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Research Article

On the Composition of Distributions $x^{-s}\ln|x|$ and $|x|^{\mu}$

Biljana Jolevska-Tuneska and Emin Özçağ

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Let F be a distribution and let f be a locally summable function. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, where $F_n(x) = F(x)*\delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The composition of the distributions $x^{-s}\text{In}|x|$ and $|x|^{\mu}$ is evaluated for $s = 1,2,\dots,\mu > 0$ and $\mu s \neq 1,2,\dots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support, let $\mathcal{D}(a,b)$ be the space of infinitely differentiable functions with support contained in the interval (a,b), and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We define the locally summable functions x_+^λ, x_-^λ, $x_+^\lambda\ln x_+$, $x_-^\lambda\ln x_-$, $|x|^\lambda$, and $|x|^\lambda\ln|x|$ for $\lambda > -1$ (see [1]) by

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^\lambda = \begin{cases} |x|^\lambda, & x < 0, \\ 0, & x > 0, \end{cases}$$
$$x_+^\lambda\ln x_+ = \begin{cases} x^\lambda\ln x, & x > 0, \\ 0, & x < 0, \end{cases} \qquad x_-^\lambda\ln x_- = \begin{cases} |x|^\lambda\ln|x|, & x < 0, \\ 0, & x > 0, \end{cases} \tag{1}$$
$$|x|^\lambda = x_+^\lambda + x_-^\lambda, \qquad |x|^\lambda\ln|x| = x_+^\lambda\ln x_+ + x_-^\lambda\ln x_-.$$

The distributions x_+^λ and x_-^λ are then defined inductively for $\lambda < -1$ and $\lambda \neq -2,-3,\dots$ by

$$(x_+^\lambda)' = \lambda x_+^{\lambda-1}, \qquad (x_-^\lambda)' = -\lambda x_-^{\lambda-1}. \tag{2}$$

It follows that if r is a positive integer and $-r-1<\lambda<-r$, then

$$\begin{aligned} \langle x_+^\lambda,\varphi(x)\rangle &= \int_0^\infty x^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx,\\ \langle x_-^\lambda,\varphi(x)\rangle &= \int_{-\infty}^0 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx \end{aligned} \tag{3}$$

for arbitrary φ in $\mathcal{D}$. In particular, if φ has its support contained in the interval $[-1,1]$, then

$$\begin{aligned} \langle x_+^\lambda,\varphi(x)\rangle &= \int_0^1 x^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!(\lambda+k+1)},\\ \langle x_-^\lambda,\varphi(x)\rangle &= \int_{-1}^0 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{(-1)^k\varphi^{(k)}(0)}{k!(\lambda+k+1)},\\ \langle |x|^\lambda,\varphi(x)\rangle &= \int_{-1}^1 |x|^\lambda\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{[1+(-1)^k]\varphi^{(k)}(0)}{k!(\lambda+k+1)}, \end{aligned} \tag{4}$$

$$\langle |x|^\lambda\ln|x|,\varphi(x)\rangle = \int_{-1}^1 |x|^\lambda\ln|x|\left[\varphi(x)-\sum_{k=1}^{r-1}\frac{\varphi^{(k)}(0)}{k!}x^k\right]dx+\sum_{k=0}^{r-1}\frac{[1+(-1)^k]\varphi^{(k)}(0)}{k!(\lambda+k+1)^2} \tag{5}$$

if $-r-1<\lambda<-r$.

We define the distribution $x^{-1}\ln|x|$ by

$$x^{-1}\ln|x| = \frac{1}{2}\left(\ln^2|x|\right)', \tag{6}$$

and we define the distribution $x^{-r-1}\ln|x|$ inductively by

$$x^{-r-1}\ln|x| = \frac{x^{-r-1}-\left(x^{-r}\ln|x|\right)'}{r} \tag{7}$$

for $r=1,2,\ldots$. It follows by induction that

$$x^{-r-1}\ln|x| = \phi(r)x^{-r-1}+\frac{(-1)^r\left(x^{-1}\ln|x|\right)^{(r)}}{r!} = \phi(r)x^{-r-1}+\frac{(-1)^r\left(\ln^2|x|\right)^{(r+1)}}{2r!}, \tag{8}$$

where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} i^{-1}, & r=1,2,\ldots,\\ 0, & r=0. \end{cases} \tag{9}$$

In the following, we let N be the neutrix, see [2], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1}n,\ \ln^{r}n,\quad \lambda>0,\ r=1,2,\dots \tag{10}$$

as well as all functions which converge to zero in the usual sense as n tends to infinity.

Now let $\rho(x)$ be an infinitely differentiable function having the following properties:

(i) $\rho(x)=0$ for $|x|\ge 1$,
(ii) $\rho(x)\ge 0$,
(iii) $\rho(x)=\rho(-x)$,
(iv) $\int_{-1}^{1}\rho(x)dx=1$.

Putting $\delta_n(x)=n\rho(nx)$ for $n=1,2,\dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Next, for an arbitrary distribution f in $\mathcal{D}'$, we define

$$f_n(x)=(f*\delta_n)(x)=\langle f(t),\delta_n(x-t)\rangle \tag{11}$$

for $n=1,2,\dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition was given in [3].

Definition 1. Let F be a distribution and let f be a locally summable function. Say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\mathrm{N}-\lim_{n\to\infty}\int_{-\infty}^{\infty}F_n(f(x))\varphi(x)dx=\langle h(x),\varphi(x)\rangle \tag{12}$$

for all test functions φ with compact support contained in (a,b).

The following theorems were proved in [4, 5] and [6], respectively.

Theorem 2. *The distribution* $(x^r)^{-s}$ *exists and*

$$(x^r)^{-s}=x^{-rs} \tag{13}$$

for $r,s=1,2,\dots$.

Theorem 3. *The distribution* $(|x|^{\mu})^{-s}$ *exists and*

$$(|x|^{\mu})^{-s}=|x|^{-\mu s} \tag{14}$$

for $s=1,2,\dots$, $\mu>0$ *and* $\mu s\neq 1,2,\dots$.

Theorem 4. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the distribution* $F_s(x^r)$ *exists and*

$$F_s(x^r)=rF_{rs}(x) \tag{15}$$

for $r,s=1,2,\dots$.

We need the following lemma which can be easily proved by induction.

Lemma 5.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\,dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases} \tag{16}$$

for $r = 0, 1, 2, \ldots$.

We now prove the following theorem on the composition of distributions in the neutrix setting.

Theorem 6. *If* $F_s(x)$ *denotes the distribution* $x^{-s}\ln|x|$, *then the distribution* $F_s(|x|^\mu)$ *exists and*

$$F_s(|x|^\mu) = \mu |x|^{-\mu s}\ln|x| \tag{17}$$

for $s = 1, 2, \ldots$, $\mu > 0$ *and* $\mu s \neq 1, 2, \ldots$.

Proof. We will suppose that $r < \mu s < r+1$ for some positive integer r. We put

$$\begin{aligned} [F_s(|x|^\mu)]_n &= F_s(|x|^\mu) * \delta_n(x) \\ &= \phi(s-1)[(|x|^\mu)^{-s}]_n - \frac{(-1)^s}{2(s-1)!}\int_{-1/n}^{1/n} \ln^2 \big|\,|x|^\mu - t\,\big|\,\delta_n^{(s)}(t)\,dt, \end{aligned} \tag{18}$$

and note that

$$\begin{aligned} &\int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2 \big|\,|x|^\mu - t\,\big|\,\delta_n^{(s)}(t)\,dt\,dx \\ &\quad = \begin{cases} 0, & k \text{ odd}, \\ 2\displaystyle\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2 \big|\,|x|^\mu - t\,\big|\,\delta_n^{(s)}(t)\,dt\,dx, & k \text{ even}. \end{cases} \end{aligned} \tag{19}$$

Then

$$\begin{aligned} &\int_0^1 x^k \int_{-1/n}^{1/n} \ln^2 |x^\mu - t|\,\delta_n^{(s)}(t)\,dt\,dx \\ &\quad = \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_0^{n^{-1/\mu}} x^k \ln^2 |x^\mu - t|\,dx\,dt \\ &\qquad + \int_{-1/n}^{1/n} \delta_n^{(s)}(t) \int_{n^{-1/\mu}}^{1} x^k \ln^2 |x^\mu - t|\,dx\,dt \\ &\quad = \frac{n^{(\mu s - k - 1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(v) \int_0^1 u^{-(\mu - k - 1)/\mu} \ln^2 \left|\frac{u - v}{n}\right| du\,dv \\ &\qquad + \frac{n^{(\mu s - k - 1)/\mu}}{\mu} \int_{-1}^{1} \rho^{(s)}(v) \int_1^n u^{-(\mu - k - 1)/\mu} \ln^2 \left|\frac{u - v}{n}\right| du\,dv \\ &\quad = I_1 + I_2, \end{aligned} \tag{20}$$

on using the substitutions $u = nx^\mu$ and $v = nt$.

It is easily seen that

$$\mathrm{N}-\lim_{n\to\infty} I_1 = 0 \tag{21}$$

for $k = 0, 1, \ldots, r-1$.

Now,

$$\begin{aligned}
I_2 &= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\left[\ln\left|\frac{1-v}{u}\right| + \ln u - \ln n\right]^2 du\,dv \\
&= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln^2\left|\frac{1-v}{u}\right| du\,dv \\
&\quad + \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln u \ln\left|\frac{1-v}{u}\right| du\,dv \\
&\quad - \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\ln n\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\ln\left|\frac{1-v}{u}\right| du\,dv \\
&= J_1 + J_2 + J_3,
\end{aligned} \tag{22}$$

since $\int_{-1}^{1}\rho^{(s)}(v)dv = 0$ for $s = 1, 2, \ldots$, by Lemma 5.

It is easily seen that

$$\mathrm{N}-\lim_{n\to\infty} J_3 = 0. \tag{23}$$

Next, we have

$$\begin{aligned}
J_1 &= \frac{n^{(\mu s-k-1)/\mu}}{\mu}\int_{-1}^{1}\rho^{(s)}(v)\int_{1}^{n}u^{-(\mu-k-1)/\mu}\left(\sum_{i=1}^{\infty}\frac{v^i}{iu^i}\right)^2 du\,dv \\
&= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)\int_{1}^{n}u^{(k+1)/\mu-i-2}du\,dv \\
&= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{\phi(i)}{i+1}\frac{\mu\left(n^{(k+1)/\mu-i-1}-1\right)}{k-\mu(i+1)+1}\int_{-1}^{1}v^{i+1}\rho^{(s)}(v)dv,
\end{aligned} \tag{24}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} J_1 = \frac{2\phi(s-1)}{s(\mu s-k-1)}\int_{-1}^{1}v^s\rho^{(s)}(v)dv = \frac{2(-1)^s\phi(s-1)(s-1)!}{\mu s-k-1}, \tag{25}$$

on using Lemma 5, for $k = 0, 1, \ldots, r-1$.

Finally,

$$\begin{aligned}
J_2 &= \frac{2n^{(\mu s-k-1)/\mu}}{\mu}\sum_{i=1}^{\infty}\frac{1}{i}\int_{-1}^{1}v^i\rho^{(s)}(v)\int_{1}^{n}u^{(k+1)/\mu-i-1}\ln u\,du\,dv \\
&= 2\sum_{i=1}^{\infty}\frac{1}{i}\left[\frac{n^{s-i}\ln n}{k-\mu i+1} - \frac{\mu\left(n^{s-i}-n^{(\mu s-k-1)/\mu}\right)}{(k-\mu i+1)^2}\right]\int_{-1}^{1}v^i\rho^{(s)}(v)dv,
\end{aligned} \tag{26}$$

and it follows that

$$\mathrm{N}-\lim_{n\to\infty} J_2 = \frac{2\mu(-1)^{s-1}(s-1)!}{(\mu s-k-1)^2}, \tag{27}$$

on using Lemma 5, for $k = 0, 1, \ldots, r-1$.

Hence,

$$\begin{aligned} &\mathrm{N}-\lim_{n\to\infty} \int_0^1 x^k \int_{-1/n}^{1/n} \ln^2 |x^\mu - t|\, \delta_n^{(s)}(t)\, dt\, dx \\ &\quad = 2(-1)^s(s-1)!\left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2}\right] \end{aligned} \tag{28}$$

for $k = 0, 1, \ldots, r-1$, on using (19) to (23). Then using (19) and (25), we see that

$$\begin{aligned} &\mathrm{N}-\lim_{n\to\infty} \int_{-1}^1 x^k \int_{-1/n}^{1/n} \ln^2 \big| |x|^\mu - t \big|\, \delta_n^{(s)}(t)\, dt\, dx \\ &\quad = 2(-1)^s(s-1)!\left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2}\right]\left((-1)^k+1\right) \end{aligned} \tag{29}$$

for $k = 0, 1, \ldots, r-1$.

When $k = r$, (18) still holds, but now we have

$$I_1 = \frac{n^{(\mu s-r-1)/\mu}}{\mu} \int_{-1}^1 \rho^{(s)}(v) \int_0^1 u^{-(\mu-r-1)/\mu} \ln^2 \left|\frac{u-v}{n}\right| du\, dv, \tag{30}$$

and it follows that for any continuous function ψ

$$\lim_{n\to\infty} \int_0^{n^{-1/\mu}} x^r \int_{-1/n}^{1/n} \ln^2 \big| |x|^\mu - t \big|\, \delta_n^{(s)}(t)\, dt\, \psi(x)\, dx = 0. \tag{31}$$

Similarly,

$$\lim_{n\to\infty} \int_{-n^{-1/\mu}}^0 x^r \int_{-1/n}^{1/n} \ln^2 \big| |x|^\mu - t \big|\, \delta_n^{(s)}(t)\, dt\, \psi(x)\, dx = 0. \tag{32}$$

Next, when $|x|^\mu \geq 1/n$, we have

$$\begin{aligned} \int_{-1/n}^{1/n} \ln^2 \big| |x|^\mu - t \big|\, \delta_n^{(s)}(t)\, dt &= n^s \int_{-1}^1 \ln^2 \big| |x|^\mu - v/n \big|\, \rho^{(s)}(v) dv \\ &= n^s \int_{-1}^1 \left[\ln|x|^\mu - \sum_{i=1}^\infty \frac{v^i}{i n^i |x|^{\mu i}}\right]^2 \rho^{(s)}(v) dv \\ &= \sum_{i=s}^\infty \frac{-2\ln|x|^\mu + 2\phi(i-1)}{i n^{i-s} |x|^{\mu i}} \int_{-1}^1 v^i \rho^{(s)}(v) dv. \end{aligned} \tag{33}$$

It follows that

$$\left| \int_{-1/n}^{1/n} \ln^2 \big| |x|^\mu - t \big|\, \delta_n^{(s)}(t)\, dt \right| \leq \sum_{i=s}^\infty \frac{(4\mu \ln|x| + 4\phi(i-1))K_s}{i n^{i-s} |x|^{\mu i}} \tag{34}$$

for $s = 1, 2, \ldots$, where

$$K_s = \int_{-1}^{1} \left| \rho^{(s)}(v) \right| dv. \tag{35}$$

If now $n^{-1/\mu} < \eta < 1$, then

$$\begin{aligned}
&\int_{n^{-1/\mu}}^{\eta} x^r \left| \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx \\
&\quad \le \sum_{i=s}^{\infty} \frac{4K_s\mu}{in^{i-s}} \int_{n^{-1/\mu}}^{\eta} x^{r-\mu i} \ln x\, dx + \sum_{i=s}^{\infty} \frac{4K_s\phi(s-1)}{in^{i-s}} \int_{n^{-1/\mu}}^{\eta} x^{r-\mu i}\, dx \\
&\quad = \sum_{i=s}^{\infty} \frac{4K_s\mu}{in^{i-s}} \left[\frac{\eta^{r+1-\mu i} \ln \eta - n^{i-(r+1)/\mu} \ln n^{-1/\mu}}{r+1-\mu i} - \frac{\eta^{r+1-\mu i} - n^{i-(r+1)/\mu}}{(r+1-\mu i)^2} \right] \\
&\qquad + \sum_{i=s}^{\infty} \frac{4K_s\phi(s-1)}{in^{i-s}} \frac{\eta^{r+1-\mu i} - n^{i-(r+1)/\mu}}{r+1-\mu i}.
\end{aligned} \tag{36}$$

It follows that

$$\lim_{n\to\infty} \int_{n^{-1/\mu}}^{\eta} x^r \left| \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx = O(\eta \ln \eta) \tag{37}$$

for $s = 1, 2, \ldots$.

Thus, if ψ is a continuous function, then

$$\lim_{n\to\infty} \left| \int_{n^{-1/\mu}}^{\eta} x^r \psi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t)\, dt\, dx \right| = O(\eta \ln \eta) \tag{38}$$

for $s = 1, 2, \ldots$.

Similarly,

$$\lim_{n\to\infty} \left| \int_{-\eta}^{-n^{-1/\mu}} x^r \psi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^\mu - t \right| \delta_n^{(s)}(t) \right| dt\, dx = O(\eta \ln \eta) \tag{39}$$

for $s = 1, 2, \ldots$.

Now let $\varphi(x)$ be an arbitrary function in $\mathscr{D}$ with support contained in the interval $[-1, 1]$. By Taylor's theorem, we have

$$\varphi(x) = \sum_{k=0}^{r-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^r}{r!} \varphi^{(r)}(\xi x), \tag{40}$$

where $0<\xi<1$. Then

$$\begin{aligned}\left\langle \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)\,dt, \varphi(x) \right\rangle &= \int_{-1}^{1} \varphi(x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)\,dt\,dx \\ &= \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-n^{-1/\mu}}^{n^{-1/\mu}} \frac{x^r}{r!} \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{n^{-1/\mu}}^{\eta} \frac{x^r}{r!} \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{\eta}^{1} \frac{x^r}{r!} \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-\eta}^{-n^{-1/\mu}} \frac{x^r}{r!} \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx \\ &+ \int_{-1}^{-\eta} \frac{x^r}{r!} \varphi^{(r)}(\xi x) \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt\,dx.\end{aligned} \tag{41}$$

Using equations (27) to (32) and noting that on the intervals $[-1,-\eta]$ and $[\eta,1]$,

$$\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!} \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt = \phi(s-1)|x|^{-\mu s} - \mu |x|^{-\mu s} \ln|x|. \tag{42}$$

Since $|x|^{\mu}$ and $F_s(x)$ are continuous on these intervals, it follows that

$$\begin{aligned}&\mathrm{N}-\lim_{n\to\infty} \frac{(-1)^s}{2(s-1)!} \left\langle \int_{-1/n}^{1/n} \ln^2 \left| |x|^{\mu}-t \right| \delta_n^{(s)}(t)dt, \varphi(x) \right\rangle \\ &\quad= \sum_{k=0}^{r-1} \left[\frac{\phi(s-1)}{\mu s-k-1} - \frac{\mu}{(\mu s-k-1)^2} \right] \frac{\varphi^{(k)}(0)}{k!} \left((-1)^k+1\right) \\ &\quad+ O(\eta|\ln\eta|) + \int_{\eta}^{1} \frac{x^{r-\mu s}}{r!} \varphi^{(r)}(\xi x)\left(\phi(s-1)-\mu \ln x\right)dx \\ &\quad+ \int_{-1}^{-\eta} \frac{|x|^{r-\mu s}}{r!} \varphi^{(r)}(\xi x)\left(\phi(s-1)-\mu \ln|x|\right)dx.\end{aligned} \tag{43}$$

Since η can be made arbitrarily small, it follows that

$$\begin{aligned}
\mathrm{N}-\lim_{n\to\infty}\frac{(-1)^s}{2(s-1)!}&\left\langle\int_{-1/n}^{1/n}\ln^2\left||x|^\mu-t\right|\delta_n^{(s)}(t)dt,\varphi(x)\right\rangle\\
&=\sum_{k=0}^{r-1}\left[\frac{\phi(s-1)}{\mu s-k-1}-\frac{\mu}{(\mu s-k-1)^2}\right]\frac{\varphi^{(k)}(0)}{k!}\left((-1)^k+1\right)\\
&\quad+\phi(s-1)\int_{-1}^{1}|x|^{-\mu s}\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx\\
&\quad-\mu\int_{-1}^{1}|x|^{-\mu s}\ln|x|\left[\varphi(x)-\sum_{k=0}^{r-1}\frac{x^k}{k!}\varphi^{(k)}(0)\right]dx\\
&=\phi(s-1)\langle|x|^{-\mu s},\varphi(x)\rangle-\mu\langle|x|^{-\mu s}\ln|x|,\varphi(x)\rangle
\end{aligned}\tag{44}$$

on using (5). This proves (15) on the interval $[-1,1]$. However, (15) clearly holds on any interval not containing the origin, and the proof is complete. □

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Biljana Jolevska-Tuneska: Faculty of Electrical Engineering and Information Technologies, Ss. Cyril and Methodius University – Skopje, Karpos II bb, 1000 Skopje, Macedonia
Email address: biljanaj@etf.ukim.edu.mk

Emin Özçağ: Department of Mathematics,, Faculty of Science, University of Hacettepe, 06532 Beytepe, Ankara, Turkey
Email address: ozcag1@hacettepe.edu.tr