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Research Article

Existence of Solutions of Nonlinear Stochastic Volterra Fredholm Integral Equations of Mixed Type

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We establish sufficient conditions for the existence and uniqueness of random solutions of nonlinear Volterra-Fredholm stochastic integral equations of mixed type by using admissibility theory and fixed point theorems. The results obtained in this paper generalize the results of several papers.

1. Introduction

Random or stochastic integral equations are important in the study of many physical phenomena in life sciences, engineering, and technology [1–13]. Currently there are two basic versions of stochastic integral equations being studied by mathematical statisticians and probabilists namely, those integral equations involving Ito-Doob type of stochastic integrals and those which can be formed as probabilistic analogues of classical deterministic integral equations whose formulation involves the usual Lebesgue integral. Equations of the later category have been studied extensively by several authors [4, 10, 14–40]. Many papers have been appeared on the problem of existence of solutions of nonlinear random integral equations and the results are established by applying various fixed point techniques. These methods are broadly classified into three categories:

(i) admissibility theory, ([2, 7, 24, 27, 41–47]),

(ii) random contractor method, ([17, 21, 35, 47–52]),

(iii) measure of noncompactness method, ([11, 53–61]).

All these methods are effectively used to study the existence of solutions for stochastic integral equations. Further asymptotic behaviour and stability of solutions of stochastic integral equations are discussed in the papers [33, 42, 50, 54, 55, 59, 61–63]. In this paper we will study the existence of random solutions of nonlinear stochastic integral equations of mixed type.

Consider a nonlinear stochastic integral equation of the form

$$x(t;w) = h(t,x(t;w)) + \int_0^t k_1(t,\tau;w) f_1(\tau, x(\tau;w))d\tau + \int_0^\infty k_2(t,\tau;w) f_2(\tau, x(\tau;w))d\tau + \int_0^t k_3(t,\tau;w) f_3(\tau, x(\tau;w))d\beta(\tau), \tag{1.1}$$

where $t \in R_+$, $\beta(t)$ is a stochastic process and

(a-i) $w \in \Omega$, the supporting set of the complete probability measure space (Ω, A, μ), with the σ-algebra A and probability measure μ,

(a-ii) $x(t;w)$ is the unknown random function for $t \in R_+$, the nonnegative real numbers,

(a-iii) $h(t,x)$ is a scalar function defined for $t \in R_+$ and $x \in R$, the real line,

(a-iv) $k_1(t,\tau;w)$ and $k_3(t,\tau;w)$ are stochastic kernels defined for t and τ satisfying $0 \le \tau \le t < \infty$,

(a-v) $k_2(t,\tau;w)$ is the stochastic kernel defined for t and τ in R_+,

(a-vi) $f_1(t,x)$, $f_2(t,x)$, $f_3(t,x)$ are scalar functions defined for $t \in R_+$ and $x \in R$, the real line.

The first and the second part of the stochastic integral (1.1) are to be understood as an ordinary Lebesque integral with probabilistic characterization, while the third part is an Ito-Doob stochastic integral. Our aim is to investigate the existence as well as uniqueness of random solutions of the stochastic integral equation (1.1) by making use of "admissibility theory" that was first introduced by Tsokos [40] and fixed point theorems due to Krasnoselskii and Banach. The results generalize the previous results of [2, 7, 24, 27, 41–46].

2. Preliminaries

Let $\beta(t;w)$ be the random process. We will assume that for each $t \in R_+$, a minimal σ-algebra A_t, $A_t \subset A$, is such that $\beta(t;w)$ is measurable with respect to A_t. In addition, we will assume that the minimal σ-algebra A_t is an increasing family such that

(H1) the random process $\{\beta(t;w), A_t : t \in R_+\}$ is a real martingale

(H2) there is a real continuous nondecreasing function, $F(t)$, such that for $s < t$ we have $E\{|\beta(t;w) - \beta(s;w)|^2\} = E\{|\beta(t;w) - \beta(s;w)|^2 : A_t\} = F(t) - F(s)\mu$- a.e. where E denotes the expected value of the random process.

In the definitions that follow, we will assume that $x(t;w)$ is A_t measurable and that $E|x(t;w)|^2 < \infty$, for each $t \in R_+$. Also we denote

$$\left\{E|x(t;w)|^2\right\}^{1/2} = \|x(t;w)\|_{L^2(\Omega,A,\mu)} = \left(\int_\Omega |x(t:w)|^2 d\mu(w)\right)^{1/2}. \tag{2.1}$$

Definition 2.1. Denote by C_c the linear space of all mean square continuous maps $x(t;w)$ on R_+ and define a topology on C_c by means of the following family of seminorms.

$$\|x(t;w)\|_n = \sup_{0\le t\le n}\left\{E|x(t;w)|^2\right\}^{1/2}. \tag{2.2}$$

It is known that such a topology is metrizable and that the metric space C_c is complete.

Definition 2.2. Define $C_g \subset C_c$ to be the space of all maps $x(t;w)$ on R_+ such that

$$\left\{E|x(t;w)|^2\right\}^{1/2} \le a g(t), \tag{2.3}$$

where $a > 0$, a constant and $g(t) > 0$, a continuous function on R_+. The norm in the space C_g is defined by

$$\|x(t;w)\|_{C_g} = \sup_{t\ge 0}\left\{\frac{1}{g(t)}\left\{E|x(t;w)|^2\right\}^{1/2}\right\}. \tag{2.4}$$

Definition 2.3. Let $C \subset C_c$ be the space of maps $x(t;w)$ on R_+ with $\{E|x(t;w)|^2\}^{1/2} < M$, for some $M > 0$. The norm in space C is defined by

$$\|x(t;w)\|_C = \sup_{t\ge 0}\left\{E|x(t;w)|^2\right\}^{1/2}. \tag{2.5}$$

Definition 2.4. The pair of Banach spaces (B, D) with $B, D \subset C_c$ is called admissible with respect to the operator $T : C_c \to C_c$ if $TB \subset D$.

Definition 2.5. We will call $x(t;w)$ a random solution of the stochastic integral equation (1.1) if $x(t;w) \in C_c$ for each $t \in R_+$ and satisfies equation (1.1) μ-a.e., for all $t > 0$.

Definition 2.6. The Banach space B is said to be stronger than C_g, if every sequence which converges in the topology of B converges also in the topology of C_g.

Finally, let $B, D \subset C_g$ be Banach spaces and T a linear operator from C_g into C_c. The following lemma is well known [13].

Lemma 2.7. *Let T be a continuous operator from C_g into C_c. If B and D are Banach spaces in C_g stronger than C_g and if the pair (B, D) is admissible with respect to T, then T is a continuous operator from B into D.*

Let us define the operators

$$(T_1x)(t;w) = \int_0^t k_1(t,\tau;w)x(\tau;w)d\tau, \tag{2.6}$$

$$(T_2x)(t;w) = \int_0^\infty k_2(t,\tau;w)x(\tau;w)d\tau, \tag{2.7}$$

$$(T_3x)(t;w) = \int_0^t k_3(t,\tau;w)x(\tau;w)d\beta(\tau), \tag{2.8}$$

for $x(t;w) \in C_g$.

We state the following assumptions for our use.

(a_1) *The functions* $f_1(t,x(t;w))$, $f_2(t,x(t;w))$, *and* $f_3(t,x(t;w))$ *are continuous functions of* $t \in R_+$ *with values in* $L_2(\Omega, A, \mu)$.

(a_2) *For each t and* τ *in* R_+, $k_2(t,\tau;w)$ *has values in the space* $L_\infty(\Omega, A, \mu)$ *and the functions* $k_1(t,\tau;w)$ *and* $k_3(t,\tau;w)$ *for each t and* τ *such that* $0 \le \tau \le t < \infty$ *has values in the space* $L_\infty(\Omega, A, \mu)$.

(a_3) *The stochastic kernels* $k_1(t,\tau;w)$ *and* $k_3(t,\tau;w)$ *are essentially a bounded function with respect to* μ *for every t and* τ *such that* $0 \le \tau \le t < \infty$ *and continuous as maps from* $\{(t,\tau) : 0 \le \tau \le t < \infty\}$ *into* $L_\infty(\Omega, A, \mu)$.

(a_4) *The stochastic kernel* $k_2(t,\tau;w)$ *is essentially a bounded function with respect to* μ *for every t and* τ *in* R_+ *and continuous as maps from* $\{(t,\tau) : 0 \le \tau \le t < \infty\}$ *into* $L_\infty(\Omega, A, \mu)$.

Define for $0 \le \tau \le t < \infty$,

$$\begin{aligned}
|||k_1(t,\tau;w)||| &= \mu - ess \sup_{w\in\Omega} |k_1(t,\tau;w)|, \\
|||k_2(t,\tau;w)||| &= \mu - ess \sup_{w\in\Omega} |k_2(t,\tau;w)|, \\
|||k_3(t,\tau;w)||| &= \mu - ess \sup_{w\in\Omega} |k_3(t,\tau;w)|.
\end{aligned} \tag{2.9}$$

The assumptions (a_1)–(a_4) *imply that if* $x(t;w) \in C_c$, *then for each* $t \in R_+$,

$$E|k_3(t,\tau;w)x(\tau;w)|^2 \le |||k_3(t,\tau;w)|||^2 E|x(t;w)|^2. \tag{2.10}$$

Because of the continuity assumptions on $|k_3(t,\tau;w)|$ *and* $E|x(\tau;w)|^2$ *it follows from the above inequality that*

$$\int_0^t E|k_3(t,\tau;w)x(\tau;w)|^2 dF(\tau) < \infty, \tag{2.11}$$

which together with (H1) and (H2) implies that the integral in (2.8) *is well defined.*

Lemma 2.8. *Under the assumptions* (a_1)–(a_4), *(H1) and (H2),* T_1, T_2, *and* T_3 *are continuous linear operators from* C_g *into* C_c *provided*

$$\int_0^\infty |||k_3(t,\tau;w)|||^2 g^2(\tau)d\tau \leq N < \infty \quad \text{for some } N > 0. \tag{2.12}$$

Proof. It is easy to show that T_1, T_2 and T_3 are linear maps from C_g into C_c. The continuity of T_1 and T_2 are also easy to prove [8, 13]. We will prove that T_3 is continuous.

Let $x(t;w) \in C_g$. Then

$$\begin{aligned} E|(T_3x)(t;w)|^2 &= E\left\{\int_0^t k_3(t,\tau;w)x(\tau;w)d\beta(\tau)\right\}^2 \\ &= \int_0^t E|k_3(t,\tau;w)x(\tau;w)|^2 dF(\tau) \\ &\leq \int_0^t |||k_3(t,\tau;w)|||^2 E|x(t;w)|^2 dF(\tau) \\ &\leq \|x(t;w)\|_{C_g}^2 \int_0^t |||k_3(t,\tau;w)|||^2 g^2(\tau)dF(\tau), \quad t < n. \end{aligned} \tag{2.13}$$

Hence, on compact intervals $[0,n]$

$$\begin{aligned} \sup_{0\leq t\leq n} \|(T_3x)(t;w)\|_{L_2(\Omega,A,\mu)} &\leq \|x(t;w)\|_{C_g} \left\{ \sup_{0\leq t\leq n} \left[\int_0^t |||k_3(t,\tau;w)|||^2 g^2(\tau)dF(\tau)\right]^{1/2} \right\} \\ &\leq N_1 \|x(t;w)\|_{C_g}, \end{aligned} \tag{2.14}$$

where N_1 is a constant depends upon n. This proves the continuity of T_3. The linearity of T_3 is obvious.

To show that T_2 maps C_g into C_c. Let $y(t;w) = \int_0^\infty k_2(t,\tau;w)x(\tau;w)d\tau$. Then

$$\left\| y(t_1;w) - y(t_2;w) \right\|_{L_2(\Omega,A,\mu)} = \|x(t;w)\|_{C_g} \int_0^\infty |||k_2(t_1,\tau;w) - k_2(t_2,\tau;w)|||^2 g^2(\tau)d\tau. \tag{2.15}$$

The right-hand side of the above inequality goes to zero as $t_2 \to t_1$, since $k_2(t,\tau;w)g(\tau) \in L_2(\Omega, A, \mu)$. Thus, this proves that T_2 maps C_g into C_c. The proof of the continuity of T_2 is similar to that of T_3.

Let the operators T_1, T_2, and T_3 be as defined in (2.6), (2.7), and (2.8) and let the assumptions of Lemma 2.8 hold. Then it follows from Lemma 2.7 that, if B and D are Banach spaces stronger than C_g and the pair (B, D) is admissible with respect to the operators T_1, T_2 and T_3, then T_1, T_2, and T_3 are continuous from B into D. Thus, there exist positive constants K_1, K_2, and K_3 such that

$$\begin{aligned} \|(T_1 x)(t;w)\|_D &\le K_1 \|x(t;w)\|_B, \\ \|(T_2 x)(t;w)\|_D &\le K_2 \|x(t;w)\|_B, \\ \|(T_3 x)(t;w)\|_D &\le K_3 \|x(t;w)\|_B. \end{aligned} \tag{2.16}$$

The constants K_1, K_2, K_3 are the bounds of the operator T_1, T_2, T_3. □

Theorem 2.9 (Krasnoselskii Theorem). *Let S be a closed, bounded and convex subset of a Banach space X and let U_1 and U_2 be operators on S satisfying the following conditions:*

(i) *$U_1(x) + U_2(y) \in S$ whenever $x, y \in S$,*

(ii) *U_1 is a contraction operator on S,*

(iii) *U_2 is completely continuous.*

Then there is at least one point $x^* \in S$ such that $U_1(x^*) + U_2(x^*) = x^*$.

3. Main Results

In this section we will prove the main result of this paper.

Theorem 3.1. *For the stochastic integral equation* (1.1) *assume the following conditions*

(i) *B and D are Banach spaces in C_g, stronger than C_g, such that (B, D) is admissible with respect to the operators T_1, T_2, and T_3 defined by* (2.6), (2.7), *and* (2.8)*;*

(ii) *$\int_0^\infty \||k_2(t,\tau;w)|^2\| g^2(\tau) d\tau \le N < \infty$ for some $N > 0$;*

(iii) *$x(t;w) \to f_1(t, x(t;w))$ is a continuous map from*

$$S = \left\{ x(t;w) : x(t;w) \in D,\ \|x(t;w)\|_D \le \rho \right\} \tag{3.1}$$

with values in B satisfying

$$\left\| f_1(t, x(t;w)) - f_1(t, y(t;w)) \right\|_B \le \lambda_1 \left\| x(t;w) - y(t;w) \right\|_D \tag{3.2}$$

for $x(t;w), y(t;w) \in S$ and $\lambda_1 \ge 0$ a constant;

(iv) *$x(t;w) \to f_2(t, x(t;w))$ is a completely continuous map from S into B;*

(v) *$x(t;w) \to f_3(t, x(t;w))$ is a continuous map from S with values in B satisfying*

$$\left\| f_3(t, x(t;w)) - f_3(t, y(t;w)) \right\|_B \le \lambda_3 \left\| x(t;w) - y(t;w) \right\|_D \tag{3.3}$$

for $x(t;w), y(t;w) \in S$ and λ_3 a constant;

(vi) $x(t;w) \to h(t, x(t;w))$ *is a continuous map from* S *into* D *such that*

$$\left\|h(t, x(t;w)) - h\left(t, y(t;w)\right)\right\|_D \leq \gamma \left\|x(t;w) - y(t;w)\right\|_D \tag{3.4}$$

for $x(t;w), y(t;w) \in S$ *and* $\gamma > 0$ *a constant.*

Then there exists a unique random solution of (1.1) *in* S *provided*

$$\begin{gathered}\gamma + K_1\lambda_1 + K_3\lambda_3 < 1, \\ \gamma\|h(t,0)\|_D + K_1\left\|f_1(t,0)\right\|_B + K_2\left\|f_2(t, x(t;w))\right\|_B + K_3\left\|f_3(t,0)\right\|_B \\ \leq \rho\left(1 - \gamma - K_1\lambda_1 - K_3\lambda_3\right),\end{gathered} \tag{3.5}$$

where K_1, K_2, *and* K_3 *are defined by* (2.16).

Proof. The set S closed, bounded, and convex in D. Let $x(t;w), y(t;w) \in S$. Then define the operator $U_1 : S \to D$ by

$$\begin{aligned}(U_1x)(t;w) = h(t, x(t;w)) &+ \int_0^t k_1(t, \tau; w) f_1(\tau, x(\tau; w)) d\tau \\ &+ \int_0^t k_3(t, \tau; w) f_3(\tau, x(\tau; w)) d\beta(\tau).\end{aligned} \tag{3.6}$$

We will show that U_1 is a contraction mapping and that $U_1S \subset S$. Let $x(t;w), y(t;w) \in S$. Then

$$\begin{aligned}(U_1x)(t;w) - \left(U_1y\right)(t;w) = h(t, x(t;w)) &- h\left(t, y(t;w)\right) \\ &+ \int_0^t k_1(t, \tau; w)\left[f_1(\tau, x(\tau;w)) - f_1\left(\tau, y(\tau;w)\right)\right] d\tau \\ &+ \int_0^t k_3(t, \tau; w)\left[f_3(\tau, x(\tau;w)) - f_3\left(\tau, y(\tau;w)\right)\right] d\beta(\tau).\end{aligned} \tag{3.7}$$

From our assumption it is clear that $(U_1x)(t;w) - (U_1y)(t;w) \in D$ and $f_1(\tau, x(\tau;w)) - f_1(\tau, y(\tau;w)), f_3(\tau, x(\tau;w)) - f_3(\tau, y(\tau;w)) \in B$. Furthermore

$$\begin{aligned}\left\|(U_1x)(t;w) - \left(U_1y\right)(t;w)\right\|_D &\leq \left\|h(t, x(t;w)) - h\left(t, y(t;w)\right)\right\|_D \\ &\quad + K_1\left\|f_1(\tau, x(\tau;w)) - f_1\left(\tau, y(\tau;w)\right)\right\|_B \\ &\quad + K_3\left\|f_3(\tau, x(\tau;w)) - f_3\left(\tau, y(\tau;w)\right)\right\|_B \\ &\leq \left(\gamma + K_1\lambda_1 + K_3\lambda_3\right)\left\|x(t;w) - y(t;w)\right\|.\end{aligned} \tag{3.8}$$

Since $(\gamma + K_1\lambda_1 + K_3\lambda_3) < 1$, U_1 is a contraction operator. Next we show that $U_1 S \subset S$. From (3.6), we have

$$\begin{aligned}\|(U_1x)(t;w)\|_D &= \|h(t,x(t;w))\|_D + \left\|\int_0^t k_1(t,\tau;w)f_1(\tau,x(\tau;w))d\tau\right\| \\ &\quad + \left\|\int_0^t k_3(t,\tau;w)f_3(\tau,x(\tau;w))d\beta(\tau)\right\| \\ &\le \|h(t,0)\|_D + \big((\gamma + K_1\lambda_1 + K_3\lambda_3)\big)\|x(t;w)\| \\ &\quad + \lambda_1\|f_1(t,0)\|_B + \lambda_3\|f(t,0)\|_B.\end{aligned} \tag{3.9}$$

Since $x(t;w) \in S$, by hypothesis, we have $\|(U_1x)(t;w)\|_D \le \rho$ which implies that $U_1 S \subset S$.

Let us define the operator $U_2 : S \to D$ as

$$(U_2x)(t;w) = \int_0^\infty k_2(t,\tau;w)f_2(\tau,x(\tau;w))d\tau. \tag{3.10}$$

It is clear that U_2 is composition of continuous map T_2 and completely continuous map f_2. Hence U_2 is completely continuous. Furthermore, if $x(t;w), y(t;w) \in S$, we have

$$\begin{aligned}\|(U_1x)(t;w) + (U_1y)(t;w)\|_D &\le \|h(t,x(t;w))\|_D \\ &\quad + K_1\|f_1(\tau,x(\tau;w))\|_B + K_2\|f_2(\tau,y(\tau;w))\|_B \\ &\quad + K_3\|f_3(\tau,x(\tau;w))\|_B \\ &\le \|h(t,0)\|_D + (\gamma + K_1\lambda_1 + K_3\lambda_3)\rho + K_1\|f_1(t,0)\|_B \\ &\quad + K_2\|f_2(t,x(t;w)\|_B + K_3\|f_3(t,0)\|_B \\ &\le \rho.\end{aligned} \tag{3.11}$$

This shows that if $x(t;w), y(t;w) \in S$, then $(U_1x)(t;w) + (U_2y)(t;w) \in S$. Hence, applying Krasnoselskii's fixed point theorem, we can conclude that there exists a random solution of (1.1) in the set S.

We will now consider the case under which the stochastic integral equation (1.1) possesses a unique solution. This will be achieved by using the Banach contraction mapping principle. □

Theorem 3.2. *For the stochastic integral equation* (1.1) *assume the following conditions*

(i) *B and D are Banach spaces in* C_g*, stronger than* C_g*, such that* (B, D) *is admissible with respect to the operators* T_1, T_2 *and* T_3 *defined by* (2.6)*,* (2.7)*, and* (2.8)*;*

(ii) $\int_0^\infty \||k_2(t,\tau;w)|^2\|g^2(\tau)d\tau \leq N < \infty$ *for some* $N > 0$*;*

(iii) $x(t;w) \rightarrow f_1(t, x(t;w))$ *is a continuous map from*

$$S = \{x(t;w) : \quad x(t;w) \in D, \ \|x(t;w)\|_D \leq \rho\} \tag{3.12}$$

with values in B satisfying

$$\|f_1(t, x(t;w)) - f_1(t, y(t;w))\|_B \leq \lambda_1 \|x(t;w) - y(t;w)\|_D \tag{3.13}$$

for $x(t;w), y(t;w) \in S$ *and* $\lambda_1 \geq 0$ *a constant;*

(iv) $x(t;w) \rightarrow f_2(t, x(t;w))$ *is a continuous map from S with values in B satisfying*

$$\|f_2(t, x(t;w)) - f_2(t, y(t;w))\|_B \leq \lambda_2 \|x(t;w) - y(t;w)\|_D \tag{3.14}$$

for $x(t;w), y(t;w) \in S$ *and* $\lambda_2 \geq$ *a constant;*

(v) $x(t;w) \rightarrow f_3(t, x(t;w))$ *is a continuous map from S with values in B satisfying*

$$\|f_3(t, x(t;w)) - f_3(t, y(t;w))\|_B \leq \lambda_3 \|x(t;w) - y(t;w)\|_D \tag{3.15}$$

for $x(t;w), y(t;w) \in S$ *and* λ_3 *a constant;*

(vi) $x(t;w) \rightarrow h(t, x(t;w))$ *is a continuous map from S into D such that*

$$\|h(t, x(t;w)) - h(t, y(t;w))\|_D \leq \gamma \|x(t;w) - y(t;w)\|_D \tag{3.16}$$

for $x(t;w), y(t;w) \in S$ *and* $\gamma > 0$ *a constant.*

Then there exists a unique random solution of (1.1) *in S provided*

$$\begin{aligned}
&\gamma + K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3 < 1, \\
&\gamma\|h(t,0)\|_D + K_1\|f_1(t,0)\|_B + K_2\|f_2(t,0)\|_B + K_3\|f_3(t,0)\|_B \\
&\qquad \leq \rho(1 - \gamma - K_1\lambda_1 - K_2\lambda_2 - K_3\lambda_3),
\end{aligned} \tag{3.17}$$

where K_1, K_2*, and* K_3 *are defined by* (2.16)*.*

Proof. Define the operator $U : S \rightarrow D$ as follows

$$\begin{aligned}
(Ux)(t;w) = h(t, x(t;w)) &+ \int_0^t k_1(t,\tau;w) f_1(\tau, x(\tau;w)) d\tau \\
&+ \int_0^\infty k_2(t,\tau;w) f_2(\tau, x(\tau;w)) d\tau + \int_0^t k_3(t,\tau;w) f_3(\tau, x(\tau;w)) d\beta(\tau).
\end{aligned} \tag{3.18}$$

We will show that U is a contraction operator on S and that $US \subset S$. Let $x(t;w), y(t;w) \in S$. Then $(Ux)(t;w) - (Uy)(t;w) \in D$ as $US \subset D$ and D is a Banach space. Also

$$\begin{aligned}
&\left\|(Ux)(t;w) - (Uy)(t;w)\right\|_D \\
&\quad \leq \left\|h(t,x(t;w)) - h(t,y(t;w))\right\|_D \\
&\qquad + \left\|\int_0^t k_1(t,\tau;w)\left[f_1(\tau,x(\tau;w)) - f_1(\tau,y(\tau;w))\right]\, d\tau\right\|_D \\
&\qquad + \left\|\int_0^\infty k_2(t,\tau;w)\left[f_2(\tau,x(\tau;w)) - f_2(\tau,y(\tau;w))\right] d\tau\right\|_D \\
&\qquad + \left\|\int_0^t k_3(t,\tau;w)\left[f_3(\tau,x(\tau;w)) - f_3(\tau,y(\tau;w))\right] d\beta(\tau)\right\|_D .
\end{aligned} \tag{3.19}$$

Thus, in view of (2.16), we have

$$\begin{aligned}
&\left\|(Ux)(t;w) - (Uy)(t;w)\right\|_D \\
&\quad \leq \gamma\left\|x(t;w) - y(t;w)\right\|_D + K_1\left\|f_1(t,x(t;w) - f_1(t,y(t;w)\right\|_B \\
&\qquad + K_2\left\|f_2(t,x(t;w)) - f_2(t,y(t;w))\right\|_B \\
&\qquad + K_3\left\|f_3(t,x(t;w)) - f_3(t,y(t;w))\right\|_B \\
&\quad \leq \left(\gamma + K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3\right)\left\|x(t;w) - y(t;w)\right\|_D.
\end{aligned} \tag{3.20}$$

Since $(\gamma + K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3) < 1$, U is a contraction operator on S.

We will now show that $US \subset S$. For any $x(t;w) \in S$, we have

$$\begin{aligned}
\|(Ux)(t;w)\|_D &\leq \|h(t,x(t;w))\|_D + \left\|\int_0^t k_1(t,\tau;w) f_1(\tau,x(\tau;w) d\tau\right\|_D \\
&\quad + \left\|\int_0^\infty k_2(t,\tau;w) f_2(\tau,x(\tau;w)) d\tau\right\|_D \\
&\quad + \left\|\int_0^t k_3(t,\tau;w) f_3(\tau,x(\tau;w)) d\beta(\tau)\right\|_D \\
&\leq \|h(t,x(t;w))\|_D + K_1\left\|f_1(t,x(t;w))\right\|_B \\
&\quad + K_2\left\|f_2(t,x(t;w))\right\|_B + K_3\left\|f_3(t,x(t;w))\right\|_B \\
&\leq \gamma\|x(t;w)\|_D + \gamma\|h(t,0)\|_D + \lambda_1 K_1\|x(t;w)\|_D + K_1\left\|f_1(t,0)\right\|_B \\
&\quad + \lambda_2 K_2\|x(t;w)\|_D + K_2\left\|f_2(t,0)\right\|_B \\
&\quad + \lambda_3 K_3\|x(t;w)\|_D + K_3\left\|f_3(t,0)\right\|_B.
\end{aligned} \tag{3.21}$$

Since $\|x(t;w)\|_D \le \rho$, it follows that

$$\begin{aligned}\|(Ux)(t;w)\|_D \le \gamma\|h(t,0)\|_D + \rho(\gamma + K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3) \\ + K_1\|f_1(t,0)\|_B + K_2\|f_2(t,0)\|_B + K_3\|f_3(t,0)\|_B.\end{aligned} \tag{3.22}$$

Using the condition that

$$\begin{aligned}\gamma\|h(t,0)\|_D + K_1\|f_1(t,0)\|_B + K_2\|f_2(t,0)\|_B + K_3\|f_3(t,0)\|_B \\ \le \rho(1 - \gamma - K_1\lambda_1 - K_2\lambda_2 - K_3\lambda_3),\end{aligned} \tag{3.23}$$

we have from (3.18)

$$\|(Ux)(t;w)\|_D \le \rho. \tag{3.24}$$

Hence $(Ux)(t;w) \in S$ for all $x(t;w) \in S$ or $US \subset S$. Thus the condition of Banach's fixed point theorem is satisfied and hence there exists a fixed point $x(t;w) \in S$ such that $(Ux)(t;w) = x(t;w)$. That is,

$$\begin{aligned}(Ux)(t;w) &= h(t,x(t;w)) + \int_0^t k_1(t,\tau;w) f_1(\tau,x(\tau;w))d\tau \\ &\quad + \int_0^\infty k_2(t,\tau;w) f_2(\tau,x(\tau;w))d\tau + \int_0^t k_3(t,\tau;w) f_3(\tau,x(\tau;w))d\beta(\tau) \\ &= x(t;w).\end{aligned} \tag{3.25}$$

□

4. Applications

In this section we will give some application of Theorem 3.2.

Theorem 4.1. *Suppose the stochastic integral equation* (1.1) *satisfies the following conditions:*

(i) *there exists a constant $A > 0$ and a continuous function $g(t)$, such that*

$$\int_0^t |||k_1(t,\tau;w)|||^2 g^2(\tau)d\tau + \int_0^\infty |||k_2(t,\tau;w)|||^2 g^2(\tau)d\tau + \int_0^t |||k_3(t,\tau;w)|||^2 g^2(\tau)d\tau < A; \tag{4.1}$$

(ii) *$f_i(t,x), i = 1,2,3$ are continuous functions on $R_+ \times R$, such that $f_i(t,0) \in C_g(R_+,R)$ and $|f_i(t,x) - f_i(t,y)| \le \lambda_i g(t)|x - y|$, for $x,y \in R$ and $0 \le \lambda_i < 1, i = 1,2,3$;*

(iii) *$h(t,x)$ is a continuous functions on $R_+ \times R$, such that $|h(t,x) - h(t,y)| \le \gamma|x - y|$, for $x,y \in R$ and $0 \le \gamma < 1$.*

Then there exists a unique random solution $x(t;w)$ of (1.1) *such that*

$$\|x(t;w)\|_C \le \rho \tag{4.2}$$

provided $\|h(t,0)\|, \|f_i(t,0)\|_{C_g}, i = 1,2,3$ are small enough.

Proof. It is easy to show that the hypothesis of Theorem 3.2 are satisfied by simply showing the pair of spaces (C_g, C_c) is admissible with respect to the operators T_1, T_2, and T_3. This follows from Lemma 2.8. □

Corollary 4.2. *Suppose the stochastic integral equation* (1.1) *satisfies the following conditions:*

(i) $\int_0^t \||k_1(t,\tau;w)|^2\| d\tau + \int_0^\infty \||k_2(t,\tau;w)|^2\| d\tau + \int_0^t \||k_3(t,\tau;w)|^2\| d\tau < A$;

(ii) $f_i(t,x), i = 1,2,3$ *are continuous functions on* $R_+ \times R$, *such that* $f_i(t,0) \in C_g(R_+, R)$ *and* $|f_i(t,x) - f_i(t,y)| \le \lambda_i g(t)|x-y|$, *for* $x, y \in R$ *and* $0 \le \lambda_i < 1, i = 1,2,3$;

(iii) $h(t,x)$ *is a continuous functions on* $R_+ \times R$, *such that* $|h(t,x) - h(t,y)| \le \gamma|x-y|$, *for* $x, y \in R$ *and* $0 \le \gamma < 1$.

Then there exists a unique random solution $x(t;w)$ *of* (1.1) *such that*

$$\|x(t;w)\|_C \le \rho \tag{4.3}$$

provided $\|h(t,0)\|, \|f_i(t,0)\|_{C_g}, i = 1,2,3$ *are small enough.*

Proof. Take $g(t) = 1$ in Theorem 4.1. □

Corollary 4.3. *Suppose the stochastic integral equation* (1.1) *satisfies the following conditions:*

(i) $\||k_i(t,\tau;w)\|| \le A$, $i = 1,2,3$ *and* $\int_0^t g^2(\tau)\tau < \infty$;

(ii) *same as conditions* $(iv), (v)$, *and* (vi) *in Theorem 3.2.*

Then there exists a unique random solution of (1.1) *provided* γ, $\|h(t,0)\|_C$ *and* $\|f_i(t,0)\|_{C_g}$ *for* $i = 1,2,3$ *small enough.*

Proof. We will show that the pair is (C_g, C_c) admissible with respect to the operator T_2. Let $x(t;w) \in C_g$. Then

$$\begin{aligned}\sup_{0\le t}\|(T_2x)(t;w)\|_{C_g} &\le \sup_{0\le t}\left\{\int_0^\infty \||k_2(t,\tau;w)\||^2 \|x(\tau;w)\|_{L_2}^2 d\tau\right\}^{1/2} \\ &\le \|x(t;w)\|_{C_g} A\int_0^\infty g^2(\tau)d\tau\end{aligned} \tag{4.4}$$

which implies that the pair (C_g, C_c) is admissible. Similarly we can show that the pair (C_g, C_c) is admissible with respect to the operators T_1, T_3. It is easy to check the other conditions of Theorem 3.2 and hence there exists a unique random solution of equation of the stochastic integral equation (1.1). □

Remark 4.4. Using the same argument one can establish the existence of a unique random solution of the following general stochastic integral equation

$$\begin{aligned}x(t;w) = h(t,x(t;w)) &+ \sum_{i=1}^n \int_0^t a_i(t,\tau;w) f_i(\tau, x(\tau;w)) d\tau \\ &+ \sum_{i=1}^n \int_0^\infty b_i(t,\tau;w) g_i(\tau, x(\tau;w)) d\tau + \sum_{i=1}^n \int_0^t c_i(t,\tau;w) k_i(\tau, x(\tau;w)) d\beta(\tau),\end{aligned} \tag{4.5}$$

where $h, k_i, a_i, b_i, c_i, g_i, f_i$, and β satisfy appropriate conditions. This general case is treated in a separate paper.

5. Example

Consider the following nonlinear stochastic integral equation:

$$\begin{aligned} x(t;w) = {} & \frac{1}{4}\sin x(t;w) + \int_0^t \frac{\sin t}{4} e^{-s-x^2(s;w)}\,ds \\ & + \int_0^\infty \frac{e^{-t-s}}{1+|x(s;w)|}\,ds + \frac{1}{8}\int_0^t \ln(1+|x(s;w)|)\,d\beta(s), \quad t \in R_+, \end{aligned} \tag{5.1}$$

where $\beta(t)$ is a stochastic process. This equation is a particular case of general stochastic integral equation occurring in mathematical biology and chemotherapy [10–13]. The above equation takes the form of (1.1) with

$$\begin{gathered} k_1(t,s,w) = \frac{\sin t}{4}e^{-s}, \qquad k_2(t,s,w) = e^{-t-s}, \qquad k_3(t,s,w) = \frac{1}{4}, \qquad h(\mathrm{t}, x(t;w)) = \frac{\sin x(t;w)}{4} \\ f_1(s,x(s;w)) = e^{-x^2(s;w)}, \qquad f_2(s,x(s;w)) = \frac{1}{1+|x(s;w)|}, \\ f_3(s,x(s;w)) = \frac{1}{2}\ln(1+|x(s;w)|). \end{gathered} \tag{5.2}$$

Take $B = D = C_g = C_c = C$ and $g(t) = 1$. It is easy to see that $\gamma = 1/4$, $K_1 = K_3 = 1/4$, $K_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 1/4$, and $\lambda_3 = 1/2$. Further $\gamma + K_1\lambda_1 + K_2\lambda_2 + K_3\lambda_3 = 7/8 < 1$ and by taking $\rho \geq 10$, the other condition of Theorem 3.2 is satisfied. It is clear that (5.1) satisfies assumptions (i) to (vi) of Theorem 3.2. Hence there exists a unique random solution for (5.1).

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