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*Research Article*

# On Maximal Ideals of Compact Connected Topological Semigroups

**Phoebe McLaughlin,[1] Shing S. So,[1] and Haohao Wang[2]**

[1] *Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, MO 64093, USA*
[2] *Department of Mathematics, Southeast Missouri State University, Cape Girardeau, MO 63701, USA*

Correspondence should be addressed to Shing S. So, so@ucmo.edu

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Several results concerning ideals of a compact topological semigroup $S$ with $S^2 = S$ can be found in the literature. In this paper, we further investigate in a compact connected topological semigroup $S$ how the conditions $S^2 = S$ and $S^2 \neq S$ affect the structure of ideals of $S$, especially the maximal ideals.

## 1. Introduction

First, we list some standard definitions which can be found in [1–3].

*Definition 1.1.* A *topological semigroup* is a topological space $S$ together with a continuous function $m : S \times S \to S$ such that $S$ is Hausdorff and $m$ is associative.

A *subsemigroup* of a semigroup $S$ is a nonvoid set $A \subset S$ such that $A^2 \subset A$, and $A$ is called a *subgroup* of $S$ if it is a group with respect to $m$.

An element $e$ of a topological semigroup $S$ is called an *idempotent* if $e^2 = e$. Similarly, an element $e$ of $S$ is called a *left identity* (*right identity*) if $ea = a$ ($ae = a$) for all $a \in S$. An element of $S$ is called an *identity* of $S$ if it is both a left and a right identity of $S$.

The set of all idempotents of $S$ will be denoted by $E$ throughout this paper. For each $e \in E$, let $H(e)$ be the union of all subgroups of $S$ containing $e$. It is shown in [3] that $H(e)$ is the maximal subgroup of $S$ containing $e$.

*Definition 1.2.* A nonempty subset $A$ of a semigroup $S$ is called a *left ideal* (*right ideal*) of $S$ if $SA \subset A$ ($AS \subset A$) and an *ideal* if it is both a left and a right ideal. A left ideal (right ideal, ideal) is said to be *proper* if it is not $S$ itself.

An (*left, right*) *ideal* $M$ of a semigroup $S$ is called *minimal* if it does not properly contain any (*left, right*) ideal of $S$. It follows that there can be at most one minimal ideal of $S$. If $S$ has a minimal ideal $K$, then $K$ is called the *kernel* of $S$.

A *maximal* (left, right) *ideal* of a semigroup $S$ is a proper (left, right) ideal of $S$ that is not properly contained in any other (left, right) ideal.

*Definition 1.3.* Let $A$ be a subset of a topological semigroup $S$, then $J_0(A)$ is defined as follows:

$$J_0(A) = \begin{cases} \emptyset & \text{if } A \text{ contains no ideal of } S, \\ \cup & \{I : I \text{is an ideal of } S \text{ and } \ I \subset A\}. \end{cases} \tag{1.1}$$

**Theorem 1.4.** *Let $S$ be a compact connected topological semigroup without zero, and let $K$ be the kernel of $S$. Then, either $K \cap E$ is infinite or $K$ is a topological subgroup of $S$.*

*Proof.* Since $S$ is a compact topological semigroup, $K = \cup\{H(e) : e \in K \cap E\}$, and $H(e) = eSe$ by [3, Theorem1.2.6]. Suppose that $K \cap E$ is finite and $K$ is not a topological subgroup of $S$. Let $e_K \in K \cap E$. Then, $K \setminus H(e_K) \neq \emptyset$. Otherwise, $K = H(e_K)$ is both the kernel and a maximal subgroup of $S$ by [3, Theorem 1.3.14], and hence $K$ is topological subgroup of $S$ with the relative topology, which contradicts our assumption.

Furthermore, since $K \cap E$ is finite and $K \setminus H(e_K) = \cup\{H(e) : e \in K \cap E, e \neq e_K\}$, it follows that $K \setminus H(e_K)$ and $H(e_K)$ form a separation of $K$. Hence, $K$ is disconnected, which contradicts [1, Theorem 1.28]. Therefore, we can deduce that either $K \cap E$ is infinite or $K$ is a maximal subgroup of $S$. □

## 2. Maximal Ideals of Compact Connected Topological Semigroups

The following theorem is a summary of the results found in [1]. It lists necessary and sufficient conditions for $S^2 = S$ in a compact topological semigroup $S$. In this section, we characterize maximal ideals in a compact connected topological semigroup $S$ with $S^2 = S$ and $S^2 \neq S$.

**Theorem 2.1.** *Let $S$ be a compact connected topological semigroup. The following are equivalent:*

(a) $S^2 = S$,

(b) $E \cap (S \setminus I) \neq \emptyset$ *for each proper ideal $I$ of $S$,*

(c) $S = SES$.

*The following theorem and corollary are results from [3], which are useful for our discussion.*

**Theorem 2.2.** *Let $S$ be a compact topological semigroup. Then, any proper (left, right) ideal of $S$ is contained in a maximal (left, right) ideal of $S$, and each maximal (left, right) ideal is open.*

**Corollary 2.3.** *If $S$ is a compact connected topological semigroup and $J$ a maximal ideal of $S$, then $J$ is dense in $S$.*

**Theorem 2.4.** *Suppose that $S$ is a compact topological semigroup and $S^2 \neq S$.*

(a) *For each $a \in S \setminus S^2$, $S \setminus \{a\}$ is a maximal ideal of $S$.*

(b) *If $S$ has more than one connected maximal ideal, then, $S$ is connected.*

*Proof.* (a) Let $a \in S \setminus S^2$. For every $x \in S \setminus \{a\}$ and $y \in S$, $\{xy, yx\} \subset S^2 \subset S \setminus \{a\}$ implies that $S \setminus \{a\}$ is a proper ideal of $S$. (b) Let $M_1$ and $M_2$ be two distinct connected maximal ideals of $S$. Suppose that $S$ is disconnected. Then, $M_1 \cup M_2 = S = P \cup Q$ such that $\overline{P} \cap Q = \emptyset = P \cap \overline{Q}$. Since $M_1$ and $M_2$ are connected, $M_1 \subset P$ and $M_2 \subset P$. It follows that $M_1 \cap M_2 = \emptyset$, and hence $M_1 \subset S \setminus M_2 = \{a_2\}$ and $M_2 \subset S \setminus M_1 = \{a_1\}$. On the other hand, since $M_1$ and $M_2$ are ideals, $a_1 a_2 = a_2 a_1 = a_2$ and $a_1 a_2 = a_2 a_1 = a_1$, and hence $M_1 = \{a_2\} = \{a_1\} = M_2$ contradicting $M_1$ and $M_2$ being distinct. Therefore, $S$ is connected, and hence $K$ is connected.

The following example shows that the condition $S$ having more than one connected maximal ideal is a necessary condition for Theorem 2.4(b). □

*Example 2.5.* Let $S = [0, 1/4] \cup \{1/2\}$ with the usual topology and the usual multiplication. Then, $S^2 = [0, 1/8] \cup \{1/4\} \neq S$, $K = \{0\}$ is connected, $M = [0, 1/4]$ is the only connected maximal ideal of $S$, and $S$ is disconnected.

The next theorem is Theorem 2.4.3 of [3], and hence the proof is omitted.

**Theorem 2.6.** *If $S$ is a connected topological semigroup and $I$ an ideal of $S$, then one and only one component of $I$ is an ideal of $S$.*

*One will call the ideal in Theorem 2.6 the component ideal of $I$.*

**Theorem 2.7.** *Let $S$ be a compact connected topological semigroup and $C = \bigcup\{M_C : M_C$ is the ideal component of a maximal proper ideal $M\}$. Then either $C = S$ or $C$ is the maximal proper connected ideal of $S$. Furthermore, if $C \neq S$, then $C$ is the component ideal of a maximal ideal of $S$.*

*Proof.* For each maximal ideal $M$ of $S$, let $M_C$ be its component ideal. Since $K$ is the kernel and $K \subset M_C$ for each $M_C$, $C = \bigcup\{M_C : M_C$ is the ideal component of a maximal proper ideal $M\}$ is a connected ideal.

Suppose that there is a connected ideal $I$ such that $C \subset I \subsetneq S$, then $I$ is contained in a maximal ideal $M$ of $S$. Since $K \subset I \cap M_C$, $I \cup M_C$ is a connected ideal of $S$ and is contained in $M$, and hence $I \cup M_C \subset M_C \subset C$, a contradiction. Thus, if $C \neq S$, then $C$ is the maximal connected proper ideal of $S$. Furthermore, there exists a maximal ideal $M$ of $S$ such that $C \subset M$. Let $M_C$ be the component ideal of $M$. Then, $M_C = C$. □

**Lemma 2.8.** *Let $S$ be a compact connected topological semigroup, $M$ a maximal ideal of $S$, and $M_C$ the component ideal of $M$. If $S^2 \neq S$, then $M_C$ is not closed in $S$.*

*Proof.* If $M_C = M$, then the result follows from Theorem 2.4(b).

If $M_C \subsetneq M$, then $M = M_C \cup K_M$ where $K_M$ is the union of all components of $M$ except $M_C$. If $M_C$ were closed in $S$, then $K_M$ is open in $S$ because $K_M = M \cap (S \setminus M_C)$ and $M$ are both open. Therefore, for $a \in S \setminus M$, $S = \overline{M} = M_C \cup (K_M \cup \{a\})$, and hence $S$ is disconnected, which is a contradiction. □

The next theorem provides a necessary and sufficient condition for a compact connected topological semigroup $S$ satisfying $S^2 \neq S$ by means of the component ideals of its maximal ideals.

**Theorem 2.9.** *Let $S$ be a compact connected topological semigroup. Then, $S^2 \neq S$ if and only if there exists a maximal ideal $M$ of $S$ with $M = S \setminus \{b\}$, $b \in S \setminus S^2$ such that $S^2 \subset M_C$ where $M_C$ is a component ideal of $M$.*

*Proof.* Suppose that $S^2 \neq S$. It follows from Theorem 2.1(a) that there exists a maximal ideal $M$ of $S$ such that $E \cap (S \setminus M) = \emptyset$. By [3, Theorem 1.3.8], $S/M$ is either the zero semigroup of order two or else completely 0-simple.

Suppose that $S/M$ is the zero semigroup of order two. Then, $S \setminus M = \{b\}$ for some $b \in S$. If $b \in S^2$, then $b = xy$ with $x, y \in S \setminus M$. It is because if $\{x, y\} \cap M \neq \emptyset$, then $b \in M$ contradicting $S \setminus M = \{b\}$. It follows that $x = y = b$, and hence $b \in E$. This contradicts $E \cap (S \setminus M) = \emptyset$. Therefore, $b \in S \setminus S^2$ and $S^2 \subset M_C \subset M \setminus \{b\}$. Note that the semigroup $S/M$ is not completely 0-simple because if $S/M$ were completely 0-simple, then $S/M$ contains a nonzero primitive idempotent, which contradicts $E \cap (S \setminus M) = \emptyset$.

The converse is obviously true.The next example shows that the component ideal $M_C$ of a maximal ideal $M$ can be $M$ itself. □

*Example 2.10.* Let $S = [0, 1/2]$ with the usual multiplication and the usual topology. Then, $S$ is a compact connected topological semigroup, and $S^2 \neq S$. Let $M = [0, 1/2)$ and $M^{\#} = S \setminus \{5/16\}$. Then, $M$ and $M^{\#}$ are maximal ideals of $S$, and $M_C = [0, 1/2) = M$ and $M^{\#}_C = [0, 5/16) \subsetneq M^{\#}$.

The next theorem is Theorem 1.40 of [1], and hence the proof is omitted.

**Theorem 2.11.** *Let $S$ be a compact connected topological semigroup. Then, $S^2 = S$ if and only if each dense (left, right) ideal (containing $K$) is connected.*

*When $S^2 = S$, it is possible that $aS = S$ for some $a \in S$. Existence of the set $P = \{\alpha \in S : \alpha S = S\}$ and its relationship to maximal ideals have been discussed in [3]. The following theorem provides a few additional properties of the set $P$ of a compact topological semigroup $S$.*

**Theorem 2.12.** *Suppose that $S$ is a compact topological semigroup such that $aS = S$ for some $a \in S$. Let $P = \{\alpha \in S : \alpha S = S\}$. Then, the following is considered.*

(a) *$P$ is a right group.*

(b) *If $P \neq S$, Then $S \setminus P$ is dense in $S$ or $S$ is disconnected.*

(c) *$J_0(S \setminus \{a\})$ is dense in $S$ for each $a \in P$ if $S$ is connected and $P \neq S$.*

*Proof.* (a) According to [3, Theorem 1.4.6], $P = \bigcup_{e \in E \cap P} H(e)$, and $P$ is a subtopological semigroup of $S$. Then, $eS = S$ for all $e \in E \cap P$, and hence $e$ is a left identity of $S$. For each $a \in P$, $a \in H(e)$ for some $e \in E \cap P$, and hence there exists $a^{-1} \in H(e)$ such that $aa^{-1} = e$. For any $x \in P$, $x = (aa^{-1})x = a(a^{-1}x) \in aP$. It follows that $P = aP$ for every $a \in P$, and hence $P$ is right simple since $S$ is compact and $P$ is closed. The result follows from Theorem 1 of [4]. □

(b) Since $P$ is a nonempty closed subtopological semigroup of $S$ and the kernel $K$ exists, $S \setminus P$ is nonempty. In fact, by [3, Theorem 1.4.7], $S \setminus P$ is the only maximal ideal of $S$ because $S \neq P \neq \emptyset$. If $\overline{S \setminus P} \neq S$, then $S \setminus P$ is both open and closed by the maximality, and hence $S$ is disconnected.

(c)The result follows immediately from part (b) and the fact that $S \setminus P \subset J_0(S \setminus \{a\})$ for every $a \in P$.

The following example shows that the condition $S \neq P$ is necessary for Theorem 2.12(b) and (c).

*Example 2.13.* Let $S = [0, 1]$ with the usual topology and the multiplication $xy = y$ for $x, y \in S$. Then, $S = P = K$.

*Definition 2.14.* A topological semigroup $S$ has the *left maximal property* (*right maximal property*) if there exists a maximal left (right) ideal $L^*$ ($R^*$) containing every proper left (right) ideal of $S$.

In [3], Paalmande Miranda presented several results showing how a compact connected topological semigroup $S$ with the left or right maximal property is related to the condition $S = aS$, where $a \in S$. In the same spirit of these results and Theorem 2.11, the following theorem characterizes a compact connected topological semigroup satisfying the maximal property and the condition $S = Sa \cup aS \cup SaS$ by means of its maximal ideals.

**Theorem 2.15.** *Let $S$ be a compact connected topological semigroup. Then, the following are equivalent.*

(a) *There is an idempotent $e$ such that $e \in S \setminus M$ for every maximal ideal $M$ of $S$.*

(b) *The semigroup $S$ has the maximal property and $S = Sa \cup aS \cup SaS$ for some $a \in S$.*

*Proof.* (a) ⇒ (b) Since $K \subset S \setminus \{e\}$ and $I \subset J_0(S \setminus \{e\})$ for every proper ideal $I$ of $S$, $S$ has the maximal property with the maximal ideal $J_0(S \setminus \{e\})$.

Let $a \in S \setminus J_0(S \setminus \{e\})$. Then, $J_0(S \setminus \{e\})$ is properly contained by the ideal $Sa \cup aS \cup SaS \cup \{a\}$. Hence, $Sa \cup aS \cup SaS \cup \{a\} = S$. Since $S$ is connected and $Sa$, $aS$, $SaS$, and $\{a\}$ are closed, $a \in Sa \cup aS \cup SaS$, and hence, $S = Sa \cup aS \cup SaS$.

(b) ⇒ (a) Suppose that $S$ has the maximal property with the maximal ideal $M^*$ and $S$ does not satisfy the condition in part (a). Then, $E \subset M^*$, and hence it follows from Theorem 2.9 that $S^2 \subset M^*$. On the other hand, $S = Sa \cup aS \cup SaS \subset S^2 \subset M^*$, which contradicts $M^*$ being the maximal ideal of $S$.

The following corollary to Theorem 1.4.12 of [3] implies that the maximal ideal $M$ in Theorem 2.9 is not unique. □

**Corollary 2.16.** *A necessary and sufficient condition that a compact connected topological semigroup $S$ has the maximal ideal property is that $S$ has at least one idempotent $e$ with $S = SeS$ and $S$ is not simple.*

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