RIEMANN-STIELTJES OPERATORS BETWEEN BERGMAN-TYPE SPACES AND α-BLOCH SPACES

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We study the following integral operators: $J_g f(z)=\int_0^z f(\xi)g'(\xi)d\xi$; $I_g f(z)=\int_0^z f'(\xi)g(\xi)d\xi$, where g is an analytic function on the open unit disk in the complex plane. The boundedness and compactness of J_g, I_g between the Bergman-type spaces and the α-Bloch spaces are investigated.

1. Introduction

Let D be the open unit disk in the complex plane. Denote by $H(D)$ the class of all analytic functions on D. An analytic function f in D is said to belong to the α-Bloch space $\mathcal{B}^\alpha$, or Bloch-type space, if

$$\|f\|_\alpha = \sup_{z\in D}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| < \infty. \tag{1.1}$$

The expression $\|f\|_\alpha$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. It makes $\mathcal{B}^\alpha$ into a Banach space.

A positive continuous function ϕ on $[0,1)$ is normal, if there exists $0<s<t$ such that (see [7])

$$\frac{\phi(r)}{(1-r)^s} \downarrow 0, \quad \frac{\phi(r)}{(1-r)^t} \uparrow \infty, \quad \text{as } r \longrightarrow 1^-. \tag{1.2}$$

For $0<p<\infty$ and a normal function ϕ, let $H(p,p,\phi)$ denote the space of all analytic functions f on D such that

$$\|f\|_{H(p,p,\phi)} = \int_D \left|f(z)\right|^p \frac{\phi^p\left(|z|\right)}{1-|z|} dA(z) < \infty. \tag{1.3}$$

Here dA denotes the normalized Lebesgue area measure on the unit disk D such that $A(D)=1$. We call $H(p,p,\phi)$ the Bergman-type space. If $1\le p<\infty$, $H(p,p,\phi)$ is a Banach

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space equipped with the norm $\|f\|_{H(p,p,\phi)}$. When $0 < p < 1$, $H(p,p,\phi)$ is a Fréchlet space. In particular, if $\phi(r) = (1-r)^{1/p}$, then $H(p,p,\phi)$ is the Bergman space A^p.

For an analytic function $f(z)$ on the unit disk D with the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the Cesàro operator acting on f is

$$\mathscr{C} f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n. \tag{1.4}$$

The integral form of $\mathscr{C}$ is

$$\mathscr{C}(f)(z) = \frac{1}{z} \int_0^z f(\zeta) \frac{1}{1-\zeta} d\zeta = \frac{1}{z} \int_0^z f(\zeta) \left(\ln \frac{1}{1-\zeta} \right)' d\zeta, \tag{1.5}$$

taking simply as a path the segment joining 0 and z, we have that

$$\mathscr{C}(f)(z) = \int_0^1 f(tz) \left(\ln \frac{1}{1-\zeta} \right)' \bigg|_{\zeta = tz} dt. \tag{1.6}$$

The following operator:

$$z\mathscr{C}(f)(z) = \int_0^z \frac{f(\zeta)}{1-\zeta} d\zeta \tag{1.7}$$

is closely related to the previous operator and on many spaces the boundedness of these two operators is equivalent. It is well known that Cesàro operator acts as a bounded linear operator on various analytic function spaces (see, e.g., [6, 9, 13, 15, 16, 18, 20], and the references therein).

Suppose that $g : \mathbb{D} \to \mathbb{C}^1$ is an analytic map, $f \in H(\mathbb{D})$. A class of integral operator introduced by Pommerenke is defined by (see [11])

$$J_g f(z) = \int_0^z f\, dg = \int_0^1 f(tz) z g'(tz) dt = \int_0^z f(\xi) g'(\xi) d\xi, \quad z \in \mathbb{D}. \tag{1.8}$$

The operator J_g can be viewed as a generalization of the Cesàro operator which was called the Riemann-Stieltjes operator (see [21]).

In [11], Pommerenke showed that J_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. Aleman and Siskakis showed that J_g is bounded (compact) on the Hardy space H^p, $1 \le p < \infty$, if and only if $g \in BMOA$ ($g \in VMOA$), and that J_g is bounded (compact) on the Bergman space A^p if and only if $g \in \mathscr{B}$ ($g \in \mathscr{B}_0$), see [2, 3]. Recently, J_g acting on various function spaces, including the Bloch space, the weighted Bergman space, the BMOA, and VMOA spaces have been studied (see [1–3, 17, 22], and the related references therein).

Another integral operator has recently been defined as the following (see [22]):

$$I_g f(z) = \int_0^z f'(\xi) g(\xi) d\xi. \tag{1.9}$$

In this paper, we study the boundedness and compactness of the operators J_g, I_g between the Bergman-type space and the α-Bloch space.

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other. $a \preceq b$ means that there is a positive constant C such that $a \leq Cb$. Moreover, if both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. $J_g, I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$

In this section, we consider the boundedness and compactness of $J_g, I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$. First, let us state some useful lemmas.

Lemma 2.1. *Assume that* $0 < p < \infty$ *and* ϕ *is normal on* $[0,1)$. *If* $f \in H(p,p,\phi)$, *then*

$$|f(z)| \leq C \frac{\|f\|_{H(p,p,\phi)}}{\phi(|z|)(1-|z|^2)^{1/p}}. \tag{2.1}$$

Proof. Let $\beta(z,w)$ denote the Bergman metric between two points z and w in D. It is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}. \tag{2.2}$$

For $a \in D$ and $r > 0$, the set $D(a,r) = \{z \in D : \beta(a,z) < r\}$ is a Bergman metric disk with center a and radius r. It is well known that (see [25])

$$\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \asymp \frac{1}{(1-|z|^2)^2} \asymp \frac{1}{(1-|a|^2)^2} \asymp \frac{1}{|D(a,r)|}, \tag{2.3}$$

when $z \in D(a,r)$. For $0 < r < 1$ and $z \in D$, by the subharmonicity of $|f(z)|^p$ and the normality of ϕ, we get

$$\begin{aligned}
|f(z)|^p &\leq \frac{C}{(1-|z|^2)^2}\int_{D(z,r)} |f(a)|^p dA(a) \\
&\leq \frac{C}{(1-|z|^2)\phi^p(|z|)}\int_{D(z,r)} (1-|a|)^{-1}\phi^p(|a|)|f(a)|^p dA(a) \\
&\leq \frac{C}{(1-|z|^2)\phi^p(|z|)}\int_{D} (1-|a|)^{-1}\phi^p(|a|)|f(a)|^p dA(a) \leq \frac{C\|f\|^p_{H(p,p,\phi)}}{(1-|z|^2)\phi^p(|z|)},
\end{aligned} \tag{2.4}$$

from which we get the desired result. □

The following lemma can be found in [7, Theorem 2].

Lemma 2.2. *Assume that* $0 < p < \infty$ *and* ϕ *is normal on* $[0,1)$. *Then for* $f \in H(D)$,

$$\|f\|^p_{H(p,p,\phi)} \asymp |f(0)|^p + \int_D |f'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{2.5}$$

From the proof of Lemmas 2.1 and 2.2, we get the following result.

Lemma 2.3. *Assume that* $0 < p < \infty$ *and* ϕ *is normal on* $[0,1)$. *If* $f \in H(p,p,\phi)$ *and* $z \in D$, *then*

$$|f'(z)| \le C\frac{\|f\|_{H(p,p,\phi)}}{\phi(|z|)(1-|z|^2)^{1/p+1}}, \quad (z \in D). \tag{2.6}$$

The following lemma can be found in [14].

Lemma 2.4. *For* $\beta > -1$ *and* $m > 1+\beta$,

$$\int_0^1 (1-\rho r)^{-m}(1-r)^{\beta}dr \le C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1. \tag{2.7}$$

The next lemma can be proved in a standard way (see [5]).

Lemma 2.5. *The operator* J_g *(or* I_g*):* $H(p,p,\phi) \to \mathcal{B}^\alpha$ *is compact if and only if for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* $H(p,p,\phi)$ *which converges to zero uniformly on compact subsets of* D, $J_g f_k$ *(or* $I_g f_k$*)* $\to 0$ *in* $\mathcal{B}^\alpha$ *as* $k \to \infty$.

Theorem 2.6. *Assume that* $0 < p < \infty$, $\alpha > 0$, *and* ϕ *is normal on* $[0,1)$. *Then the operator* $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)}|g'(z)| < \infty. \tag{2.8}$$

Moreover, the following relationship:

$$||J_g||_{H(p,p,\phi)\to\mathcal{B}^\alpha} \asymp \sup_{z\in D}\frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)}|g'(z)| \tag{2.9}$$

holds.

Proof. By (1.8), it is easy to see that $(J_g f)'(z) = f(z)g'(z)$, $(J_g f)(0) = 0$. Let $f(z) \in H(p,p,\phi)$. We have

$$(1-|z|^2)^{\alpha}|(J_g f)'(z)| \le C\|f\|_{H(p,p,\phi)}\frac{(1-|z|^2)^{\alpha}}{\phi(|z|)(1-|z|^2)^{1/p}}|g'(z)|. \tag{2.10}$$

Taking supremum over the unit disk in this inequality, we obtain

$$||J_g f||_{\mathcal{B}^\alpha} \le C\|f\|_{H(p,p,\phi)}\sup_{z\in D}\frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)}|g'(z)|. \tag{2.11}$$

Therefore, (2.8) implies that $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded.

Conversely, suppose J_g is a bounded operator from $H(p,p,\phi)$ to $\mathcal{B}^\alpha$. For $w \in D$, let

$$f_w(z) = \frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-\overline{w}z)^{1/p+t+1}}. \tag{2.12}$$

It is easy to see that

$$f_w(w) = \frac{1}{\phi(|w|)(1-|w|^2)^{1/p}}, \qquad |f_w'(w)| = \left(\frac{1}{p}+t+1\right)\frac{|\overline{w}|}{\phi(|w|)(1-|w|^2)^{1/p+1}}. \tag{2.13}$$

By [12], we get

$$M_p(f_w,r) \le C\frac{(1-|w|^2)^{t+1}}{\phi(|w|)(1-r|w|)^{t+1}}. \tag{2.14}$$

Since ϕ is normal, by Lemma 2.4,

$$\begin{aligned}
||f_w||_{p,p,\phi}^p &= \int_0^1 r(1-r)^{-1}\phi^p(r)M_p^p(f_w,r)\,dr \preceq \int_0^1 (1-r)^{-1}\phi^p(r)\frac{(1-|w|^2)^{p(t+1)}}{\phi^p(|w|)(1-r|w|)^{p(t+1)}}dr \\
&\preceq \int_0^{|w|}(1-r)^{-1}\frac{\phi^p(r)(1-|w|^2)^{p(t+1)}dr}{\phi^p(|w|)(1-r|w|)^{p(t+1)}} + \int_{|w|}^1 (1-r)^{-1}\frac{\phi^p(r)(1-|w|^2)^{p(t+1)}dr}{\phi^p(|w|)(1-r|w|)^{p(t+1)}} \\
&\preceq (1-|w|^2)^p\int_0^{|w|}\frac{(1-r)^{pt-1}dr}{(1-r|w|)^{p(t+1)}} + (1-|w|^2)^{p(t+1)-ps}\int_{|w|}^1\frac{(1-r)^{ps-1}dr}{(1-r|w|)^{p(t+1)}} \\
&\preceq (1-|w|^2)^p\int_0^1\frac{(1-r)^{pt-1}dr}{(1-r|w|)^{p(t+1)}} + (1-|w|^2)^{p(t+1)-ps}\int_0^1\frac{(1-r)^{ps-1}dr}{(1-r|w|)^{p(t+1)}} \le C.
\end{aligned} \tag{2.15}$$

Therefore, $f_w \in H(p,p,\phi)$ (or see [23]). Moreover, there is a positive constant C such that $\|f_w\|_{H(p,p,\phi)} \le C$. Hence

$$(1-|z|^2)^\alpha|f_w(z)g'(z)| \le ||J_g f_w||_{\mathcal{B}^\alpha} \le ||J_g||_{H(p,p,\phi)\to\mathcal{B}^\alpha}||f_w||_{H(p,p,\phi)} \tag{2.16}$$

for every $z,w \in D$.

From this and (2.13), we have

$$\frac{(1-|w|^2)^\alpha}{\phi(|w|)(1-|w|^2)^{1/p}}|g'(w)| \le (1-|w|^2)^\alpha|f_w(w)g'(w)| \le C||J_g||_{H(p,p,\phi)\to\mathcal{B}^\alpha}, \tag{2.17}$$

from which (2.8) follows. Combining (2.11) with (2.17), we get (2.9). □

THEOREM 2.7. *Assume that* $0<p<\infty$, $\alpha>0$, *and* ϕ *is normal on* $[0,1)$. *Then* $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in D}\frac{(1-|z|^2)^{\alpha-1/p-1}}{\phi(|z|)}|g(z)| < \infty. \tag{2.18}$$

Moreover, the following relationship:

$$||I_g||_{H(p,p,\phi)\to\mathcal{B}^\alpha} \asymp \sup_{z\in D}\frac{(1-|z|^2)^{\alpha-1/p-1}}{\phi(|z|)}|g(z)| \tag{2.19}$$

holds.

Proof. Similar to the case of J_g, we have $(I_g f)' = f'(z)g(z)$, $(I_g f)(0) = 0$. Assume (2.18) holds. Let $f(z) \in H(p,p,\phi)$. Then

$$(1-|z|^2)^\alpha |(I_g f)'(z)| \le C\|f\|_{H(p,p,\phi)}\frac{(1-|z|^2)^\alpha}{\phi(|z|)(1-|z|^2)^{1/p+1}}|g(z)|. \tag{2.20}$$

It follows that $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded.

Conversely, if $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded. For $w \in D$, let $f_w(z)$ be defined by (2.12). From (2.3) and (2.13),

$$\begin{aligned}
&\left(\frac{1}{p}+t+1\right)^2\frac{|\overline{w}|^2}{\phi^2(|w|)(1-|w|^2)^{2(1/p+1)}}|g(w)|^2\\
&\quad = |f_w'(w)g(w)|^2 \le \frac{C}{(1-|w|^2)^2}\int_{D(w,r)}|f_w'(z)|^2|g(z)|^2 dA(z)\\
&\quad \le \frac{C}{(1-|w|^2)^2}\int_{D(w,r)}|f_w'(z)|^2|g(z)|^2(1-|z|^2)^{2\alpha}\frac{1}{(1-|z|^2)^{2\alpha}}dA(z)\\
&\quad \le C\int_{D(w,r)}\frac{dA(z)}{(1-|z|^2)^{2\alpha+2}}\sup_{z\in D(w,r)}(1-|z|^2)^{2\alpha}|f_w'(z)|^2|g(z)|^2\\
&\quad \le \frac{C}{(1-|w|^2)^{2\alpha}}||I_g f_w||^2_{\mathcal{B}^\alpha},
\end{aligned} \tag{2.21}$$

that is,

$$\frac{|\overline{w}|(1-|w|^2)^\alpha}{\phi(|w|)(1-|w|^2)^{1/p+1}}|g(w)| \le C||I_g f_w||_{\mathcal{B}^\alpha} \le C||I_g||_{H(p,p,\phi)\to\mathcal{B}^\alpha}. \tag{2.22}$$

Taking supremum in the last inequality over the set $1/2 \le |w| < 1$ and noticing that by the maximum modulus principle there is a positive constant C independent of $g \in H(D)$ such that

$$\sup_{|w|\le 1/2}\frac{(1-|w|^2)^\alpha}{\phi(|w|)(1-|w|^2)^{1/p+1}}|g(w)| \le C\sup_{1/2\le|w|<1}\frac{|\overline{w}|(1-|w|^2)^\alpha}{\phi(|w|)(1-|w|^2)^{1/p+1}}|g(w)|, \tag{2.23}$$

the result follows. From (2.20) and (2.22), we obtain (2.19). $\square$

Theorem 2.8. *Assume that* $0 < p < \infty$, $\alpha > 0$, *and* ϕ *is normal on* $[0,1)$. *Then the operator* $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| = 0. \tag{2.24}$$

Proof. First, we assume that (2.24) holds. In order to prove that J_g is compact, by Lemma 2.5, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H(p,p,\phi)$ that converges to 0 uniformly on compact subsects of D, then $\|J_g f_n\|_{\mathcal{B}^\alpha} \to 0$. Let $\{f_n\}$ be a sequence in $H(p,p,\phi)$ with $\|f_n\|_{H(p,p,\phi)} \le 1$ and $f_n \to 0$ uniformly on compact subsets of D. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |z| < 1$ implies

$$\frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| < \frac{\epsilon}{2}. \tag{2.25}$$

Let $K = \{z \in D : |z| \le \delta\}$. Note that K is a compact subsect of D and ϕ is normal, we have

$$\begin{aligned} ||J_g f_n||_{\mathcal{B}^\alpha} &= \sup_{z\in D} (1-|z|^2)^\alpha |(J_g f_n)'(z)| \\ &\le \sup_{z\in K} (1-|z|^2)^\alpha |g'(z) f_n(z)| + \sup_{z\in D\setminus K} \frac{(1-|z|^2)^\alpha}{\phi(|z|)(1-|z|^2)^{1/p}} |g'(z)| \, \|f\|_{H(p,p,\phi)} \\ &\le CN \sup_{z\in K} (1-|z|^2)^{s+1/p} |f_n(z)| + \frac{C\epsilon}{2}, \end{aligned} \tag{2.26}$$

where

$$N = \sup_{z\in D} \frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)|. \tag{2.27}$$

By the assumption and Theorem 2.6, we obtain $\|J_g f_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. Therefore, $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact.

Conversely, suppose $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact. Let $\{z_n\}$ be a sequence in D such that $|z_n| \to 1$ as $n \to \infty$. Let

$$f_n(z) = \frac{\left(1-|z_n|^2\right)^{t+1}}{\phi(|z_n|)(1-\overline{z_n}z)^{1/p+t+1}}. \tag{2.28}$$

Then $f_n \in H(p,p,\phi)$ and f_n converges to 0 uniformly on compact subsets of D (see [7]). Since J_g is compact, by Lemma 2.5, $\|J_g f_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. In addition,

$$||J_g f_n||_{\mathcal{B}^\alpha} = \sup_{z\in D} (1-|z|^2)^\alpha |(J_g f_n)'(z)| \ge \frac{\left(1-|z_n|^2\right)^\alpha |g'(z_n)|}{\phi(|z_n|)\left(1-|z_n|^2\right)^{1/p}}, \tag{2.29}$$

from which the result follows. ∎

Theorem 2.9. *Assume that* $0 < p < \infty$, $\alpha > 0$, *and* ϕ *is normal on* $[0,1)$. *Then* $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| = 0. \tag{2.30}$$

Proof. Suppose that $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact. Let $\{z_n\}$ be a sequence in D such that $|z_n| \to 1$ as $n \to \infty$. Let $f_n(z)$ be defined by (2.28). Then from the proof of Theorem 2.8 and the compactness of I_g, $\|I_g f_n\|_{\mathcal{B}^\alpha} \to 0$ as $n \to \infty$. In addition,

$$||I_g f_n||_{\mathcal{B}^\alpha} = \sup_{z\in D} (1-|z|^2)^\alpha |(I_g f_n)'(z)| \geq \left(\frac{1}{p}+t+1\right) \frac{\left(1-|z_n|^2\right)^\alpha |g(z_n)\overline{z_n}|}{\phi(|z_n|)\left(1-|z_n|^2\right)^{1+1/p}}, \tag{2.31}$$

from which we get the desired result. □

Assume (2.30) holds, in order to prove that I_g is compact, it suffices to show that if $\{f_n\}$ is a bounded sequence in $H(p,p,\phi)$ that converges to 0 uniformly on compact subsects of D, then $\|I_g f_n\|_{\mathcal{B}^\alpha} \to 0$. Let $\{f_n\}$ be a sequence in $H(p,p,\phi)$ with $\|f_n\|_{H(p,p,\phi)} \leq 1$ and $f_n \to 0$ uniformly on compact subsets of D. By the assumption, for any $\epsilon > 0$, there is a constant δ, $0 < \delta < 1$, such that $\delta < |z| < 1$ implies

$$\frac{(1-|z|^2)^\alpha |g(z)|}{\phi(|z|)(1-|z|^2)^{1+1/p}} < \frac{\epsilon}{2}. \tag{2.32}$$

Similar to the proof of Theorem 2.8, we have

$$||I_g f_n||_{\mathcal{B}^\alpha} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.33}$$

Therefore, $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is compact.

From the introduction, we can easily get the following corollary.

Corollary 2.10. *Let* $0 < p < \infty$, $\alpha \geq 1 + 2/p$. *Then*

(1) $J_g : A^p \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in D} (1-|z|^2)^{\alpha-2/p} |g'(z)| < \infty, \tag{2.34}$$

(2) $I_g : A^p \to \mathcal{B}^\alpha$ *is bounded if and only if*

$$\sup_{z\in D} (1-|z|^2)^{\alpha-2/p-1} |g(z)| < \infty, \tag{2.35}$$

(3) $J_g : A^p \to \mathcal{B}^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha-2/p} |g'(z)| = 0, \tag{2.36}$$

(4) $I_g : A^p \to \mathcal{B}^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} (1-|z|^2)^{\alpha-2/p-1} |g(z)| = 0. \tag{2.37}$$

3. $J_g, I_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$

In this section, we characterize the boundedness and compactness of $J_g, I_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$. For this purpose, we need Lemma 3.1. When $\alpha = 1$, Lemma 3.1 was proved in [8]. For the general case, the proof is similar to the proof of the case $\alpha = 1$. We omit the details.

Lemma 3.1. *A closed set K in $\mathcal{B}_0^\alpha$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)^\alpha |f'(z)| = 0. \tag{3.1}$$

Theorem 3.2. *Assume that* $0 < p < \infty$, $\alpha > 0$, *and* ϕ *is normal on* $[0,1)$. *Then the following statements hold.*

(i) $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is bounded if and only if* $g \in \mathcal{B}_0^\alpha$ *and* $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is bounded.*

(ii) $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^{\alpha-1/p}}{\phi(|z|)} |g'(z)| = 0. \tag{3.2}$$

Proof. (i) It is clear to see that $g \in \mathcal{B}_0^\alpha$ and $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded if $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is bounded.

Conversely, suppose that $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded and $g \in \mathcal{B}_0^\alpha$. For any polynomial $p(z)$, since $g \in \mathcal{B}_0^\alpha$ and

$$(1-|z|^2)^\alpha (J_g p)'(z) = (1-|z|^2)^\alpha p(z) g'(z), \tag{3.3}$$

we know that $J_g p \in \mathcal{B}_0^\alpha$. For any $f \in H(p,p,\phi)$, there exists a sequence of polynomials $\{p_n\}$ such that $\|f - p_n\|_{H(p,p,\phi)} \to 0$ as $n \to \infty$. Since $\mathcal{B}_0^\alpha$ is closed, we get

$$J_g f = \lim_{n\to\infty} J_g p_n \in \mathcal{B}_0^\alpha. \tag{3.4}$$

In addition, $J_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ is bounded. Therefore, $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is bounded.

(ii) If $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is compact, then by Theorem 2.8, we get (3.2).

Conversely, assume that (3.2) holds. It follows from Lemma 3.1 that $J_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ is compact if and only if

$$\lim_{|z|\to 1} \sup_{\|f\|_{H(p,p,\phi)}\le 1} (1-|z|^2)^\alpha |(J_g f)'(z)| = 0. \tag{3.5}$$

In fact,

$$(1-|z|^2)^\alpha |(J_g f)'(z)| = \frac{(1-|z|^2)^\alpha |g'(z)|}{\phi(|z|)(1-|z|^2)^{1/p}} \phi(|z|)(1-|z|^2)^{1/p} |f(z)|. \tag{3.6}$$

By Lemma 2.1, the result follows. □

Similarly, we have the following results.

THEOREM 3.3. *Assume that* $0 < p < \infty$, $\alpha > 0$, *and* ϕ *is normal on* $[0,1)$. *Then the following statements hold.*

(i) $I_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is bounded if and only if* $I_g : H(p,p,\phi) \to \mathcal{B}^\alpha$ *is bounded and*

$$\lim_{|z|\to 1} |g(z)|\left(1-|z|^2\right)^\alpha = 0. \tag{3.7}$$

(ii) $I_g : H(p,p,\phi) \to \mathcal{B}_0^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)^{\alpha-1/p-1}}{\phi(|z|)} |g(z)| = 0. \tag{3.8}$$

COROLLARY 3.4. *Let* $0 < p < \infty$, $\alpha > 0$. *Then the following statements hold.*

(i) $J_g : A^p \to \mathcal{B}_0^\alpha$ *is bounded if and only if* $J_g : A^p \to \mathcal{B}^\alpha$ *is bounded and* $g \in \mathcal{B}_0^\alpha$.

(ii) $J_g : A^p \to \mathcal{B}_0^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^{\alpha-2/p} |g'(z)| = 0. \tag{3.9}$$

COROLLARY 3.5. *Let* $0 < p < \infty$, $\alpha > 0$. *Then the following statements hold.*

(i) $I_g : A^p \to \mathcal{B}_0^\alpha$ *is bounded if and only if* $I_g : A^p \to \mathcal{B}^\alpha$ *is bounded and*

$$\lim_{|z|\to 1} |g(z)|\left(1-|z|^2\right)^\alpha = 0. \tag{3.10}$$

(ii) $I_g : A^p \to \mathcal{B}_0^\alpha$ *is compact if and only if*

$$\lim_{|z|\to 1} \left(1-|z|^2\right)^{\alpha-2/p-1} |g(z)| = 0. \tag{3.11}$$

Remark 3.6. In Corollary 3.5, if $\alpha \le 2/p+1$, then by the maximum modulus principle, it is easy to see that $g \equiv 0$.

4. $J_g, I_g : \mathcal{B}^\alpha \to H(p,p,\phi)$

The following lemma is well known(e.g., see [19]).

LEMMA 4.1. *Let* $f \in \mathcal{B}^\alpha(D)$, $0 < \alpha < \infty$. *Then*

$$|f(z)| \le \begin{cases} |f(0)| + \|f\|_{\mathcal{B}^\alpha} \dfrac{1-(1-|z|)^{1-\alpha}}{1-\alpha}, & \alpha \ne 1, \\ |f(0)| + \|f\|_{\mathcal{B}^\alpha} \ln \dfrac{2}{1-|z|}, & \alpha = 1. \end{cases} \tag{4.1}$$

The following lemma can be found in [10].

LEMMA 4.2. *Let* μ *be a positive measure on* D *and* $0 < p < \infty$. *Let either* $0 < \alpha < \infty$ *and* $n \in \mathbb{N}$, *or* $1 < \alpha < \infty$ *and* $n = 0$. *Then*

$$\int_D \frac{d\mu(z)}{\left(1-|z|^2\right)^{\alpha p}} < \infty \tag{4.2}$$

if and only if there is a positive constant C such that

$$\left(\int_D |f^{(n)}(z)|^p (1-|z|^2)^{p(n-1)} d\mu(z)\right)^{1/p} \le C\|f\|_{\mathcal{B}^\alpha} \tag{4.3}$$

for all analytic functions f in D, in particular, for all $f \in \mathcal{B}^\alpha$.

Let $0 < p < \infty$, let μ be a positive Borel measure on D. Define

$$D_p(\mu) = \left\{ f \in H(D),\ \|f\|^p_{D_p(\mu)} = \int_D |f'(z)|^p d\mu(z) < \infty \right\}. \tag{4.4}$$

Lemma 4.3. *Let μ be a positive measure on D and $0 < p, \alpha < \infty$. Then the following statements are equivalent.*

(1) $i : \mathcal{B}^\alpha \mapsto D_p(\mu)$ *is bounded.*

(2) $i : \mathcal{B}^\alpha \mapsto D_p(\mu)$ *is compact.*

(3) $i : \mathcal{B}^\alpha_0 \mapsto D_p(\mu)$ *is bounded.*

(4) $i : \mathcal{B}^\alpha_0 \mapsto D_p(\mu)$ *is compact.*

(5)

$$\int_D \frac{d\mu(z)}{(1-|z|^2)^{\alpha p}} < \infty. \tag{4.5}$$

Remark 4.4. The above lemma was obtained by Zhao when $0 < \alpha \le 1$ (see [24]). In fact, his proof implies that the result also holds for $\alpha > 1$. Partial results can also be found in [4] when $\alpha = 1$.

Theorem 4.5. *Assume that $0 < \alpha < \infty$, $0 < p < \infty$, and ϕ is normal on $[0,1)$. Then the following statements are equivalent.*

(1) $I_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is bounded.*

(2) $I_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is compact.*

(3) $I_g : \mathcal{B}^\alpha_0 \to H(p,p,\phi)$ *is bounded.*

(4) $I_g : \mathcal{B}^\alpha_0 \to H(p,p,\phi)$ *is compact.*

(5)

$$\int_D |g(z)|^p (1-|z|^2)^{p-p\alpha} \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{4.6}$$

Proof. Since

$$\begin{aligned} ||I_g f||^p_{H(p,p,\phi)} &\asymp \int_D |(I_g f)'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) \\ &= \int_D |g(z)|^p |f'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) = \int_D |f'(z)|^p d\mu(z), \end{aligned} \tag{4.7}$$

where

$$d\mu(z) = |g(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{4.8}$$

By Lemma 4.3, we know that $I_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to H(p,p,\phi)$ is bounded (or compact) if and only if

$$\infty > \int_D \frac{d\mu}{(1-|z|^2)^{\alpha p}} = \int_D |g(z)|^p (1-|z|^2)^{p-p\alpha} \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{4.9}$$

□

THEOREM 4.6. *Assume that* $\alpha > 1$, $0 < p < \infty$, *and* ϕ *is normal on* $[0,1)$. *Then the following statements are equivalent.*

(i) $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is bounded.*

(ii) $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is compact.*

(iii)

$$\int_D |g'(z)|^p (1-|z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1-|z|} dA(z) < \infty. \tag{4.10}$$

Proof. Since

$$\begin{aligned} \|J_g f\|^p_{H(p,p,\phi)} &\asymp \int_D |(J_g f)'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) \\ &= \int_D |g'(z)|^p |f(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) \\ &= \int_D |f(z)|^p (1-|z|^2)^{-p} d\mu(z), \end{aligned} \tag{4.11}$$

where

$$d\mu = |g'(z)|^p (1-|z|^2)^{2p} \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{4.12}$$

Similar to the proof of Theorem 4.5, we get (i)⇔(iii) by Lemma 4.2.

(ii)⇒(i) is clear. Next we prove that (iii)⇒(ii). Assume (iii) holds, we obtain that J_g is bounded and so $g \in H(p,p,\phi)$. In addition to this, we also find that for any $\varepsilon > 0$, there is an $r \in (0,1)$ such that

$$\int_{|z|>r} |g'(z)|^p (1-|z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1-|z|} dA(z) < \varepsilon. \tag{4.13}$$

Let $\{f_k\}$ be any sequence in the unit ball of $\mathcal{B}^\alpha$ and converges to 0 uniformly on compact subsets of D. For the above ε, there exists a $k_0 > 0$ such that $\sup_{|z|\le r} |f_k(z)| < \varepsilon$ as $k > k_0$. Hence we have

$$\begin{aligned} \|J_g f_k\|^p_{H(p,p,\phi)} &\asymp \left(\int_{|z|\le r} + \int_{|z|>r}\right) |(J_g f_k)'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) \\ &\le C\varepsilon \|g\|^p_{H(p,p,\phi)} + \|f_k\|^p_{\mathcal{B}^\alpha} \int_{|z|>r} |g'(z)|^p (1-|z|^2)^{(2-\alpha)p} \frac{\phi^p(|z|)}{1-|z|} dA(z) \\ &\le C\varepsilon \|g\|^p_{H(p,p,\phi)} + \varepsilon \|f_k\|^p_{\mathcal{B}^\alpha}. \end{aligned} \tag{4.14}$$

In other words, we obtain $\lim_{k\to\infty}\|J_g f\|_{H(p,p,\phi)} = 0$ and so $J_p : \mathcal{B}^\alpha \to H(p,p,\phi)$ is compact. □

Theorem 4.7. *Assume that* $0<\alpha<1$, $0<p<\infty$, *and* ϕ *is normal on* $[0,1)$. *Then the following statements are equivalent.*

(i) $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is bounded.*

(ii) $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ *is compact.*

(iii)

$$\int_D |g'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z) < \infty. \tag{4.15}$$

Proof. (ii)⇒(i) is clear.

(i)⇒(iii). Assume that $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ is bounded. Hence

$$\|J_g f\|^p_{H(p,p,\phi)} \asymp \int_D |g'(z)|^p |f(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z). \tag{4.16}$$

Taking $f=1$, we get (iii).

Conversely, we assume that (iii) holds. Let $f\in\mathcal{B}^\alpha$, then $|f(z)|\le C\|f\|_{\mathcal{B}^\alpha}$. Therefore, by (4.16), we see that $J_g : \mathcal{B}^\alpha \to H(p,p,\phi)$ is bounded. Similar to the proof of Theorem 4.6, we obtain that (iii)⇒(ii). □

Theorem 4.8. *Assume that* $0<p<\infty$ *and* ϕ *is normal on* $[0,1)$. *Then the following statements hold.*

(i) *If the operator* $J_g : \mathcal{B} \to H(p,p,\phi)$ *is bounded, then*

$$\sup_{z\in D} \phi(|z|)(1-|z|^2)^{1+1/p} |g'(z)| \ln\frac{2}{1-|z|^2} < \infty. \tag{4.17}$$

(ii) *If*

$$\sup_{z\in D} \phi(|z|)(1-|z|^2)^{1-1/p} |g'(z)| \ln\frac{2}{1-|z|^2} < \infty, \tag{4.18}$$

then $J_g : \mathcal{B} \to H(p,p,\phi)$ *is bounded.*

Proof. Assume that (4.18) holds. For any $f\in\mathcal{B}$, by Lemmas 2.2, 4.1 and the fact $(J_g f)'(z) = f(z)g'(z)$, $(J_g f)(0)=0$, we have

$$\begin{aligned}
\|J_g f\|^p_{H(p,p,\phi)} &\asymp \int_D |(J_g f)'(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z)\\
&= \int_D |g'(z)|^p |f(z)|^p (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z)\\
&\le C\|f\|^p_{\mathcal{B}} \int_D |g'(z)|^p \ln^p\frac{2}{1-|z|^2} (1-|z|^2)^p \frac{\phi^p(|z|)}{1-|z|} dA(z)
\end{aligned}$$

$$\begin{aligned}&\le C\|f\|_{\mathcal{B}}^p \sup_{z\in D}|g'(z)|^p\,|\ln^p\frac{2}{1-|z|^2}(1-|z|^2)^p\frac{\phi^p(|z|)}{1-|z|}\int_D dA(z)\\&\le C\|f\|_{\mathcal{B}}^p \sup_{z\in D}|g'(z)|^p\,|\ln^p\frac{2}{1-|z|^2}(1-|z|^2)^{p-1}\phi^p(|z|).\end{aligned}\tag{4.19}$$

Therefore, $J_g:\mathcal{B}\to H(p,p,\phi)$ is bounded.

Assume that $J_g:\mathcal{B}\to H(p,p,\phi)$ is bounded. For $w\in D$, put $f_w(z)=\ln 2/(1-\overline{w}z)$. Since

$$(1-|z|^2)\,|f_w'(z)|\le(1-|z|^2)\frac{|w|}{|1-\overline{w}z|}\le\frac{1-|z|^2}{|1-\overline{w}z|}\le 2,\tag{4.20}$$

we have $\|f_w\|_{\mathcal{B}}\le\ln 2+2$. By the subharmonicity, we have

$$\begin{aligned}C||J_g||^p&\ge C||J_g||^p||f_w||_{\mathcal{B}}^p\ge C||J_gf_w||_{H(p,p,\phi)}^p\\&\ge C\int_D|(J_gf_w)'(z)|^p(1-|z|^2)^p\frac{\phi^p(|z|)}{1-|z|}dA(z)\\&\ge C\int_{D(w,r)}|g'(z)|^p|f_w(z)|^p(1-|z|^2)^p\frac{\phi^p(|z|)}{1-|z|}dA(z)\\&\ge C|g'(w)|^p|f_w(w)|^p(1-|w|^2)^{p+1}\phi^p(|w|)\\&\ge C(1-|w|)^{p+1}\phi^p(|w|)\,|g'(w)|^p\left(\ln\frac{2}{1-|w|^2}\right)^p.\end{aligned}\tag{4.21}$$

Therefore, we get the desired result. □

Remark 4.9. We use another method to prove the necessary condition of the boundedness of $J_g:\mathcal{B}\to H(p,p,\phi)$.

Since $J_gf\in H(p,p,\phi)$, by Lemma 2.3, we have

$$|(J_gf)'(z)|\le C\frac{||J_gf||_{H(p,p,\phi)}}{(1-|z|^2)^{1/p+1}\phi(|z|)}\le C\frac{||J_g||_{\mathcal{B}\to H(p,p,\phi)}\|f\|_{\mathcal{B}}}{(1-|z|^2)^{1/p+1}\phi(|z|)}.\tag{4.22}$$

For any $w\in D$, let $f_w(z)=\ln 2/(1-z\overline{w})$. Then we get

$$|g'(z)|\,|f_w(z)|\phi(|z|)(1-|z|^2)^{1/p+1}\le C||J_g||_{\mathcal{B}\to H(p,p,\phi)}||f_w||_{\mathcal{B}}.\tag{4.23}$$

Let $z=w$, we have

$$|g'(w)|\ln\frac{2}{1-|w|^2}\phi(|w|)(1-|w|^2)^{1/p+1}\le C||J_g||_{\mathcal{B}\to H(p,p,\phi)}||f_w||_{\mathcal{B}}.\tag{4.24}$$

Therefore, we get the desired result.

Theorem 4.10. *Assume that* $0 < p < \infty$ *and* ϕ *is normal on* $[0,1)$. *Then the following statements hold.*

(i) *If the operator* $J_g : \mathcal{B} \to H(p,p,\phi)$ *is compact, then*

$$\lim_{|z|\to 1} \phi(|z|)(1-|z|^2)^{1+1/p} |g'(z)| \ln \frac{2}{1-|z|^2} = 0. \tag{4.25}$$

(ii) *If*

$$\lim_{|z|\to 1} \phi(|z|)(1-|z|^2)^{1-1/p} |g'(z)| \ln \frac{2}{1-|z|^2} = 0, \tag{4.26}$$

then $J_g : \mathcal{B} \to H(p,p,\phi)$ *is compact.*

Proof. Suppose the operator $J_g : \mathcal{B} \to H(p,p,\phi)$ is compact. Let z_n be a sequence in D such that $|z_n| \to 1$ as $n \to \infty$. Take

$$f_n(z) = \left(\ln \frac{2}{1-|z_n|^2}\right)^{-1} \left(\ln \frac{2}{1-\overline{z_n}z}\right)^2. \tag{4.27}$$

Then

$$f_n'(z) = 2\left(\ln \frac{2}{1-|z_n|^2}\right)^{-1} \left(\ln \frac{2}{1-z\overline{z_n}}\right) \frac{\overline{z_n}}{1-z\overline{z_n}}. \tag{4.28}$$

Thus for any $z \in D$,

$$(1-|z|^2)|f_n'(z)| \le 2(1-|z|^2) \left| \frac{\ln 2/(1-z\overline{z_n})}{\ln 2/\left(1-|z_n|^2\right)} \right| \frac{1}{1-|z|} \le 4 \frac{C + \ln 2/(1-|z_n|)}{\ln 2/\left(1-|z_n|^2\right)} \le C. \tag{4.29}$$

On the other hand,

$$|f_n(0)| \le \left(\ln \frac{2}{1-|z_n|^2}\right)^{-1} (\ln 2)^2 \le \ln 2. \tag{4.30}$$

Thus $\|f_n\|_{\mathcal{B}} \le M$, where M is a constant independent of n. Since for $|z| = r < 1$, we have

$$|f_n(z)| = \frac{|\ln 2/(1-z\overline{z_n})|^2}{\ln 2/\left(1-|z_n|^2\right)} \le \frac{(\ln 2/(1-r) + C)^2}{\ln 2/\left(1-|z_n|^2\right)} \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{4.31}$$

that is, $f_n \to 0$ uniformly on compact subsets of D as $n \to \infty$. By the proof of Theorem 4.8, we obtain

$$\phi(|z_n|)(1-|z_n|^2)^{1+1/p} |g'(z_n)| \ln \frac{2}{1-|z_n|^2} \le \|J_g f_n\| \longrightarrow 0 \tag{4.32}$$

as $n \to \infty$. Therefore, we get (4.25).

From (4.26), for any $\varepsilon > 0$, there exists an r, $0 < r < 1$, such that

$$\phi(|z|)\left(1-|z|^2\right)^{1-1/p}|g'(z)|\ln\frac{2}{1-|z|^2} < \varepsilon, \tag{4.33}$$

when $|z| > r$. Also, from (4.26), we see that there exist $C > 0$ such that

$$\sup_{|z|\le r}\phi(|z|)\left(1-|z|^2\right)^{1-1/p}|g'(z)| < C. \tag{4.34}$$

Let $\{f_k\}$ be any sequence in the unit ball of $\mathcal{B}$ and converges to 0 uniformly on compact subsets of D. For the above ε, there exists a $k_0 > 0$ such that $\sup_{|z|\le r}|f_k(z)| < \varepsilon$ as $k > k_0$. Hence we have

$$\begin{aligned}
\|J_g f_k\|^p_{H(p,p,\phi)} &\asymp \left(\int_{|z|\le r}+\int_{|z|>r}\right)|(J_g f_k)'(z)|^p(1-|z|^2)^p\frac{\phi^p(|z|)}{1-|z|}dA(z)\\
&\le C\varepsilon+\|f_k\|^p_{\mathcal{B}}\int_{|z|>r}|g'(z)|^p(1-|z|^2)^p\left(\ln\frac{2}{1-|z|^2}\right)^p\frac{\phi^p(|z|)}{1-|z|}dA(z)\\
&\le C\varepsilon+\varepsilon\|f_k\|^p_{\mathcal{B}},
\end{aligned} \tag{4.35}$$

as $k > k_0$, from which we get the desired result. □

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