TAUBERIAN CONDITIONS FOR A GENERAL LIMITABLE METHOD

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Let (u_n) be a sequence of real numbers, L an additive limitable method with some property, and $\mathcal{U}$ and $\mathcal{V}$ different spaces of sequences related to each other. We prove that if the classical control modulo of the oscillatory behavior of (u_n) in $\mathcal{U}$ is a Tauberian condition for L, then the general control modulo of the oscillatory behavior of integer order m of (u_n) in $\mathcal{U}$ or $\mathcal{V}$ is also a Tauberian condition for L.

1. Introduction

In this paper, $u_n = O(1)$ and $u_n = o(1)$ denote $O(1)$ as $n \to \infty$ and $o(1)$ as $n \to \infty$, respectively. Let $\mathcal{N}$, $\mathcal{B}$, $\mathcal{S}$, and $\mathcal{M}$ denote the space of sequences converging to 0, bounded, slowly oscillating, and moderately oscillating, respectively.

The classical control modulo of the oscillatory behavior of (u_n) is denoted by $\omega_n^{(0)}(u) = n\Delta u_n$ and the general control modulo of the oscillatory behavior of order m of (u_n) is defined by $\omega_n^{(m)}(u) = \omega_n^{(m-1)}(u) - \sigma_n^{(1)}(\omega^{(m-1)}(u))$, where

$$\Delta u_n = \begin{cases} u_n - u_{n-1}, & n \geq 1, \\ u_0, & n = 0, \end{cases} \qquad \sigma_n^{(1)}(u) = \frac{1}{n+1}\sum_{k=0}^{n} u_k. \tag{1.1}$$

Tauber [10] proved that if (u_n) is Abel limitable and

$$(\omega_n^{(0)}(u)) \in \mathcal{N}, \tag{1.2}$$

then (u_n) is convergent. The condition (1.2) on the sequence (u_n) is called a Tauberian condition for Abel limitable method and the resulting theorem is called a Tauberian theorem.

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Tauber [10] further proved that the condition

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \mathcal{N} \tag{1.3}$$

is also a Tauberian condition. It was shown by Littlewood [6] that the condition (1.2) could be replaced by

$$(\omega_n^{(0)}(u)) \in \mathcal{B}. \tag{1.4}$$

Hardy and Littlewood [5] improved Littlewood's theorem replacing (1.4) by onesided boundedness of $(\omega_n^{(0)}(u))$.

Stanojević [9] reformulated the definition of slow oscillation given by Schmidt [8] in a more suitable form and then proved that the conditions (1.2) and (1.3) could be replaced by

$$(\omega_n^{(0)}(u)) \in \mathcal{S}, \tag{1.5}$$

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \mathcal{S}, \tag{1.6}$$

respectively.

A generalization of slow oscillation, moderate oscillation, was introduced by Stanojević and it was proved by Dik [4] that (1.5) could be replaced by

$$(\omega_n^{(0)}(u)) \in \mathcal{M}, \tag{1.7}$$

and (1.6) could not be replaced by

$$(\sigma_n^{(1)}(\omega^{(0)}(u))) \in \mathcal{M}. \tag{1.8}$$

Recently, Çanak and Totur [3] have shown that for any nonnegative integer $m \geq 1$,

$$(\omega_n^{(m)}(u)) \in \mathcal{M} \tag{1.9}$$

is a Tauberian condition for Abel limitable method.

Meyer-König and Tietz [7] proved that if (1.2) is a Tauberian conditions for an additive and regular limitability method, then (1.3) is a Tauberian condition for L. Çanak et al. [1] extended and generalized Meyer-König and Tietz's [7] result and obtained the following theorems for an additive and $(C,1)$ regular method L.

Theorem 1.1. *If $(\omega_n^{(0)}(u)) \in \mathcal{S}$ is a Tauberian condition for an additive and $(C,1)$ regular limitable method L, then $(\omega_n^{(1)}(u)) \in \mathcal{S}$ is a Tauberian condition for L.*

Theorem 1.2. *If $(\omega_n^{(0)}(u)) \in \mathcal{B}$ is a Tauberian condition for an additive and $(C,1)$ regular limitable method L, then $(\omega_n^{(1)}(u)) \in \mathcal{B}$ is a Tauberian condition for L.*

Let $\mathcal{U}$ and $\mathcal{V}$ be distinct spaces of sequences related to each other. In this paper, we prove that if the classical control modulo of the oscillatory behavior of (u_n) in $\mathcal{U}$ is a Tauberian condition for an additive and $(C,1)$ limitable method L, then the general control modulo of the oscillatory behavior of integer order m of (u_n) in $\mathcal{U}$ or $\mathcal{V}$ is also a Tauberian condition for L.

2. Notations and definitions

Throughout this paper, let $u = (u_n)$ be a sequence of real numbers. For each integer $m \geq 0$ and for all nonnegative integers n denote $\sigma_n^{(m)}(u)$ by

$$\sigma_n^{(m)}(u) = \begin{cases} \dfrac{1}{n+1}\displaystyle\sum_{k=0}^{n}\sigma_k^{(m-1)}(u) = u_0 + \sum_{k=1}^{n}\frac{V_k^{(m-1)}(\Delta u)}{k}, & m \geq 1, \\ u_n, & m = 0, \end{cases} \tag{2.1}$$

where

$$V_n^{(m)}(\Delta u) = \begin{cases} \sigma_n^{(1)}\left(V^{(m-1)}(\Delta u)\right), & m \geq 1, \\ \dfrac{1}{n+1}\displaystyle\sum_{k=0}^{n} k\Delta u_k, & m = 0. \end{cases} \tag{2.2}$$

The identity

$$u_n - \sigma_n^{(1)}(u) = V_n^{(0)}(\Delta u) \tag{2.3}$$

is well known and will be extensively used. We define inductively for each integer $m \geq 1$ and for all nonnegative integers n,

$$(n\Delta)_m u_n = n\Delta\left((n\Delta)_{m-1}u_n\right), \quad \text{where } (n\Delta)_0 u_n = u_n. \tag{2.4}$$

It is proved in [2] that for each integer $m \geq 1$,

$$\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u). \tag{2.5}$$

Definition 2.1. A sequence $u = (u_n)$ is Abel limitable to s if the limit $\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty}u_n x^n = s$.

Definition 2.2. A sequence $u = (u_n)$ is L limitable to s if $L - \lim_n u_n = s$.

A limitation method L is called additive if $L - \lim_n u_n = s$ and $L - \lim_n v_n = t$ imply that $L - \lim_n (u_n + v_n) = s + t$. A limitation method L is called regular if the $L -$ limit of every convergent sequence is equal to its limit. L is called $(C,1)$ regular if $L - \lim_n u_n = s$ implies $L - \lim_n \sigma_n^{(1)}(u) = s$. It is clear that every regular limitable method is $(C,1)$ regular.

Definition 2.3. A sequence $u = (u_n)$ is one-sidedly bounded if for some $C \geq 0$ and for each nonnegative integer n, $u_n \geq -C$.

Definition 2.4. A sequence $u = (u_n)$ is slowly oscillating [9] if

$$\lim_{\lambda\to 1^+}\overline{\lim_{n}}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right| = 0, \tag{2.6}$$

where $[\lambda n]$ denotes the integer part of λn.

A sequence $u=(u_n)\in\mathcal{S}$ if and only if $(V_n^{(0)}(\Delta u))\in\mathcal{S}$ and $(V_n^{(0)}(\Delta u))\in\mathcal{B}$ (see [4]). The next definition is a generalization of slow oscillation.

Definition 2.5. A sequence $u=(u_n)$ is moderately oscillating [9] if for $\lambda>1$,

$$\overline{\lim_n}\max_{n+1\le k\le[\lambda n]}\left|\sum_{j=n+1}^{k}\Delta u_j\right|<\infty. \tag{2.7}$$

A sequence $(u_n)\in\mathcal{M}$ if and only if $(V_n^{(0)}(\Delta u))\in\mathcal{B}$ (see [4]).

3. Results and proofs

Theorem 3.1. *If $(\omega_n^{(0)}(u))\in\mathcal{M}$ is a Tauberian condition for L, then for any integer $m\ge 1$, $(\omega_n^{(m)}(u))\in\mathcal{M}$ is also a Tauberian condition for L.*

Proof. Assume that $(\omega_n^{(0)}(u))\in\mathcal{M}$ is a Tauberian condition for L. Let $L-\lim_n u_n=s$. Since L is $(C,1)$ regular, it follows by (2.3) that $L-\lim_n V_n^{(0)}(\Delta u)=0$. It is obvious that $L-\lim_n u_n=s$ implies $L-\lim_n(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u)=0$. Since

$$(\omega_n^{(m)}(u))=\left(n\Delta\left((n\Delta)_{m-1}V_n^{(m-1)}(\Delta u)\right)\right)\in\mathcal{M}, \tag{3.1}$$

by assumption, we have

$$(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u)=o(1). \tag{3.2}$$

By the same reasoning, we deduce that

$$(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u)=n\Delta\left((n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)\right)=o(1) \tag{3.3}$$

and $L-\lim_n(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)=0$. Again by assumption, we have

$$(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u)=o(1). \tag{3.4}$$

From the identity

$$(n\Delta)_{m-1}V_n^{(m-1)}(\Delta u)=(n\Delta)_{m-2}V_n^{(m-2)}(\Delta u)-(n\Delta)_{m-2}V_n^{(m-1)}(\Delta u), \tag{3.5}$$

(3.2), and (3.4), we have

$$(n\Delta)_{m-2}V_n^{(m-2)}(\Delta u)=o(1). \tag{3.6}$$

Continuing in this vein, we have

$$n\Delta V_n^{(1)}(\Delta u)=o(1). \tag{3.7}$$

Since $L-\lim_n V_n^{(1)}(\Delta u)=0$, it follows by (3.7) that

$$V_n^{(1)}(\Delta u)=o(1). \tag{3.8}$$

From (3.7) and (3.8), we have $V_n^{(0)}(\Delta u) = o(1)$. $L - \lim_n \sigma_n^{(1)}(u) = s$ and $V_n^{(0)}(\Delta u) = n\Delta\sigma_n^{(1)}(u) = o(1)$ imply that $\lim_n \sigma_n^{(1)}(u) = s$. Hence, by (2.3), (u_n) converges to s. □

THEOREM 3.2. *If $(\omega_n^{(0)}(u)) \in \mathcal{B}$ is a Tauberian condition for L, then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{B}$ is also a Tauberian condition for L.*

Proof. Assume that $\omega_n^{(0)}(u) = O(1)$ is a Tauberian condition for L. Let $L - \lim_n u_n = s$. Since $L - \lim_n (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = 0$ and $\omega_n^{(m)}(u) = n\Delta((n\Delta)_{m-1} V_n^{(m-1)}(\Delta u)) = O(1)$, $(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = o(1)$ by assumption. The rest of the proof is as in the proof of Theorem 3.1. □

THEOREM 3.3. *If for some $C \geq 0$, $\omega_n^{(0)}(u) \geq -C$ is a Tauberian condition for L, then for any integer $m \geq 1$, $\omega_n^{(m)}(u) \geq -C$ is also a Tauberian condition for L.*

Proof. Assume that $\omega_n^{(0)}(u) \geq -C$ for some $C \geq 0$ is a Tauberian condition for L. Let $L - \lim_n u_n = s$. Since $L - \lim_n (n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = 0$ and $\omega_n^{(m)}(u) = n\Delta((n\Delta)_{m-1} V_n^{(m-1)}(\Delta u)) \geq -C$, $(n\Delta)_{m-1} V_n^{(m-1)}(\Delta u) = o(1)$ by assumption. The rest of the proof is as in the proof of Theorem 3.1. □

We now prove that if $(\omega_n^{(0)}(u)) \in \mathcal{M}$ (or $\in \mathcal{B}$) is a Tauberian condition for L, then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{B}$ (or $\in \mathcal{M}$) is a Tauberian condition for L, respectively.

THEOREM 3.4. *If $(\omega_n^{(0)}(u)) \in \mathcal{M}$ is a Tauberian condition for L, then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{B}$ is also a Tauberian condition for L.*

Proof. It is sufficient to note that $\omega_n^{(m)}(u) = (n\Delta)_m V_n^{(m-1)}(\Delta u) = V_n^{(0)}(\Delta\omega^{(m-1)}(u)) = O(1)$ implies $(\omega_n^{(m-1)}(u)) \in \mathcal{M}$. Proof now follows from Theorem 3.1. □

THEOREM 3.5. *If $(\omega_n^{(0)}(u)) \in \mathcal{B}$ is a Tauberian condition for L, then for any integer $m \geq 1$, $(\omega_n^{(m)}(u)) \in \mathcal{M}$ is also a Tauberian condition for L.*

Proof. It is sufficient to note that $(\omega_n^{(m)}(u)) \in \mathcal{M}$ implies $V_n^{(0)}(\Delta\omega^{(m)}(u)) = \omega_n^{(m+1)}(u) = O(1)$. Proof now follows from Theorem 3.4. □

Remark 3.6. Because of the inclusion $\mathcal{N} \subset \mathcal{S} \subset \mathcal{M}$, the condition "belonging to $\mathcal{M}$" can be replaced by "belonging to $\mathcal{S}$" or "belonging to $\mathcal{N}$."

In Theorems 3.1, 3.2, and 3.3, taking $m = 1$ and replacing $\mathcal{M}$ by $\mathcal{S}$, we have [1, Theorems 4.1, 4.2, and 4.4] by Çanak et al.

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