(G^2)$ satisfies the hypothesis of Theorem 4.13.

Example 4.15. Let G be a finite simple nonabelian additive group. Now by Pilz [2, Corollary 7.48], $E(G) = M_0(G)$. So, $M_0(G)$ is a finite simple d.g. near-ring with identity. Moreover, $J_2(M_0(G)) = \{0\}$ and each minimal left ideal of $M_0(G)$ is isomorphic to G as left $M_0(G)$-groups. Since each distributive element of $M_0(G)$ is an endomorphism of $(G,+)$, 0 and the automorphisms of $(G,+)$ are the only distributive elements of $M_0(G)$. Therefore, $M_0(G)$ has no nontrivial matrix units. Hence, $M_0(G)$ is not isomorphic to a matrix near-ring $M_n(S)$, where $n > 1$.

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