Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2009, Article ID 950797, 14 pages
doi:10.1155/2009/950797

Research Article

Existence and Uniqueness of Periodic Solutions for a Second-Order Nonlinear Differential Equation with Piecewise Constant Argument

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Received 4 July 2009; Revised 1 October 2009; Accepted 2 November 2009

Recommended by Enrico Obrecht

Based on a continuation theorem of Mawhin, a unique periodic solution is found for a second-order nonlinear differential equation with piecewise constant argument.

1. Introduction

Qualitative behaviors of first-order delay differential equations with piecewise constant arguments are the subject of many investigations (see, e.g., [1–19]), while those of higher-order equations are not.

However, there are reasons for studying higher-order equations with piecewise constant arguments. Indeed, as mentioned in [10], a potential application of these equations is in the stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant and with a discrete (sampled) controller. As an example, suppose that a moving particle with time variable mass $r(t)$ is subjected to a restoring controller $-\phi(x[t])$ which acts at sampled time $[t]$. Then Newton's second law asserts that

$$\left(r(t)x'(t)\right)' = -\phi(x[t]). \tag{1.1}$$

Since this equation is "similar" to the harmonic oscillator equation

$$\left(r(t)x'(t)\right)' + \kappa x(t) = 0, \tag{1.2}$$

we expect that the well-known qualitative behavior of the later equation may also be found in the former equation, provided appropriate conditions on $r(t)$ and $\phi(x)$ are imposed.

In this paper we study a slightly more general second-order delay differential equation with piecewise constant argument:

$$\left(r(t)x'(t)\right)' + f(t, x([t])) = p(t), \tag{1.3}$$

where $f(t,x)$ is a real continuous function defined on R^2 with positive *integer* period ω for $t; r(t)$ and $p(t)$ are continuous function defined on R with period $\omega, r(t) > 0$ for $t \in R$ and $\int_0^{\omega} p(t)dt = 0$.

By a solution of (1.3) we mean a function $x(t)$ which is defined on R and which satisfies the following conditions: (i) $x'(t)$ is continuous on R, (ii) $r(t)x'(t)$ is differentiable at each point $t \in R$, with the possible exception of the points $[t] \in R$ where one-sided derivatives exist, and (iii) substitution of $x(t)$ into (1.3) leads to an identity on each interval $[n, n+1) \subset R$ with integral endpoints.

In this note, existence and uniqueness criteria for periodic solutions of (1.3) will be established. For this purpose, we will make use of a continuation theorem of Mawhin. Let X and Y be two Banach spaces and $L : \operatorname{Dom} L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$, and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L_{|\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem A (Mawhin's continuation theorem [18]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose that*

(i) *for each* $\lambda \in (0,1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$;

(ii) *for each* $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ *and* $\deg(JQN, \Omega \cap \operatorname{Ker}, 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{dom} L$.

2. Existence and Uniqueness Criteria

Our main results of this paper are as follows.

Theorem 2.1. *Suppose that there exist constants $D > 0$ and $\delta \geqslant 0$ such that*

(i) $f(t,x) \operatorname{sgn} x > 0$ *for* $t \in R$ *and* $|x| > D$,

(ii) $\lim_{x \to -\infty} \max_{0 \le t \le \omega} (f(t,x)/x) \le \delta$ *(or* $\lim_{x \to +\infty} \max_{0 \le t \le \omega} (f(t,x)/x) \le \delta$*).*

If $\omega^2 \delta (\max_{0 \le t \le \omega}(1/r(t))) < 1$, then (1.3) has an ω-periodic solution. Furthermore, the ω-periodic solution is unique if in addition one has the following.

(iii) *$f(t,x)$ is strictly monotonous in x and there exists nonnegative constant $b < (4/\omega^2) \min_{0 \le t \le \omega} r(t)$ such that*

$$\left| f(t,x_1) - f(t,x_2) \right| \le b|x_1 - x_2|, \quad (t,x_1), (t,x_2) \in R^2. \tag{2.1}$$

Theorem 2.2. *Suppose that there exist constants $D > 0$ and $\delta \geqslant 0$ such that*

(i′) $f(t,x)\operatorname{sgn} x < 0$ *for* $t \in R$ *and* $|x| > D$,

(ii′) $\lim_{x\to-\infty}\max_{0\le t\le\omega}(f(t,x)/x) \geqslant -\delta$ *(or* $\lim_{x\to+\infty}\max_{0\le t\le\omega}(f(t,x)/x) \geqslant -\delta$*).*

If $\omega^2\delta(\max_{0\le t\le\omega}(1/r(t))) < 1$, *then* (1.3) *has an* ω*-periodic solution. Furthermore, the* ω*-periodic solution is unique if in addition one has the following.*

(iii) $f(t,x)$ *is strictly monotonous in* x *and there exists nonnegative constant* $b < (4/\omega^2)\min_{0\le t\le\omega} r(t)$ *such that* (2.1) *holds.*

We only give the proof of Theorem 2.1, as Theorem 2.2 can be proved similarly.

First we make the simple observation that $x(t)$ is an ω-periodic solution of the following equation:

$$r(t)x'(t) = r(0)x'(0) - \int_0^t (f(s,x([s])) - p(s))ds, \tag{2.2}$$

if, and only if, $x(t)$ is an ω-periodic solution of (1.3). Next, let X_ω be the Banach space of all real ω-periodic continuously differentiable functions of the form $x = x(t)$ which is defined on R and endowed with the usual linear structure as well as the norm $\|x\|_1 = \sum_{i=0}^{1}\max_{0\le i\le\omega}|x^{(i)}(t)|$. Let Y_ω be the Banach space of all real continuous functions of the form $y = \alpha t + h(t)$ such that $y(0) = 0$, where $\alpha \in R$ and $h(t) \in X_\omega$, and endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$, respectively, by

$$Lx(t) = r(t)x'(t) - r(0)x'(0), \tag{2.3}$$

$$Nx(t) = -\int_0^t (f(s,x([s])) - p(s))ds. \tag{2.4}$$

Let

$$\overline{h}(t) = -\int_0^t (f(s,x([s])) - p(s))ds + \frac{t}{\omega}\int_0^\omega f(s,x([s]))ds. \tag{2.5}$$

Since $\overline{h} \in X_\omega$ and $\overline{h}(0) = 0$, N is a well-defined operator from X_ω to Y_ω. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$, respectively, by

$$Px(t) = x(0), \quad n \in Z \tag{2.6}$$

for $x = x(t) \in X_\omega$ and

$$Qy(t) = \alpha t \tag{2.7}$$

for $y(t) = \alpha t + h(t) \in Y_\omega$.

Lemma 2.3. *Let the mapping L be defined by* (2.3). *Then*

$$\operatorname{Ker} L = R. \tag{2.8}$$

Proof. It suffices to show that if $x(t)$ is a real ω-periodic continuously differentiable function which satisfies

$$r(t)x'(t) = r(0)x'(0), \quad t \in R, \tag{2.9}$$

then $x(t)$ is a constant function. To see this, note that for such a function $x = x(t)$,

$$x'(t) = \frac{r(0)x'(0)}{r(t)}, \quad t \in R. \tag{2.10}$$

Hence by integrating both sides of the above equality from 0 to t, we see that

$$x(t) = x(0) + r(0)x'(0)\int_0^t \frac{ds}{r(s)}, \quad t \in R. \tag{2.11}$$

Since $r(t)$ is positive, continuous, and periodic,

$$\int_0^\infty \frac{ds}{r(s)} = \infty. \tag{2.12}$$

Since $x(t)$ is bounded, we may infer from (2.11) that $x'(0) = 0$. But then (2.9) implies $x'(t) = 0$ for $t \in R$. The proof is complete. □

Lemma 2.4. *Let the mapping L be defined by* (2.3). *Then*

$$\operatorname{Im} L = \{y \in X_\omega \mid y(0) = 0\} \subset Y_\omega. \tag{2.13}$$

Proof. It suffices to show that for each $y = y(t) \in X_\omega$ that satisfies $y(0) = 0$, there is a $x = x(t) \in X_\omega$ such that

$$y(t) = r(t)x'(t) - r(0)x'(0), \quad t \geq 0. \tag{2.14}$$

But this is relatively easy, since we may let

$$\alpha = \frac{1}{\int_0^\omega (ds/r(s))}, \tag{2.15}$$

$$x(t) = \int_0^t \frac{y(s)}{r(s)}ds - \alpha\int_0^\omega \frac{y(s)}{r(s)}ds\int_0^t \frac{ds}{r(s)}, \quad t \geq 0. \tag{2.16}$$

Then it may easily be checked that (2.14) holds. The proof is complete. □

Lemma 2.5. *The mapping L defined by* (2.3) *is a Fredholm mapping of index zero.*

Indeed, from Lemmas 2.3 and 2.4 and the definition of Y_ω, $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1 < +\infty$. From (2.13), we see that $\operatorname{Im} L$ is closed in Y_ω. Hence L is a Fredholm mapping of index zero.

Lemma 2.6. *Let the mapping L, P, and Q be defined by* (2.3), (2.6), *and* (2.7), *respectively. Then* $\operatorname{Im} P = \operatorname{Ker} L$ *and* $\operatorname{Im} L = \operatorname{Ker} Q$.

Indeed, from Lemmas 2.3 and 2.4 and defining conditions (2.6) and (2.7), it is easy to see that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q$.

Lemma 2.7. *Let L and N be defined by* (2.3) *and* (2.4), *respectively. Suppose that Ω is an open and bounded subset of X_ω. Then N is L-compact on $\overline{\Omega}$.*

Proof. It is easy to see that for any $x \in \overline{\Omega}$,

$$QNx(t) = -\frac{t}{\omega}\int_0^\omega f(s, x([s]))ds, \tag{2.17}$$

so that

$$\|QNx\|_2 = \left|\frac{1}{\omega}\int_0^\omega f(s, x([s]))ds\right|, \tag{2.18}$$

$$(I-Q)Nx(t) = -\int_0^t \big(f(s, x([s])) - p(s)\big)ds + \frac{t}{\omega}\int_0^\omega f(s, x([s]))ds, \quad t \geq 0. \tag{2.19}$$

These lead us to

$$\begin{aligned} K_P(I-Q)Nx(t) = &-\int_0^t \frac{1}{r(v)}dv\int_0^v \big(f(s, x([s])) - p(s)\big)ds \\ &+ \alpha\left(\int_0^\omega \frac{dv}{r(v)}\int_0^v \big(f(s, x([s])) - p(s)\big)ds\right)\int_0^t \frac{1}{r(v)}dv \\ &+ \frac{1}{\omega}\int_0^t \frac{v}{r(v)}dv\int_0^\omega f(s, x([s]))ds \\ &- \frac{\alpha}{\omega}\left(\int_0^\omega \frac{vdv}{r(v)}\int_0^\omega f(s, x([s]))ds\right)\int_0^t \frac{1}{r(v)}dv, \end{aligned} \tag{2.20}$$

where α is defined by (2.15). By (2.18), we see that $QN(\overline{\Omega})$ is bounded. Noting that (2.7) holds and N is a completely continuous mapping, by means of the Arzela-Ascoli theorem we know that $K_P(I-Q)N(\overline{\Omega})$ is relatively compact. Thus N is L-compact on $\overline{\Omega}$. The proof is complete. □

Lemma 2.8. *Suppose that* $g(t)$ *is a real, bounded and continuous function on* $[a,b)$ *and* $\lim_{x\to b^-} g(t)$ *exists. Then there is a point* $\xi \in (a,b)$ *such that*

$$\int_a^b g(s)ds = g(\xi)(b-a). \tag{2.21}$$

The above result is only a slight extension of the integral mean value theorem and is easily proved.

Lemma 2.9. *Suppose that condition (i) in Theorem 2.1 holds. Suppose further that* $x(t) \in X_\omega$ *satisfies*

$$\int_0^\omega f(s, x([s]))ds = 0. \tag{2.22}$$

Then there is $t_1 \in [0,\omega]$ *such that* $|x(t_1)| \leq D$.

Proof. From (2.22) and Lemma 2.8, we have $\xi_i \in (i-1, i)$ for $i = 1, \ldots, \omega$ such that

$$\sum_{i=1}^{\omega} f(\xi_i, x(i-1)) = \sum_{i=1}^{\omega} \int_{i-1}^{i} f(s, x([s]))ds \int_0^\omega f(s, x([s]))ds = 0. \tag{2.23}$$

In case $\omega = 1$, from the condition (i) in Theorem 2.1 and (2.23), we know that $|x(0)| \leq D$. Suppose $\omega \geq 2$. Our assertion is true if one of $x(0), x(1), \ldots, x(\omega-1)$ has absolute value less than or equal to D. Otherwise, there should be $x(\eta_1)$ and $x(\eta_2)$ among $x(0), x(1), \ldots$ and $x(\omega-1)$ such that $x(\eta_1) > D$ and $x(\eta_2) < -D$. Since $x(t)$ is continuous, in view of the intermediate value theorem, there is $x(\eta_3)$ such that $-D \leq x(\eta_3) \leq D$, (here $\eta_1 > \eta_3 > \eta_2$ or $\eta_2 > \eta_3 > \eta_1$). Since $x(t)$ is periodic, there is $t_1 \in [0,\omega]$ such that $|x(t_1)| = |x(\eta_3)| \leq D$. The proof is complete. □

Now, we consider that following equation:

$$r(t)x'(t) - r(0)x'(0) = -\lambda \int_0^t (f(s, x([s])) - p(s))ds, \tag{2.24}$$

where $\lambda \in (0,1)$.

Lemma 2.10. *Suppose that conditions (i) and (ii) of Theorem 2.1 hold. If* $\omega^2 \delta(\max_{0\leq t\leq\omega}(1/r(t))) < 1$, *then there are positive constants* D_0 *and* D_1 *such that for any* ω*-periodic solution* $x(t)$ *of* (2.24),

$$\left|x^{(i)}(t)\right| \leq D_i, \quad t \in [0,\omega]; \ i = 0, 1. \tag{2.25}$$

Proof. Let $x(t)$ be a ω-periodic solution of (2.24). By (2.24) and our assumption that $\int_0^\omega p(s)ds = 0$, we have

$$\int_0^\omega f(s, x([s]))ds = 0. \tag{2.26}$$

By Lemma 2.9, there is $t_1 \in [0,\omega]$ such that

$$|x(t_1)| \le D. \tag{2.27}$$

Since $x(t)$ and $x'(t)$ are with period ω, thus for any $t \in [t_1, t_1+\omega]$, we have

$$\begin{gathered} x(t) = x(t_1) + \int_{t_1}^{t} x'(s)ds, \\ x(t) = x(t_1+\omega) + \int_{t_1+\omega}^{t} x'(s)ds = x(t_1) + \int_{t_1+\omega}^{t} x'(s)ds. \end{gathered} \tag{2.28}$$

From (2.28), we see that for any $t \in [t_1, t_1+\omega]$,

$$|x(t)| \le |x(t_1)| + \frac{1}{2}\int_{t_1}^{t_1+\omega} \left|x'(s)\right|ds = |x(t_1)| + \frac{1}{2}\int_{0}^{\omega} \left|x'(s)\right|ds. \tag{2.29}$$

It is easy to see from (2.27) and (2.29) that for any $t \in [0,\omega]$

$$|x(t)| \le |x(t_1)| + \frac{1}{2}\int_{0}^{\omega} \left|x'(s)\right|ds \le D + \frac{1}{2}\int_{0}^{\omega} \left|x'(s)\right|ds. \tag{2.30}$$

In view of the condition $\omega^2\delta(\max_{0\le t\le\omega}(1/r(t))) < 1$, we know that there is a positive number ε such that

$$\eta_1 := \omega^2(\delta+\varepsilon)\left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right) < 1. \tag{2.31}$$

From condition (ii), we see that there is a $\rho > D$ such that for $t \in R$ and $x < -\rho$,

$$\frac{f(t,x)}{x} < \delta + \varepsilon. \tag{2.32}$$

Let

$$E_1 = \{t \mid t \in [0,\omega], x([t]) < -\rho\}, \tag{2.33}$$

$$E_2 = \{t \mid t \in [0,\omega], |x([t])| \le \rho\}, \tag{2.34}$$

$$E_3 = [0,\omega] \setminus (E_1 \cup E_2), \tag{2.35}$$

$$M_0 = \max_{0\le t\le\omega, |x|\le\rho}\left|f(t,x)\right|. \tag{2.36}$$

By (2.32) and (2.33), we have

$$\int_{E_1}\left|f(s,x([s]))\right|ds \le (\delta+\varepsilon)\int_{E_1}|x([s])|ds \le (\delta+\varepsilon)\omega\max_{0\le t\le\omega}|x(t)|. \tag{2.37}$$

From (2.34) and (2.36), we have

$$\int_{E_2} \left| f(s, x([s])) \right| ds \le \omega M_0. \tag{2.38}$$

In view of condition (i), (2.26), (2.37), and (2.38), we get

$$\begin{aligned}
\int_{E_3} \left| f(s, x([s])) \right| ds &= \int_{E_3} f(s, x([s])) ds \\
&= -\int_{E_1} f(s, x([s])) ds - \int_{E_2} f(s, x([s])) ds \\
&\le \int_{E_1} \left| f(s, x([s])) \right| ds + \int_{E_2} \left| f(s, x([s])) \right| ds \\
&\le (\delta + \varepsilon)\omega \max_{0 \le t \le \omega} |x(t)| + \omega M_0.
\end{aligned} \tag{2.39}$$

It follows from (2.37), (2.38), and (2.39) that

$$\begin{aligned}
\int_0^{\omega} \left| f(s, x([s])) \right| ds &= \int_{E_1} \left| f(s, x([s])) \right| ds + \int_{E_2} \left| f(s, x([s])) \right| ds + \int_{E_3} \left| f(s, x([s])) \right| ds \\
&\le 2(\delta + \varepsilon)\omega \max_{0 \le t \le \omega} |x(t)| + 2\omega M_0.
\end{aligned} \tag{2.40}$$

Since $x(0) = x(\omega)$, thus there is a $t_1 \in (0, \omega)$ such that $x'(t_1) = 0$. In view of (2.24) and the fact that $x'(t_1) = 0$, we conclude that for any $t \in [t_1, t_1 + \omega]$,

$$\begin{aligned}
\left| r(t)x'(t) \right| &= \left| r(t_1)x'(t_1) - \lambda \int_{t_1}^{t} \left(f(s, x([s])) - p(s) \right) ds \right| \\
&= \left| -\lambda \int_{t_1}^{t} \left(f(s, x([s])) - p(s) \right) ds \right| \\
&\le \left| \int_{t_1}^{t} \left(f(s, x([s])) - p(s) \right) ds \right| \\
&\le \int_{t_1}^{t_1+\omega} \left| f(s, x([s])) \right| ds + \int_{t_1}^{t_1+\omega} \left| p(s) \right| ds \\
&\le \int_0^{\omega} \left| f(s, x([s])) \right| ds + \int_0^{\omega} \left| p(s) \right| ds.
\end{aligned} \tag{2.41}$$

From (2.40) and (2.41), we see that

$$\max_{0\le t\le\omega}\left|x'(t)\right| \le \left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)\left\{2(\delta+\varepsilon)\omega\max_{0\le t\le\omega}|x(t)| + 2\omega M_0 + \max_{0\le t\le\omega}\left|p(t)\right|\right\}. \tag{2.42}$$

It follows from (2.30), (2.31), and (2.42) that

$$\begin{aligned}\max_{0\le t\le\omega}|x(t)| &\le D + \frac{1}{2}\int_0^\omega \left|x'(s)\right|ds\\ &\le \omega^2\left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)(\delta+\varepsilon)\max_{0\le t\le\omega}|x(t)| + M_1\\ &= \eta_1\max_{0\le t\le\omega}|x(t)| + M_1,\end{aligned} \tag{2.43}$$

where

$$M_1 = D + \left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)\left(2\omega M_0 + \max_{0\le t\le\omega}\left|p(t)\right|\right). \tag{2.44}$$

Let $D_0 = M_1/(1-\eta_1)$, then from (2.43) we have

$$\max_{0\le t\le\omega}|x(t)| \le D_0. \tag{2.45}$$

From (2.42) and (2.45), for any $t \in [0,\omega]$, we have

$$\max_{0\le t\le\omega}\left|x'(t)\right| \le D_1, \tag{2.46}$$

where

$$D_1 = \left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)\left\{2(\delta+\varepsilon)\omega D_0 + 2\omega M_0 + \max_{0\le t\le\omega}\left|p(t)\right|\right\}. \tag{2.47}$$

The proof is complete. □

Lemma 2.11. *Suppose that condition (iii) of Theorem 2.1 is satisfied. Then* (1.3) *has at most one ω-periodic solution.*

Proof. Suppose that $x_1(t)$ and $x_2(t)$ are two ω-periodic solutions of (1.3). Set $z(t) = x_1(t) - x_2(t)$. Then we have

$$\left(r(t)z'(t)\right)' + f(t, x_1([t])) - f(t, x_2([t])) = 0. \tag{2.48}$$

Case (i). For all $t \in [0,\omega]$, $z(t) \neq 0$. Without loss of generality, we assume that $z(t) > 0$, that is, $x_1(t) > x_2(t)$ for $t \in [0,\omega]$. Integrating (2.48) from 0 to ω, we have

$$\int_0^{\omega} [f(t, x_1(x_1([t]))) - f(t, x_2([t]))]\, dt = 0. \tag{2.49}$$

Combining condition (iii) and $x_1(t) > x_2(t)$, either

$$f(t, x_1([t])) - f(t, x_2([t])) > 0, \quad t \in [0,\omega] \tag{2.50}$$

or

$$f(t, x_1([t])) - f(t, x_2([t])) < 0, \quad t \in [0,\omega] \tag{2.51}$$

holds. This is contrary to (2.49).

Case (ii). There exist $\xi \in [0,\omega]$ such that $z(\xi) = 0$. As in the proof of (2.30) in Lemma 2.10, we have

$$\max_{0 \le t \le \omega} |z(t)| \le |z(\xi)| + \frac{1}{2}\int_0^{\omega} |z'(s)|\, ds = \frac{1}{2}\int_0^{\omega} |z'(s)|\, ds. \tag{2.52}$$

On the other hand, since $z(0) = z(\omega)$, thus there is a $t_1 \in (0,\omega)$ such that $z'(t_1) = 0$. In view of (2.48), we conclude that for any $t \in [t_1, t_1 + \omega]$,

$$\begin{aligned}
r(t)z'(t) &= r(t_1)z'(t_1) - \int_{t_1}^{t} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds, \\
r(t)z'(t) &= r(t_1 + \omega)z'(t_1 + \omega) - \int_{t_1+\omega}^{t} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds \\
&= r(t_1)z'(t_1) - \int_{t_1+\omega}^{t} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds.
\end{aligned} \tag{2.53}$$

By (2.53) and the fact that $z'(t_1) = 0$, we have for any $t \in [t_1, t_1 + \omega]$,

$$\begin{aligned}
r(t)z'(t) &= r(t_1)z'(t_1) - \frac{1}{2}\int_{t_1}^{t} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds \\
&\quad + \frac{1}{2}\int_{t}^{t_1+\omega} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds. \\
&= -\frac{1}{2}\int_{t_1}^{t} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds \\
&\quad + \frac{1}{2}\int_{t}^{t_1+\omega} \big(f(s, (x_1([s]))) - f(s, x_2([s]))\big)\, ds.
\end{aligned} \tag{2.54}$$

It follows that for any $t \in [t_1, t_1 + \omega]$,

$$\begin{aligned}\left|r(t)z'(t)\right| &\le \frac{1}{2}\int_{t_1}^{t_1+\omega}\left|f(s,(x_1([s]))) - f(s,x_2([s]))\right|ds \\ &\le \frac{1}{2}\int_{0}^{\omega}\left|f(s,(x_1([s]))) - f(s,x_2([s]))\right|ds \\ &\le \frac{1}{2}b\omega\max_{0\le t\le\omega}|z(t)|.\end{aligned} \tag{2.55}$$

We know that for any $t \in [0, \omega]$,

$$\left|r(t)z'(t)\right| \le \frac{1}{2}b\omega\max_{0\le t\le\omega}|z(t)|. \tag{2.56}$$

From (2.56), we have

$$\max_{0\le t\le\omega}\left|z'(t)\right| \le \frac{b\omega}{2}\left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)\max_{0\le t\le\omega}|z(t)|. \tag{2.57}$$

By (2.52), we get

$$\max_{0\le t\le\omega}|z(t)| \le \frac{\omega}{2}\max_{0\le t\le\omega}\left|z'(t)\right|. \tag{2.58}$$

It is easy to see from (2.57) and (2.58) that

$$\max_{0\le t\le\omega}|z(t)| \le \frac{b\omega^2}{4}\left(\max_{0\le t\le\omega}\frac{1}{r(t)}\right)\max_{0\le t\le\omega}|z(t)|. \tag{2.59}$$

By condition (iii) of Theorem 2.1, we see that $(b\omega^2/4)(\max_{0\le t\le\omega}(1/r(t))) < 1$. Thus (2.58) leads us to $\max_{0\le t\le\omega}|z(t)| = 0$, which is contrary to $x_1 \neq x_2$. So (1.3) has at most one ω-periodic solution. The proof is complete. □

We now turn to the proof of Theorem 2.1. Suppose $\omega^2\delta(\max_{0\le t\le\omega}(1/r(t))) < 1$. Let L, N, P, and Q be defined by (2.3), (2.4), (2.6), and (2.7), respectively. By Lemma 2.10, there are positive constants D_0 and D_1 such that for any ω-periodic solution $x(t)$ of (2.24) such that (2.25) holds. Set

$$\Omega = \left\{x \in X_\omega \mid \|x\|_1 < \overline{D}\right\}, \tag{2.60}$$

where $\overline{D}$ is a fixed number which satisfies $\overline{D} > D + D_0 + D_1$. It is easy to see that Ω is an open and bounded subset of X_ω. Furthermore, in view of Lemmas 2.5 and 2.7, L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. Noting that $\overline{D} > D_0 + D_1$, by Lemma 2.10, for each $\lambda \in (0,1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Next note that a function $x \in \partial\Omega \cap \operatorname{Ker} L$ must be constant: $x(t) \equiv \overline{D}$ or $x(t) \equiv -\overline{D}$. Hence by (i) and (2.17), $x(t) \equiv -\overline{D}$. Hence by conditions (i), (iii) and (2.17),

$$QNx(t) = -\frac{t}{\omega}\int_0^{\omega} f(s, x([s]))ds = -\frac{t}{\omega}\int_0^{\omega} f(s, x)ds, \tag{2.61}$$

so $QNx \neq \theta_2$. The isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is defined by $J(t\alpha) = \alpha$ for $\alpha \in R$ and $t \in R$. Then

$$JQNx = -\frac{1}{\omega}\int_0^{\omega} f(s, x)ds\frac{1}{\omega} \neq 0. \tag{2.62}$$

In particular, we see that if $x = \overline{D}$, then

$$JQNx = -\frac{1}{\omega}\int_0^{\omega} f\left(s, \overline{D}\right)ds < 0, \tag{2.63}$$

and if $x = -\overline{D}$, then

$$JQNx = -\frac{1}{\omega}\int_0^{\omega} f\left(s, -\overline{D}\right)ds > 0. \tag{2.64}$$

Consider the mapping

$$H(x, \mu) = \mu x + (1-\mu)JQNx, \quad 0 \leq \mu \leq 1. \tag{2.65}$$

From (2.63) and (2.65), for each $\mu \in [0, 1]$ and $x = \overline{D}$, we have

$$H(x, \mu) = \mu\overline{D} + (1-\mu)\frac{-1}{\omega}\int_0^{\omega} f\left(s, \overline{D}\right)ds < 0. \tag{2.66}$$

Similarly, from (2.64) and (2.65), for each $\mu \in [0, 1]$ and $x = -\overline{D}$, we have

$$H(x, \mu) = \mu\overline{D} + (1-\mu)\frac{-1}{\omega}\int_0^{\omega} f\left(s, -\overline{D}\right)ds < 0. \tag{2.67}$$

By (2.66) and (2.67), $H(x, \mu)$ is a homotopy. This shows that

$$\deg(JQNx, \Omega \cap \operatorname{Ker} L, \theta_1) = \deg(-x, \Omega \cap \operatorname{Ker} L, \theta_1) \neq 0. \tag{2.68}$$

By Theorem A, we see that equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$. In other words, (1.3) has an ω-periodic solution $x(t)$. Furthermore, if (iii) is satisfied, from Lemma 2.11, we know that (1.3) has an ω-periodic solution only. The proof is complete.

3. Example

Consider the equation

$$\left(x'(t)\exp\left(-2-\cos\frac{2\pi t}{5}\right)\right)' + \left(3-\sin\frac{2\pi t}{5}\right)\arctan x([t]) = \cos\frac{2\pi t}{5}, \tag{3.1}$$

and we can show that it has a nontrivial 5-periodic solution. Indeed, take

$$\begin{gathered} r(t) = \exp\left(2-\cos\frac{2\pi t}{5}\right), \qquad p(t) = \cos\frac{2\pi t}{5}, \\ f(t,x) = \frac{1}{100}\left(3-\sin\frac{2\pi t}{5}\right)\arctan x. \end{gathered} \tag{3.2}$$

We see that $\min_{0\le t\le 5} r(t) = e$. Let $D > 0$ and $\delta = b = 1/25$. Then condition (i) of Theorem 2.1 is satisfied:

$$\lim_{x\to-\infty}\max_{0\le t\le\omega}\frac{f(t,x)}{x} = \frac{1}{25}. \tag{3.3}$$

Let $D > 0$ and $\delta = b = 1/25$. Then conditions (i), (ii) and (iii), of Theorem 2.1 are satisfied. Note further that $5^2\delta(\max_{0\le t\le\omega}(1/r(t))) = e^{-1} < 1$. Therefore (3.1) has exactly one 5-periodic solution. Furthermore, it is easy to see that any solution of (3.1) must be nontrivial. We have thus shown the existence of a unique nontrivial 5-periodic solution of (3.1).

Acknowledgment

The first author is supported by the Natural Science Foundation of Guangdong Province of China under Grant no. 9151008002000012.

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