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*Research Article*

# Viscosity Approximation Methods for Nonexpansive Nonself-Mappings in Hilbert Spaces

Rabian Wangkeeree

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Viscosity approximation methods for nonexpansive nonself-mappings are studied. Let $C$ be a nonempty closed convex subset of Hilbert space $H$, $P$ a metric projection of $H$ onto $C$ and let $T$ be a nonexpansive nonself-mapping from $C$ into $H$. For a contraction $f$ on $C$ and $\{t_n\} \subseteq (0,1)$, let $x_n$ be the unique fixed point of the contraction $x \mapsto t_n f(x) + (1-t_n)(1/n)\sum_{j=1}^{n}(PT)^j x$. Consider also the iterative processes $\{y_n\}$ and $\{z_n\}$ generated by $y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)(1/(n+1))\sum_{j=0}^{n}(PT)^j y_n, n \geq 0$, and $z_{n+1} = (1/(n+1))\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), n \geq 0$, where $y_0, z_0 \in C, \{\alpha_n\}$ is a real sequence in an interval $[0,1]$. Strong convergence of the sequences $\{x_n\}, \{y_n\}$, and $\{z_n\}$ to a fixed point of $T$ which solves some variational inequalities is obtained under certain appropriate conditions on the real sequences $\{\alpha_n\}$ and $\{t_n\}$.



## 1. Introduction

Throughout this paper, we denote the set of all nonnegative integers by $\mathbb{N}$. Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $C$ be a closed convex subset of $H$, and $T$ a nonself-mapping from $C$ into $H$. We denote the set of all fixed points of $T$ by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. $T$ is said to be *nonexpansive mapping* if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in C$. From condition on $C$, there is a mapping $P$ from $H$ onto $C$ which satisfies

$$\left\|x - P_C x\right\| = \min_{y \in C}\|x - y\| \tag{1.2}$$

for all $x \in C$. This mapping $P$ is said to be *the metric projection* from $H$ onto $C$. We know that the metric projection is nonexpansive. Recall that a self-mapping $f : C \to C$ is a *contraction* on $C$ if there exists a constant $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\| \quad \forall x, y \in C. \tag{1.3}$$

We use $\Pi_C$ to denote the collection of all contractions on $C$. That is,

$$\Pi_C = \{f : f : C \longrightarrow C \text{ a contraction}\}. \tag{1.4}$$

Note that each $f \in \Pi_C$ has a unique fixed point in $C$.

Given a real sequence $\{t_n\} \subseteq (0,1)$ and a contraction $f \in \Pi_C$, define another mapping $T_n : C \to C$ by

$$T_n x = t_n f(x) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x \quad \forall n \geq 1. \tag{1.5}$$

It is not hard to see that $T_n$ is a contraction on $C$. Indeed, for $x, y \in C$, we have

$$\begin{aligned}
\|T_n x - T_n y\| &= \left\| t_n(f(x) - f(y)) + (1 - t_n)\frac{1}{n}\left(\sum_{j=1}^{n}(PT)^j x - \sum_{j=1}^{n}(PT)^j y\right)\right\| \\
&\leq t_n\|f(x) - f(y)\| + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}\|(PT)^j x - (PT)^j y\| \\
&\leq t_n\alpha\|x - y\| + (1 - t_n)\|x - y\| \\
&= (1 - t_n(1 - \alpha))\|x - y\|.
\end{aligned} \tag{1.6}$$

For each $n$, let $x_n \in C$ be the unique fixed point of $T_n$. Thus $x_n$ is the unique solution of fixed point equation

$$x_n = t_n f(x_n) + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \quad \forall n \geq 1. \tag{1.7}$$

One of the purposes of this paper is to study the convergence of $\{x_n\}$ when $t_n \to 0$ as $n \to \infty$ in Hilbert spaces. Fix $u \in C$ and define a contraction $S_n$ on $C$ by

$$S_n x = t_n u + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x \quad \forall n \geq 1. \tag{1.8}$$

Let $s_n \in C$ be the unique fixed point of $S_n$. Thus

$$s_n = t_n u + (1 - t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j s_n \quad \forall n \geq 1. \tag{1.9}$$

Shimizu and Takahashi [1] studied the strong convergence of the sequence $\{s_n\}$ defined by (1.9) for asymptotically nonexpansive mappings in Hilbert spaces.

We also study the convergence of the following iteration schemes: for $y_0, z_0 \in C$, compute the sequences $\{y_n\}$ and $\{z_n\}$ by the iterative schemes

$$y_{n+1} = \alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j yn, \quad n \geq 0, \tag{1.10}$$

$$z_{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n f(z_n) + (1-\alpha_n)(TP)^j z_n), \quad n \geq 0, \tag{1.11}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$, $f : C \to C$ is a contraction mapping on $C$, and $P$ is the metric projection of $H$ onto $C$. The first special case of (1.10) was considered by Shimizu and Takahashi [2] who introduced the following iterative process:

$$y_{n+1} = \alpha_n y + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j y_n, \quad n \geq 0, \tag{1.12}$$

where $y, y_0$ are arbitrary (but fixed) and $\{\alpha_n\} \subseteq [0,1]$ and then they proved the following theorem.

Theorem 1.1 [2]. *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $T$ be a nonexpansive self-mapping of $C$ such that $F(T)$ is nonempty, and let $P_{F(T)}$ be the metric projection from $C$ onto $F(T)$. Let $\{\alpha_n\}$ be a real sequence which satisfies $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $y$ and $y_0$ be element of $C$ and let $\{y_n\}$ be the sequence defined by (1.12). Then $\{y_n\}$ converges strongly to $P_{F(T)}y$.*

The second special case of (1.10) and (1.11) was considered by Matsushita and Kuroiwa [3] who introduced the following iterative process:

$$\begin{aligned} y_{n+1} &= \alpha_n y + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n, \quad n \geq 0, \\ z_{n+1} &= \frac{1}{n+1}\sum_{j=0}^{n} P(\alpha_n z + (1-\alpha_n)(TP)^j z_n), \quad n \geq 0, \end{aligned} \tag{1.13}$$

where $y$, $z$, $y_0$, $z_0$ are arbitrary (but fixed) in $C$ and $\{\alpha_n\} \subseteq [0,1]$. More precisely, they proved the following theorem.

Theorem 1.2 [3]. *Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, $P$ the metric projection of $H$ onto $C$, and let $T$ be a nonexpansive nonself-mapping from $C$ into $H$ such that $F(T)$ is nonempty, and $\{\alpha_n\}$ a sequence of real numbers such that $0 \leq \alpha_n \leq 1$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{y_n\}$ and $\{z_n\}$ are defined by (1.13), respectively. Then $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(T)}y$ and $P_{F(T)}z$ in $F(T)$, respectively, where $P_{F(T)}$ is the metric projection from $C$ onto $F(T)$.*

The purpose of this paper is twofold. First, we study the convergence of the sequence $\{x_n\}$ defined by (1.7) in Hilbert spaces. Second, we prove the strong convergence of the iteration schemes $\{y_n\}$ and $\{z_n\}$ defined by (1.10) and (1.11), respectively, in Hilbert spaces. Our results extend and improve the corresponding ones announced by Shimizu and Takahashi [2], Matsushita and Kuroiwa [3], and others.

## 2. Preliminaries

For the sake of convenience, we restate the following concepts and results.

LEMMA 2.1. *Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$, and $P_C : H \to C$ the metric (nearest point) projection. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C. \tag{2.1}$$

*Definition 1.* A mapping $T : C \to H$ is said to satisfy *nowhere normal outward* ($NNO$) *condition* if and only if for each $x \in C$, $Tx \in S_x^C$, where $S_x = \{y \in H : y \neq x,\ Py = x\}$ and $P$ is the metric projection from $H$ onto $C$.

The following results were proved by Matsushita and Kuroiwa [4].

LEMMA 2.2 (see [4, Proposition 2, page 208]). *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $P$ the metric projection of $H$ onto $C$, and $T : C \to H$ a nonexpansive nonself-mapping. If $F(T)$ is nonempty, then $T$ satisfies $NNO$ condition.*

LEMMA 2.3 (see [4, Proposition 1, page 208]). *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $P$ the metric projection of $H$ onto $C$, and $T : C \to H$ a nonself-mapping. Suppose that $T$ satisfies NNO condition. Then $F(PT) = F(T)$.*

LEMMA 2.4 (see [4]). *Let $H$ be a Hilbert space, $C$ a closed convex subset of $H$, and $T : C \to C$ a nonexpansive self-mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ such that $\{x_{n+1} - (1/(n+1))\sum_{i=1}^{n+1} T^i x_n\}$ converges strongly to 0 as $n \to \infty$ and let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $x$. Then $x$ is a fixed point of $T$.*

Finally, the following two lemmas are useful for the proof of our main theorems.

LEMMA 2.5 (see [5]). *Let $\{\alpha_n\}$ be a sequence in $[0,1]$ that satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{a_n\}$ be a sequence of nonnegative real numbers such that for all $\epsilon > 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \epsilon. \tag{2.2}$$

*Then* $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.6 (see [5]). *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, and $f : C \to C$ a contraction with coefficient $\alpha < 1$. Then*

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq (1 - \alpha)\|x - y\|^2, \quad x, y \in C. \tag{2.3}$$

*Remark 2.7.* As in Lemma 2.6, if $f$ is a nonexpansive mapping, then

$$\langle x - y, (I - f)x - (I - f)y \rangle \geq 0 \quad \forall x, y \in C. \tag{2.4}$$

## 3. Main results

THEOREM 3.1. *Let $H$ be a Hilbert space, $C$ a nonempty closed convex subset of $H$, $P$ the metric projection of $H$ onto $C$, and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ be sequence in $(0,1)$ which satisfies $\lim_{n\to\infty} t_n = 0$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{x_n\}$ defined by (1.7) converges strongly to $z$, where $z$ is the unique solution in $F(T)$ to the variational inequality*

$$\langle (I-f)z,\, x-z\rangle \geq 0, \quad x \in F(T), \tag{3.1}$$

*or equivalently $z = P_{F(T)} f(z)$, where $P_{F(T)}$ is a metric projection mapping from $H$ onto $F(T)$.*

*Proof.* Since $F(T)$ is nonempty, it follows that $T$ satisfies $NNO$ condition by Lemma 2.2. We first show that $\{x_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned} ||x_n - q|| &= \left|\left| t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - q\right|\right| \\ &\leq \left|\left| t_n (f(x_n) - q) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}\left((PT)^j x_n - (PT)^j q\right)\right|\right| \\ &\leq t_n ||f(x_n) - q|| + (1-t_n)||x_n - q|| \quad \forall n \geq 1. \end{aligned} \tag{3.2}$$

So we get

$$\begin{aligned} ||x_n - q|| \leq ||f(x_n) - q|| &\leq ||f(x_n) - f(q)|| + ||f(q) - q|| \\ &\leq \alpha ||x_n - q|| + ||f(q) - q|| \quad \forall n \geq 1. \end{aligned} \tag{3.3}$$

Hence

$$||x_n - q|| \leq \frac{1}{1-\alpha}||f(q) - q|| \quad \forall n \geq 1. \tag{3.4}$$

This shows that $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{(1/n)\sum_{j=1}^{n}(PT)^j x_n\}$. Further, we note that

$$\begin{aligned} \left|\left| x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right|\right| &= \left|\left| t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right|\right| \\ &= t_n \left|\left| f(x_n) - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right|\right| \\ &\leq t_n \left( ||f(x_n)|| + \left|\left|\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right|\right|\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.5}$$

Thus $\{x_n - (1/n)\sum_{j=1}^{n}(PT)^j x_n\}$ converges strongly to 0. Since $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $z \in C$. By Lemmas 2.3 and 2.4, we have $z \in F(T)$. For each $n \geq 1$, since

$$x_n - z = t_n\big(f(x_n) - z\big) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}\big((PT)^j x_n - z\big), \tag{3.6}$$

we get

$$\begin{aligned} \|x_n - z\|^2 &= (1-t_n)\left\langle \frac{1}{n}\sum_{j=1}^{n}\big((PT)^j x_n - z\big),\, x_n - z \right\rangle + t_n\langle f(x_n) - z,\, x_n - z\rangle \\ &\leq (1-t_n)\|x_n - z\|^2 + t_n\langle f(x_n) - z,\, x_n - z\rangle. \end{aligned} \tag{3.7}$$

Hence

$$\begin{aligned} \|x_n - z\|^2 &\leq \langle f(x_n) - z,\, x_n - z\rangle \\ &= \langle f(x_n) - f(z),\, x_n - z\rangle + \langle f(z) - z,\, x_n - z\rangle \\ &\leq \alpha\|x_n - z\|^2 + \langle f(z) - z,\, x_n - z\rangle. \end{aligned} \tag{3.8}$$

This implies that

$$\|x_n - z\|^2 \leq \frac{1}{1-\alpha}\langle x_n - z,\ f(z) - z\rangle. \tag{3.9}$$

In particular, we have

$$\|x_{n_j} - z\|^2 \leq \frac{1}{1-\alpha}\langle x_{n_j} - z,\ f(z) - z\rangle. \tag{3.10}$$

Since $x_{n_j} \rightharpoonup z$, it follows that

$$x_{n_j} \longrightarrow z \quad \text{as } j \longrightarrow \infty. \tag{3.11}$$

Next we show that $z \in C$ solves the variational inequality (3.1). Indeed, we note that

$$x_n = t_n f(x_n) + (1-t_n)\frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n \quad \forall n \geq 1, \tag{3.12}$$

we have

$$(I - f)x_n = -\frac{1-t_n}{t_n}\left(x_n - \frac{1}{n}\sum_{j=1}^{n}(PT)^j x_n\right). \tag{3.13}$$

Thus for any $q \in F(T)$, we infer by Remark 2.7 that

$$\begin{aligned}\langle (I-f)x_n, x_n - q\rangle &= -\frac{1-t_n}{t_n}\left\langle \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)x_n, x_n - q\right\rangle \\ &= -\frac{1-t_n}{t_n}\left\langle \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)x_n - \left(I - \frac{1}{n}\sum_{j=1}^{n}(PT)^j\right)z, x_n - q\right\rangle \\ &\leq 0 \quad \forall n \geq 1.\end{aligned} \tag{3.14}$$

In particular

$$\langle (I-f)x_{n_j}, x_{n_j} - q\rangle \leq 0 \quad \forall j \geq 1. \tag{3.15}$$

Taking $j \to \infty$, we obtain

$$\langle (I-f)z, z-q\rangle \leq 0 \quad \forall q \in F(T), \tag{3.16}$$

or equivalent to $z = P_{F(T)}f(z)$ as required. Finally, we will show that the whole sequence $\{x_n\}$ converges strongly to $z$. Let another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ be such that $x_{n_k} \to z' \in C$ as $k \to \infty$. Then $z' \in F(T)$, it follows from the inequality (3.16) that

$$\langle (I-f)z, z-z'\rangle \leq 0. \tag{3.17}$$

Interchange $z$ and $z'$ to obtain

$$\langle (I-f)z', z'-z\rangle \leq 0. \tag{3.18}$$

Adding (3.17) and (3.18) and by Lemma 2.6, we get

$$(1-\alpha)\|z-z'\|^2 \leq \langle z-z', (I-f)z-(I-f)z'\rangle \leq 0. \tag{3.19}$$

This implies that $z = z'$. Hence $\{x_n\}$ converges strongly to $z$. This completes the proof. □

THEOREM 3.2. *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $P$ the metric projection of $H$ onto $C$, and $T: C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then for a contraction mapping $f: C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{y_n\}$ defined by (1.10) converges strongly to $z$, where $z$ is the unique solution in $F(T)$ of the variational inequality (3.1).*

*Proof.* Since $F(T)$ is nonempty, it follows that $T$ satisfies $NNO$ condition by Lemma 2.2. We first show that $\{y_n\}$ is bounded. Let $q \in F(T)$. We note that

$$\begin{aligned}
||y_{n+1}-q|| &= \left\|\alpha_n f(y_n)+(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - q\right\| \\
&\le \alpha_n||f(y_n)-q|| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}||(PT)^j y_n - q|| \\
&\le \alpha_n||f(y_n)-f(q)|| + \alpha_n||f(q)-q|| + (1-\alpha_n)||y_n-q|| \\
&\le \alpha_n\alpha||y_n-q|| + \alpha_n||f(q)-q|| + (1-\alpha_n)||y_n-q|| \\
&= (1-\alpha_n(1-\alpha))||y_n-q|| + \alpha_n||f(q)-q|| \\
&\le \max\left\{||y_n-q||, \frac{1}{1-\alpha}||f(q)-q||\right\} \quad \forall n \ge 1.
\end{aligned} \tag{3.20}$$

So by induction, we get

$$||y_n-q|| \le \max\left\{||y_0-q||, \frac{1}{1-\alpha}||f(q)-q||\right\}, \quad n \ge 0. \tag{3.21}$$

This shows that $\{y_n\}$ is bounded, so are $\{f(y_n)\}$ and $\{(1/(n+1))\sum_{j=0}^{n}(PT)^j y_n\}$. We observe that

$$\begin{aligned}
\left\|y_{n+1}-\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| &= \left\|\alpha_n f(y_n)+(1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - \frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| \\
&= \alpha_n\left\|f(y_n)-\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\| \\
&\le \alpha_n\left(||f(y_n)|| + \left\|\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n\right\|\right).
\end{aligned} \tag{3.22}$$

Hence $\{y_{n+1}-(1/(n+1))\sum_{j=0}^{n}(PT)^j y_n\}$ converges strongly to 0. We next show that

$$\limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle \le 0. \tag{3.23}$$

Let $\{y_{n_j}\}$ be a subsequence of $\{y_n\}$ such that

$$\lim_{j\to\infty}\langle z-y_{n_j}, z-f(z)\rangle = \limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle, \tag{3.24}$$

and $y_{n_j} \rightharpoonup q \in C$. It follows by Lemmas 2.3 and 2.4 that $q \in F(PT) = F(T)$. By the inequality (3.1), we get

$$\limsup_{n\to\infty}\langle z-y_n, z-f(z)\rangle = \langle z-q, z-f(z)\rangle \le 0 \tag{3.25}$$

as required. Finally, we will show that $y_n \to z$. For each $n \geq 0$, we have

$$\begin{aligned}
\|y_{n+1} - z\|^2 &= \|y_{n+1} - z + \alpha_n(z - f(z)) - \alpha_n(z - f(z))\|^2 \\
&\leq \|y_{n+1} - z + \alpha_n(z - f(z))\|^2 + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&= \left\|\alpha_n f(y_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}(PT)^j y_n - (\alpha_n f(z) + (1-\alpha_n)z)\right\|^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&= \left\|\alpha_n(f(y_n) - f(z)) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}((PT)^j y_n - z)\right\|^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z,\ f(z) - z\rangle \\
&\leq \left[\alpha_n\|f(y_n) - f(z)\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|(PT)^j y_n - z\|\right]^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq \left[\alpha_n\alpha\|y_n - z\| + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n}\|y_n - z\|\right]^2 \\
&\quad + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&= (1 - \alpha_n(1-\alpha))^2\|y_n - z\|^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle \\
&\leq (1 - \alpha_n(1-\alpha))\|y_n - z\|^2 + 2\alpha_n\langle y_{n+1} - z, f(z) - z\rangle.
\end{aligned} \tag{3.26}$$

Now, let $\epsilon > 0$ be arbitrary. Then, by the fact (3.23), there exists a natural number $N$ such that

$$\langle z - y_n,\ z - f(z)\rangle \leq \frac{\epsilon}{2} \quad \forall n \geq N. \tag{3.27}$$

From (3.26), we get

$$\|y_{n+1} - z\|^2 \leq (1 - \alpha_n(1-\alpha))\|y_n - z\|^2 + \alpha_n\epsilon. \tag{3.28}$$

By Lemma 2.5, the sequence $\{y_n\}$ converges strongly to a fixed point $z$ of $T$. This completes the proof. □

By using the same arguments and techniques as those of Theorem 3.2, we have also the following main theorem.

Theorem 3.3. *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $P$ the metric projection of $H$ onto $C$, and $T : C \to H$ a nonexpansive nonself-mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be sequence in $[0,1]$ which satisfies $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then for a contraction mapping $f : C \to C$ with coefficient $\alpha \in (0,1)$, the sequence $\{z_n\}$ defined by (1.11) converges strongly to $z$, where $z$ is the unique solution in $F(T)$ of the variational inequality (3.1).*

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Rabian Wangkeeree: Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand
*Email address*: rabianw@nu.ac.th