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*Research Article*

# Harmonic Maps and Stability on $f$-Kenmotsu Manifolds

**Vittorio Mangione**

*Universitá degli Studi di Parma, Viale G.P. Usberti 53-A, 43100 Parma, Italy*

Correspondence should be addressed to Vittorio Mangione, vittorio.mangione@unipr.it

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The purpose of this paper is to study some submanifolds and Riemannian submersions on an $f$-Kenmotsu manifold. The stability of a $\varphi$-holomorphic map from a compact $f$-Kenmotsu manifold to a Kählerian manifold is proven.



## 1. Introduction

In Section 2, we give preliminaries on $f$-Kenmotsu manifolds. The concept of $f$-Kenmotsu manifold, where $f$ is a real constant, appears for the first time in the paper of Jannsens and Vanhecke [1]. More recently, Olszak and Roşca [2] defined and studied the $f$-Kenmotsu manifold by the formula (2.3), where $f$ is a function on $M$ such that $df \wedge \eta = 0$. Here, $\eta$ is the dual 1-form corresponding to the characteristic vector field $\xi$ of an almost contact metric structure on $M$. The condition $df \wedge \eta = 0$ follows in fact from (2.3) if $\dim M \geq 5$. This does not hold in general if $\dim M = 3$.

A 1-Kenmotsu manifold is a Kenmotsu manifold (see Kenmotsu [3, 4]. Theorem 2.1 provides a geometric interpretation of an $f$-Kenmotsu structure.

In Section 3, we initiate a study of harmonic maps when the domain is a compact $f$-Kenmotsu manifold and the target is a Kähler manifold.

Ianus and Pastore [5, 6] defined a $(\varphi, J)$-holomorphic map between an almost contact metric manifold $M(\varphi, \eta, \xi, g)$ and an almost Hermitian manifold $N(J, h)$ as a smooth map $F : M \to N$ such that the condition $F_\star \circ \varphi = J \circ F_\star$ is satisfied. Then, the formula $J(\tau(F)) = F_\star(\operatorname{div}\varphi) - \operatorname{Tr}_g \beta$ holds, where $\tau(F)$ is the tension field of $F$ and $\beta(X, Y) = (\widetilde{\nabla}_X J)(F_\star Y)$, $\widetilde{\nabla}$ being the connection induced in the pull-back bundle $F^\star(TN)$ (see [7]). It is easy to see that in our assumptions $\operatorname{div}\varphi = 0$ and $\operatorname{Tr}_g \beta = 0$ so that a $(\varphi, J)$-holomorphic map between an

$f$-Kenmotsu manifold $M$ and a Kähler manifold $N$ is a harmonic map. If $M$ is a compact manifold, a second-order elliptic operator $J_F$, called the Jacobi operator, is associated to the harmonic map $F$. It is well known that the spectrum of $J_F$ consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded by the first one. We define the *Morse index* of the harmonic map $F$ as the sum of multiplicities of negative eigenvalues of the Jacobi operator $J_F$ [8, 9]. A harmonic map is called *stable* if the Morse index is zero. We have proven that any $(\varphi, J)$-holomorphic map from a compact $f$-Kenmotsu manifold to a Kähler manifold is a stable harmonic map (see [10]).

## 2. $f$-Kenmotsu manifolds

A differentiable $(2n+1)$-dimensional manifold $M$ is said to have a $(\varphi,\xi,\eta)$-structure or an almost contact structure if there exist a tensor field $\varphi$ of type (1, 1), a vector field $\xi$, and a 1-form $\eta$ on $M$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

where $I$ denotes the identity transformation.

It seems natural to include also $\varphi\xi = 0$ and $\eta \circ \varphi = 0$; both can be derived from (2.1).

Let $g$ be an associated Riemannian metric on $M$ such that

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y). \tag{2.2}$$

Putting $Y = \xi$ in (2.2) and using (2.1), we get $\eta(X) = g(X,\xi)$, for any vector field $X$ on $M$.

In this paper, we denote by $C^\infty(M)$ and $\Gamma(E)$ the algebra of smooth functions on $M$ and the $C^\infty(M)$-module of smooth sections of a vector bundle $E$, respectively. All manifolds are assumed to be connected and of class $C^\infty$. Tensors fields, distribution, and so on are assumed to be of class $C^\infty$ if not stated otherwise.

We say that $M$ is an *$f$-Kenmotsu manifold* if there exists an almost contact metric structure $(\varphi,\xi,\eta,g)$ on $M$ satisfying

$$(\widetilde{\nabla}_X\varphi)Y = f\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big) \tag{2.3}$$

for $X, Y \in \Gamma(TM)$, where $f$ is a smooth function on $M$ such that $df \wedge \eta = 0$.

A 1-Kenmotsu manifold is a Kenmotsu manifold [2, 3].

The following theorem provides a geometric interpretation of any $f$-Kenmotsu structure.

**Theorem 2.1** (Olszak-Roşca). *Let $M$ be an almost contact metric manifold. Then, $M$ is $f$-Kenmotsu if and only if it satisfies the following conditions:*

(a) *the distribution $D = \operatorname{Ker}\eta$ is integrable and any leaf of the foliation $\mathcal{F}$ corresponding to $D$ is a totally umbilical hypersurface with constant mean curvature;*

(b) *the almost Hermitian structure $(J, g)$ induced on an arbitrary leaf is Kähler;*

(c) $\nabla_\xi\xi = 0$ *and* $L_\xi\varphi = 0$.

Moreover, we have

$$\tilde{\nabla}_X \xi = f\left(X - \eta(X)\xi\right) \tag{2.4}$$

which gives $\operatorname{div} \xi = 2nf$.

The characteristic vector field of an $f$-Kenmotsu manifold also satisfies

$$R(X,Y)\xi = f^2\left(\eta(X)Y - \eta(Y)X\right). \tag{2.5}$$

Levy proven that a second-order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor [11]. On the other hand, Sharma proven that there is no nonzero skew-symmetric second-order parallel tensor on a Sasakian manifold [12]. For an $f$-Kenmotsu manifold we have the following theorem.

**Theorem 2.2.** *There is no nonzero parallel* 2*-form on an* $f$*-Kenmotsu manifold.*

*Proof.* We omit it. □

A plane section $p$ in $T_x\widetilde{M}$, $x \in \widetilde{M}$, of a Kenmotsu manifold ($f = 1$) is called a $\varphi$-section if it spanned by a vector $X$ orthogonal to $\xi$ and $\varphi X$. A connected Kenmotsu manifold $\widetilde{M}$ is called a *Kenmotsu space form* and it is denoted by $\widetilde{M}(c)$ if it has the constant $\varphi$-sectional curvature $c$. The curvature tensor of a Kenmotsu space form $\widetilde{M}(c)$ is given by

$$\begin{aligned} 4R(X,Y)Z &= (c-3)\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + (c+1)\{\eta(X)\eta(Z)Y + -\eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\eta \\ &\quad + g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z\} \end{aligned} \tag{2.6}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$.

Now, let $M(J, g')$ be a $2m$-dimensional almost Hermitian manifold. A surjective map $\pi : \widetilde{M} \to M$ is called a *contact-complex Riemannian submersion* if it is a Riemannian submersion and satisfies [10]

$$\pi_\star \circ \varphi = J \circ \pi_\star. \tag{2.7}$$

In [13], we have proven the following theorem.

**Theorem 2.3.** *Let* $\pi : \widetilde{M} \to M$ *be a contact-complex Riemannian submersion from a* $(2m+1)$*-dimensional Kenmotsu manifold* $\widetilde{M}$ *to a* $2m$*-dimensional almost Hermitian manifold* $M$*. Then,* $M$ *is a Kählerian manifold. Moreover,* $\widetilde{M}$ *is a Kenmotsu space form if and only if* $M$ *is a complex space form .*

## 3. Harmonic maps and stability

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds and $F : M \to N$ a differentiable map. Then, the second fundamental form $\alpha_F$ of $F$ is defined by

$$\alpha_F(X,Y) = \tilde{\nabla}_X F_\star Y - F_\star(\nabla_X Y), \tag{3.1}$$

where $\nabla$ is the Levi-Civita connection on $M$ and $\widetilde{\nabla}$ is the connection induced by $F$ on the bundle $F^{-1}(TN)$, which is the pull-back of the Levi-Civita connection $\nabla'$ on $N$, and satisfies the following formula (see [8]):

$$\widetilde{\nabla}_X F_\star Y - \widetilde{\nabla}_Y F_\star X = F_\star([X, Y]), \quad X, Y \in \Gamma(TM). \tag{3.2}$$

The tension field $\tau(F)$ of $F$ is defined as the trace of the second fundamental form $\alpha_F$, that is $\tau(F)_x = \sum \alpha_F(e_i, e_i)(x)$, where $(e_1, \ldots, e_m)$ is an orthonormal basis for $T_x M$ at $x \in M$.

In what follows, we will use Einstein summation convention, so we will omit the sigma symbol.

We say that a map $F : M \to N$ is a *harmonic map* $\tau(F)$ $x \in M$.

*Examples.* (1) If $M$ is the circle $S^1$, a map $F : S^1 \to (N, g)$ is harmonic if and only if it is a geodesic parametrized proportionally to arc length. (2) If $N = \mathbb{R}$, a harmonic map $F : (M, g) \to \mathbb{R}$ is a harmonic function. (3) A holomorphic map between two Kähler manifolds is harmonic [8]. For examples in the contact metric geometry, see [5, 6, 14].

Now let us consider a variation $F_{s,t} \in C^\infty(M, N)$, with $s, t \in (-\varepsilon, \varepsilon)$ and $F_{0,0} = F$. If the corresponding variation vector fields are denoted by $V$ and $W$, the Hessian of $F$ is given by

$$H_F(V, W) = \int_M h(J_F(V), W)\mathcal{V}_g, \tag{3.3}$$

where $\mathcal{V}_g$ is the canonical measure associated to the Riemannian metric $g$ and $J_F(V)$ is a second-order self-adjoint operator acting on $\Gamma(F^{-1}(TN))$ by

$$J_F(V) = \sum_i \left(\widetilde{\nabla}_{\nabla_{e_i}} e_i - \widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i}\right)V - \sum_i R'(V, F_\star e_i)F_\star e_i, \tag{3.4}$$

where $R'$ is the curvature operator on $(N, h)$.

We say that a map $f : (M, \varphi, \xi, \eta, g) \to (N, J, h)$ from an almost contact metric manifold to an almost Hermitian manifold is a $(\varphi, J)$*-holomorphic map if and only if* $F_\star \circ \varphi = J \circ F_\star$.

If $M(\varphi, \xi, \eta, g)$ is a Sasaki manifold and $N(J, h)$ is a Kähler manifold, then any $(\varphi, J)$-holomorphic map from $M$ to $N$ is a harmonic map [14].

Then, we can prove the same result for any $(\varphi, J)$-holomorphic map from an $f$-Kenmotsu manifold to a Kähler manifold (see also [15]).

Our main result is the following.

**Theorem 3.1.** *Let $M(\varphi, \xi, \eta, g)$ be a compact $f$-Kenmotsu manifold and let $N(J, h)$ be a Kähler manifold. Then, any $(\varphi, J)$-holomorphic map $F : M \to N$ is stable.*

If $M$ is compact, the spectrum of $J_F$ consists only of a discrete set of an infinite number of eigenvalues with finite multiplicities, bounded below by the first one. We define the *Morse index* of the harmonic map $F : M \to N$ as the *sum of multiplicities of negative eigenvalues of the Jacobi operator* $J_F$. Equivalently, the Morse index of $F$ equals the dimension of the largest subspace of $\Gamma(f^{-1}(\mathrm{TN}))$ on which the Hessian $H_F$ is negative definite (see [8, 9]).

We recall the following formula (see [5, 9]):

$$H_F(V,W) = \int_M \left(h\left(\tilde{\nabla}_{e_a}V, \tilde{\nabla}_{e_a}W\right) + h\left(R'\left(F_\star e_a, V\right)F_* e_a, W\right)\right)\mho_g, \tag{3.5}$$

where we omitted the summation symbol for repeated indices $a = 1, \ldots, n$, $n = \dim M$ [5].

Now, let $(e_1, \ldots, e_m; f_1, \ldots, f_m, \xi)$ be a local orthonormal $\varphi$-basis on $M(\varphi, \xi, \eta, g)$ such that $f_i = \varphi e_i$, $i = 1, \ldots, m$.

From the $(\varphi, J)$-holomorphicity of $F$ and by $\varphi\xi = 0$, we have $F_\star\xi = 0$. Thus, from (3.5), we obtain the following.

**Lemma 3.2.** *Let $F : M \to N$ be a $(\varphi, J)$-holomorphic map from an $f$-Kenmotsu manifold $M$ to a Kähler manifold $N$. Then, one has*

$$\begin{aligned} &H_F(V,V) \\ &= \int_M \left(h\left(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{e_i}V\right) + h\left(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{f_i}V\right)\right)\mho_g + \int_M \left(h\left(R'\left(F_\star e_i, V\right)F_\star e_i, V\right) + h\left(R'\left(F_\star f_i, V\right)F_\star f_i, V\right)\right)\mho_g. \end{aligned} \tag{3.6}$$

**Lemma 3.3.** *Let $T$ be a vector field on $M$ such that*

$$g(T, X) = h\left(\tilde{\nabla}_{\varphi X}V, JV\right) \tag{3.7}$$

*for any $X \in \Gamma(D)$, where $D = \operatorname{Ker}\eta$ and $g(T, \xi) = 0$. Then,*

$$\operatorname{div}(T) = h\left(R'\left(F_\star e_i, F_\star f_i\right)V, JV\right) + 2h\left(\tilde{\nabla}_{e_i}JV, \tilde{\nabla}_{f_i}V\right). \tag{3.8}$$

*Proof.* Let

$$\begin{aligned} h\left(R'\left(F_\star e_i, F_\star f_i\right)V, JV\right) &= h\left(\tilde{\nabla}_{e_i}\tilde{\nabla}_{f_i}V - \tilde{\nabla}_{f_i}\tilde{\nabla}_{e_i}V - \tilde{\nabla}_{[e_i,f_i]}V, JV\right) \\ &= e_i h\left(\tilde{\nabla}_{f_i}V, JV\right) - h\left(\tilde{\nabla}_{f_i}V, \tilde{\nabla}_{e_i}JV\right) - f_i h\left(\tilde{\nabla}_{e_i}V, JV\right) \\ &\quad + h\left(\tilde{\nabla}_{e_i}V, \tilde{\nabla}_{f_i}JV\right) - h\left(\tilde{\nabla}_{\nabla_{e_i}f_i}V, JV\right) + h\left(\tilde{\nabla}_{\nabla_{f_i}e_i}V, JV\right). \end{aligned} \tag{3.9}$$

By using (3.7) and (2.3), we obtain

$$\begin{aligned} \operatorname{div}(T) &= g\left(\nabla_{e_i}T, e_i\right) + g\left(\nabla_{f_i}T, f_i\right) + g\left(\nabla_\xi T, \xi\right) \\ &= e_i g\left(T, e_i\right) - g\left(T, \nabla_{e_i}e_i\right) + f_i g\left(T, f_i\right) - g\left(T, \nabla_{f_i}f_i\right) \\ &= e_i h\left(\tilde{\nabla}_{f_i}V, JV\right) - f_i h\left(\tilde{\nabla}_{e_i}V, JV\right) + h\left(\tilde{\nabla}_{\nabla_{f_i}e_i}V, JV\right) + h\left(\tilde{\nabla}_{\nabla_{e_i}f_i}V, JV\right) \end{aligned} \tag{3.10}$$

and (3.8) follows. □

**Proposition 3.4.** *Let $M(\varphi, \xi, \eta, g)$ be a compact $f$-Kenmotsu manifold. Then, the function $f$ satisfies*

$$\int_M f\mho_g = 0. \tag{3.11}$$

*Proof.* We have

$$\operatorname{div}(\xi) = g(e_i, \nabla_{e_i}\xi) + g(f_i, \nabla_{f_i}\xi) + g(\xi, \nabla_{\xi}\xi). \tag{3.12}$$

Using (2.1)–(2.4), we obtain $\operatorname{div}(\xi) = -2nf$. Since $M$ is a compact manifold (without boundary), using Stokes's theorem, we have

$$\int_M \operatorname{div}(\xi)\mathcal{V}_g = 0, \tag{3.13}$$

so that (3.11) follows from (3.13). □

Now we are ready to prove Theorem 3.1. Since $F$ is a $(\varphi, J)$-holomorphic map, by using the curvature Kähler identity $R'(U,V)JW = JR'(U,V)W$ on $N(J,h)$ and Bianchi's identity, we have

$$R'(F_\star e_i, V)F_\star e_i + R'(F_\star f_i, V)F_\star f_i = -JR'(F_\star e_i, F_\star f_i V). \tag{3.14}$$

For any $V \in \Gamma(F^{-1}(TN))$, we define the operator

$$DV : \Gamma(TM) \longrightarrow \Gamma(F^{-1}(TN)) \tag{3.15}$$

by the formula

$$DV(X) = \widetilde{\nabla}_{\varphi X}V - J\widetilde{\nabla}_X V, \tag{3.16}$$

for any $X \in \Gamma(TM)$ (see [5]).

Using Lemmas 3.2, 3.3, and (3.14), by a straightforward calculation, we obtain

$$H_F(V,V) = \frac{1}{2}\int_M \left(h(DV(e_i), DV(e_i)) + h(DV(f_i), DV(f_i))\right)\mathcal{V}_g \tag{3.17}$$

because $\int_M \operatorname{div}(T)\mathcal{V}_g = 0$.

Thus, we have $H_F(V,V) \geq 0$ for any $V \in \Gamma(F^{-1}(TN))$, so that $F$ is a stable harmonic map.

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