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Research Article

Subordination Properties for Certain Analytic Functions

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The purpose of the present paper is to derive a subordination result for functions in the class $H_n^*(\alpha,\lambda,b)$ of normalized analytic functions in the open unit disk $\mathbb{U}$. A number of interesting applications of the subordination result are also considered.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. We also denote by K the class of functions $f \in A$ that are convex in $\mathbb{U}$.

Given two functions $f, g \in A$, where f is given by (1.1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

the Hadamard product (or convolution) $f*g$ is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in \mathbb{U}). \tag{1.3}$$

By using the Hadamard product, Ruscheweyh [1] defined

$$D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z) \quad (\alpha \geq -1). \tag{1.4}$$

From the definition of (1.4), we observe that

$$D^{n} f(z) = \frac{z\left(z^{n-1} f(z)\right)^{(n)}}{n!}, \tag{1.5}$$

when $n = \alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$. The symbol $D^n f(z)$ $(n \in \mathbb{N}_0)$ was called the nth-order Ruscheweyh derivative of f by Al-Amiri [2]. We also note that $D^0 f(z) = f(z)$ and $D^1 f(z) = z f'(z)$.

Definition 1.1. Suppose that $f \in A$. Then the function f is said to be a member of the class $H_n(\alpha, \lambda, b)$ if it satisfies

$$\left| \frac{\lambda(n+1)\left(D^{n+1} f(z)/z\right) + \left[1 - \lambda(n+1)\right]\left(D^n f(z)/z\right) - 1}{\lambda(n+1)\left(D^{n+1} f(z)/z\right) + \left[1 - \lambda(n+1)\right]\left(D^n f(z)/z\right) + 2b(1-\alpha) - 1} \right| < 1$$
$$\left(z \in \mathbb{U};\ 0 \leq \alpha < 1;\ \lambda \geq 0;\ b \in \mathbb{C} \setminus \{0\};\ n \in \mathbb{N}_0\right). \tag{1.6}$$

By specializing α, λ, b, and n, one can obtain various subclasses studied by many authors (see, e.g., [3–11]).

Definition 1.2. Let g be analytic and univalent in $\mathbb{U}$. If f is analytic in $\mathbb{U}$, $f(0) = g(0)$, and $f(\mathbb{U}) \subset g(\mathbb{U})$, then one says that f is subordinate to g in $\mathbb{U}$, and we write $f \prec g$ or $f(z) \prec g(z)$. One also says that g is superordinate to f in $\mathbb{U}$.

Definition 1.3. An infinite sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers will be called a subordinating factor sequence if for every univalent function f in K, one has

$$\sum_{k=1}^{\infty} b_k a_k z^k \prec f(z) \quad (z \in \mathbb{U};\ a_1 = 1). \tag{1.7}$$

Lemma 1.4 (see [12]). *The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.8}$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $H_n(\alpha, \lambda, b)$.

Lemma 1.5. *If the function f which is defined by* (1.1) *satisfies the following condition:*

$$\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)\left|a_k\right| \leq (1-\alpha)|b| \quad \left(0 \leq \alpha < 1;\ \lambda \geq 0;\ b \in \mathbb{C}\setminus\{0\};\ n \in \mathbb{N}_0\right), \tag{1.9}$$

where

$$C(n,k)=\prod_{j=2}^{k}\frac{(j+n-1)}{(k-1)!} \quad (k=2,3,\dots), \tag{1.10}$$

then $f \in H_n(\alpha,\lambda,b)$.

Proof. Suppose that the inequality (1.9) holds. Using the identity

$$z\left(D^n f(z)\right)' = (n+1)D^{n+1}f(z) - nD^n f(z), \tag{1.11}$$

we have for $z \in \mathbb{U}$,

$$\begin{aligned}
&\left|(1-\lambda)\frac{D^n f(z)}{z}+\lambda\left(D^n f(z)\right)'-1\right| - \left|2b(1-\alpha)+(1-\lambda)\frac{D^n f(z)}{z}+\lambda\left(D^n f(z)\right)'-1\right| \\
&\quad= \left|\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)a_k z^{k-1}\right| - \left|2b(1-\alpha)+\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)a_k z^{k-1}\right| \\
&\quad\leq \sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)\left|a_k\right||z|^{k-1} \\
&\qquad - \left\{2|b|(1-\alpha)-\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)\left|a_k\right||z|^{k-1}\right\} \\
&\quad\leq 2\left\{\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)\left|a_k\right| - |b|(1-\alpha)\right\} \leq 0,
\end{aligned} \tag{1.12}$$

which shows that f belongs to $H_n(\alpha,\lambda,b)$. □

Let $H_n^*(\alpha,\lambda,b)$ denote the class of functions f in A whose Taylor-Maclaurin coefficients a_k satisfy the condition (1.9).

We note that

$$H_n^*(\alpha,\lambda,b) \subseteq H_n(\alpha,\lambda,b). \tag{1.13}$$

Example 1.6. (i) For $0 \leq \alpha < 1$, $\lambda > 0$, $b \in \mathbb{C}\setminus\{0\}$, and $n \in \mathbb{N}_0$, the following function defined by:

$$f_0(z) = z + \frac{2b(1-\alpha)}{(n+1)(\lambda+1)}z^2 {}_3F_2\left(1,2,1+\frac{1}{\lambda};2+\frac{1}{\lambda},n+2;z\right) \quad (z \in \mathbb{U}), \tag{1.14}$$

is in the class $H_n(\alpha,\lambda,b)$.

(ii) For $0 \leq \alpha < 1$, $\lambda > 0$, $b \in \mathbb{C}\setminus\{0\}$, and $n \in \mathbb{N}_0$, the following functions defined by:

$$\begin{aligned}
f_1(z) &= z \pm \frac{(1-\alpha)|b|}{(\lambda+1)(n+1)}z^2 \quad (z \in \mathbb{U}), \\
f_2(z) &= z \pm \frac{(1-\alpha)|b|}{(2\lambda+1)(n+1)(n+2)}z^3 \quad (z \in \mathbb{U}), \\
f_3(z) &= z \pm \frac{1}{(\lambda+1)(n+1)}z^2 \pm \frac{2\left[(1-\alpha)|b|-1\right]}{(2\lambda+1)(n+1)(n+2)}z^3 \quad (z \in \mathbb{U})
\end{aligned} \tag{1.15}$$

are in the class $H_n^*(\alpha,\lambda,b)$.

In this paper, we obtain a sharp subordination result associated with the class $H_n^*(\alpha, \lambda, b)$ by using the same techniques as in [13] (see also [14–16]). Some applications of the main result which give important results of analytic functions are also investigated.

2. Main theorem

Theorem 2.1. *Let* $f \in H_n^*(\alpha,\lambda,b)$. *Then*

$$\frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}) \tag{2.1}$$

for every function g *in* K, *and*

$$\operatorname{Re} f(z) > -\frac{(\lambda+1)(n+1)+|b|(1-\alpha)}{(\lambda+1)(n+1)}. \tag{2.2}$$

The constant $(\lambda+1)(n+1)/2[(\lambda+1)(n+1)+|b|(1-\alpha)]$ *cannot be replaced by a larger one.*

Proof. Let $f \in H_n^*(\alpha,\lambda,b)$ and let

$$g(z) = z + \sum_{k=2}^{\infty} c_k z^k \tag{2.3}$$

be any function in the class K. Then we readily have

$$\frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}(f*g)(z) = \frac{(\lambda+1)(n+1)}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}\left(z+\sum_{k=2}^{\infty}a_k c_k z^k\right). \tag{2.4}$$

Thus, by Definition 1.2, the subordination result (2.1) will hold true if the sequence

$$\left\{\frac{(\lambda+1)(n+1)a_k}{2[(\lambda+1)(n+1)+|b|(1-\alpha)]}\right\}_{k=1}^{\infty} \tag{2.5}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1.4, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(\lambda+1)(n+1)}{[(\lambda+1)(n+1)+|b|(1-\alpha)]}a_k z^k\right\} > 0 \quad (z \in \mathbb{U}). \tag{2.6}$$

Now, since

$$[1+\lambda(k-1)]C(n,k) \quad (\lambda \ge 0,\ n \in \mathbb{N}_0) \tag{2.7}$$

is an increasing function of k, we have

$$\begin{aligned}
&\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}a_k z^k\right\}\\
&\quad=\operatorname{Re}\left\{1+\frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}z\right.\\
&\qquad\left.+\frac{1}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}\sum_{k=2}^{\infty}(\lambda+1)(n+1)a_k z^k\right\}\\
&\quad>1-\frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}r\\
&\qquad-\frac{1}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]C(n,k)\left|a_k\right|r^k\\
&\quad>1-\frac{(\lambda+1)(n+1)}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}r-\frac{|b|(1-\alpha)}{\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}r>0\quad(|z|=r).
\end{aligned}\tag{2.8}$$

This proves the inequality (2.6), and hence also the subordination result (2.1) asserted by Theorem 2.1. The inequality (2.2) follows from (2.1) by taking

$$g(z)=\frac{z}{1-z}\in K.\tag{2.9}$$

Next, we consider the function

$$f_1(z)=z-\frac{|b|(1-\alpha)}{(\lambda+1)(n+1)}z^2\quad\left(0\le\alpha<1;\ \lambda\ge 0;\ b\in\mathbb{C}\setminus\{0\};\ n\in\mathbb{N}_0\right)\tag{2.10}$$

which is a member of the class $H_n^*(\alpha,\lambda,b)$. Then by using (2.1), we have

$$\frac{(\lambda+1)(n+1)}{2\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}f_1(z)\prec\frac{z}{1-z}\quad(z\in\mathbb{U}).\tag{2.11}$$

It can be easily verified for the function $f_1(z)$ defined by (2.10) that

$$\inf_{z\in\mathbb{U}}\left\{\operatorname{Re}\left(\frac{(\lambda+1)(n+1)}{2\left[(\lambda+1)(n+1)+|b|(1-\alpha)\right]}f_1(z)\right)\right\}=-\frac{1}{2}\quad(z\in\mathbb{U})\tag{2.12}$$

which completes the proof of Theorem 2.1. □

3. Some applications

Taking $n=0$ in Theorem 2.1, we obtain the following.

Corollary 3.1. *If the function f defined by* (1.1) *satisfies*

$$\sum_{k=2}^{\infty}\left[1+\lambda(k-1)\right]\left|a_k\right|\le m\quad(\lambda\ge 0,\ m>0)\tag{3.1}$$

then for every function g *in* K, *one has*

$$\frac{(\lambda+1)}{2(\lambda+m+1)}(f*g)(z) \prec g(z), \quad (z \in \mathbb{U}),$$
$$\operatorname{Re} f(z) > -\left(1+\frac{m}{\lambda+1}\right). \tag{3.2}$$

The constant $(\lambda+1)/2(\lambda+m+1)$ *cannot be replaced by larger one.*

Putting $\lambda = 0$ in Theorem 2.1, we have the following corollary.

Corollary 3.2. *If the function* f *defined by* (1.1) *satisfies*

$$\sum_{k=2}^{\infty} C(n,k)\left|a_k\right| \leq m, \quad m > 0,\ n \in \mathbb{N}_0, \tag{3.3}$$

where $C(n,k)$ *is defined by* (1.10), *then for every function* g *in* K, *one has*

$$\frac{(n+1)}{2(n+m+1)}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}),$$
$$\operatorname{Re} f(z) > -\left(1+\frac{m}{n+1}\right). \tag{3.4}$$

The constant $(n+1)/2(n+m+1)$ *cannot be replaced by larger one.*

Next, letting $\lambda = 1$ and $n = 0$, in Theorem 2.1, we obtain the following corollary.

Corollary 3.3. *If the function* f *satisfies*

$$\sum_{k=2}^{\infty} k\left|a_k\right| \leq m \quad (m > 0), \tag{3.5}$$

then for every function g *in* K, *one has*

$$\frac{1}{(m+2)}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}),$$
$$\operatorname{Re} f(z) > -\left(1+\frac{m}{2}\right). \tag{3.6}$$

The constant $1/(m+2)$ *cannot be replaced by a larger one.*

Remark 3.4. Putting $\lambda = 1$, $m = 1$, and $n = 0$, in Theorem 2.1, we obtain the result due to Singh [17].

Also, by taking $\lambda = 0$ and $n = 0$, in Theorem 2.1, we have the following.

Corollary 3.5. *If the function* f *satisfies*

$$\sum_{k=2}^{\infty} \left|a_k\right| \leq m \quad (m > 0), \tag{3.7}$$

then for every function g in K, one has

$$\frac{1}{2(m+1)}(f*g)(z) \prec g(z) \quad (z \in \mathbb{U}), \qquad \operatorname{Re} f(z) > -(1+m). \tag{3.8}$$

The constant $1/2(m+1)$ cannot be replaced by a larger one.

It is clearly from the proof of Theorem 2.1 that the function $f(z) = z - mz^2$ $(m > 0,\ z \in \mathbb{U})$ is the extremal function of Corollary 3.5. Also, the following example gives a nonpolynomial extremal function for the same corollary.

Example 3.6. Let the function h be defined by

$$h(z) = \frac{(m+1)z}{(m+1)+mz} \quad (m > 0,\ z \in \mathbb{U}), \tag{3.9}$$

the above function is analytic in $\mathbb{U}$ and it is equivalent to

$$h(z) = z + \sum_{k=2}^{\infty} \left(\frac{-m}{m+1}\right)^{k-1} z^k. \tag{3.10}$$

Then we have

$$\sum_{k=2}^{\infty} \left| \left(\frac{-m}{m+1}\right)^{k-1} \right| = m. \tag{3.11}$$

Therefore, the Taylor-Maclaurin coefficients of the function h satisfy the condition in Corollary 3.5. Moreover, it can be easily verified that

$$\inf_{z \in \mathbb{U}} \operatorname{Re} h(z) = h(-1) = -(m+1). \tag{3.12}$$

Then, the constant $-(m+1)$ cannot be replaced by a larger one. Therefore, the function h is the extremal function of Corollary 3.5.

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