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Research Article

On Epiorthodox Semigroups

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It has been well known that the band of idempotents of a naturally ordered orthodox semigroup satisfying the "strong Dubreil-Jacotin condition" forms a normal band. In the literature, the naturally ordered orthodox semigroups satisfying the strong Dubreil-Jacotin condition were first considered by Blyth and Almeida Santos in 1992. Based on the name "epigroup" in the paper of Blyth and Almeida Santos and also the name "epigroups" proposed by Shevrin in 1955; we now call the naturally ordered orthodox semigroups satisfying the Dubreil-Jacotin condition the *epiorthodox semigroups*. Because the structure of this kind of orthodox semigroups has not yet been described, we therefore give a structure theorem for the epi-orthodox semigroups.

1. Introduction

We recall that an ordered semigroup S is an algebraic system $(S, \cdot, \leq)$ in which the following conditions are satisfied: (1) $(S, \cdot)$ is a semigroup, (2) $(S, \leq)$ is a poset, and (3) $a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb$ for all $a, b \in S$. An ordered semigroup $(S, \cdot, \leq)$ is said to satisfy the Dubreil-Jacotin condition if there exists an isotone epimorphism which is a surjective homomorphism θ from the semigroup S onto an ordered group G such that $\{x \in S : \theta(x^2) \leq \theta(x)\}$ has the greatest element. This kind of ordered semigroups was first studied by Dubreil and Jacotin. We notice that Blyth and Giraldes [1] first investigated the perfect elements of Dubreil-Jacotin regular semigroups in 1992. In this paper, we call the naturally ordered orthodox semigroups satisfying the Dubreil-Jacotin condition the "*epiorthodox semigroups.*" Recall that a semigroup S is called an orthodox semigroup if the set of its idempotents E forms a subsemigroup of the semigroup S (see [2, 3]). It is well known in the theory of semigroups that the class of orthodox semigroups played an important role in the class of regular semigroups. The structure of some special orthodox semigroup has been investigated and studied by Ren, Shum et al. in [4–6]. The so-called super $\mathcal{R}^*$-unipotent semigroups have been particularly studied by Ren et al. in [6]. In this paper, we call an ordered semigroup $(S, \cdot, \leq)$ a naturally ordered semigroup if

$$(\forall e, f \in E) \quad e \preceq f \Longrightarrow e \leq f, \tag{1.1}$$

where "$\preceq$" is a natural order on the subset E of S.

We notice here that the class of naturally ordered regular semigroups with the greatest idempotent was first considered by Blyth and McFadden [7] and Blyth and Almeida Santos in 1992 [8]. A well-known generalized class of regular semigroups is the class of rpp semigroups. For rpp semigroups and their generalizations, the reader is referred to [9]. It was observed by McAlister [10] that each element in a naturally ordered regular semigroup with the greatest idempotent has the greatest inverse.

An ordered semigroup S is said to satisfy the strong Dubreil-Jacotin condition if there exists an epimorphism f from S onto an ordered group G such that $f\colon S \to G$ is residuated in the sense that the preimage under f of every principal order ideal of G is a principal order ideal of S. The class of orthodox semigroups which are naturally ordered satisfying the strong Dubreil-Jacotin condition was first studied by Blyth and Almeida Santos in 1992 (see [8, 11]). In their paper [8], they first named a naturally ordered semigroup satisfying the strong Dubreil-Jacotin condition an "epigroup." However, a semigroup was also called an "epigroup" by Shevrin since 1955 (see [12, 13]). An epigroup means a semigroup in which some power of each of its element lies in a subgroup of a given semigroup. Thus, an epigroup can be regarded as a unary semigroup with the unary operation of pseudoinversion (see the articles of Shevrin [14, 15] for more information of epigroups). We emphasize here that the concept of epigroups initiated by Shevrin is quite different from the naturally ordered semigroup satisfying the strong Dubreil-Jacotin condition described by Blyth and Almeida Santos. For the lattice properties of epigroups, the readers are referred to the recent articles of Shevrin and Ovsyannikov in 2008 [16, 17]. In this paper, our purpose is to establish a structure theorem of an epiorthodox semigroup. Concerning the regular semigroups and their generalizations, the reader is referred to [9, 18]. For other notations and terminologies not mentioned in this paper, the reader is referred to Shum and Guo [19] and Howie [18].

Throughout this paper, following the terminology "epigroups" proposed by Shevrin and Blyth and Almeida Santos, we call an orthodox semigroup which is naturally ordered satisfying the Dubreil-Jacotin condition an "epiorthodox semigroup."

2. Preliminaries

Let S be a naturally ordered regular semigroup. We first assume that every element $x \in S$ has the greatest inverse in S. Denote this element by x°. Then, we call Green's relation $\mathcal{R}$ on S the *left regular* relation if $x \leq y \Rightarrow xx^\circ \leq yy^\circ$, for all $x, y \in S$. Similarly, Green's relation $\mathcal{L}$ on a semigroup S is called the *right regular* relation on S if $x \leq y \Rightarrow x^\circ x \leq y^\circ y$, for all $x, y \in S$.

It was shown by McAlister [10] (see [10, Proposition 1.9]) that if S is an ordered regular semigroup with the greatest idempotent u, then S is a naturally ordered orthodox semigroup if and only if u is a *middle unit*, that is, $xy = xuy$ for all $x, y \in S$. In addition, it has been stated in [18] that if u is a middle unit then every $x \in S$ has the greatest inverse, say, $x^\circ = ux'u$ for every inverse element x' of x.

Consider an epiorthodox semigroup S with $\max\{x \in S : \theta(x^2) \le \theta(x)\} = \xi$. Because S is a regular semigroup, if $\xi' \in V(\xi)$, $\xi\xi'$ is an idempotent then $\xi\xi' \le \xi$ in S. Consequently, because the semigroup S satisfies the Dubreil-Jacotin condition, we have $1 = \theta(\xi\xi') \le \theta(\xi)$ and

$$\theta\left(\xi^2\right) \le \theta(\xi) \Longrightarrow \theta(\xi) = \theta\left(\xi^2\xi'\right) \le \theta(\xi\xi') = 1. \tag{2.1}$$

Thus, $1 \le \theta(\xi) \le 1$ so that $\theta(\xi) = 1$, where 1 is the identity element of the group G. It hence follows that $\xi \in 1\theta^{-1}$ and so the semigroup S has the greatest element ξ. By Lemma 1.7 in [10], McAlister noticed that if an ordered regular semigroup has the greatest element then its greatest element must be an idempotent. It follows that the ξ is the greatest idempotent of the epiorthodox semigroup S.

In view of the above results, we have the following lemma.

Lemma 2.1. *Let S be an epiorthodox semigroup in which* $\max\{x \in S : \theta(x^2) \le \theta(x)\} = \xi$. *Then*

(1) *ξ is the greatest idempotent of S and is a middle unit;*

(2) *the set E of idempotents of S forms a normal band.*

Proof. Part (1) of the above lemma follows easily from observation. To prove part (2) of the lemma, we first recall a result of Blyth and Almeida Santos [11] (see [11, Theorem 2]) that if T is an ordered regular semigroup with the greatest idempotent α, then T is naturally ordered if and only if α is a normal medial idempotent (in the sense that $\overline{e}\alpha\overline{e} = \overline{e}$ for all $\overline{e} \in \overline{E}$, where $\overline{E}$ is the subsemigroup generated by E, and $\alpha\overline{E}\alpha$ is a semilattice). Since S is orthodox, we have that $E = \overline{E}$. Also, since S is naturally ordered semigroup, $\xi E \xi$ is a semilattice. The concept of middle unit in an orthodox semigroup was first introduced by Blyth [20]. Because ξ is a middle unit, for any $e,\ f,\ g,$ and h in E, we have that

$$efgh = e \cdot \xi f \xi \cdot \xi \cdot h = e \cdot \xi g \xi \cdot \xi f \xi \cdot h = egfh. \tag{2.2}$$

This shows that E is a normal band. □

Lemma 2.2. *Let S be an epiorthodox semigroup. Suppose that* $\max\{x \in S : \theta(x^2) \le \theta(x)\} = \xi$ *in S. Then the following properties hold:*

(1) *$x^\circ = \xi x' \xi$ is the greatest inverse of any $x \in S$, with $x' \in V(x)$;*

(2) *$x^{\circ\circ} = \xi x \xi$, for every $x \in S$;*

(3) *$\xi x^\circ = x^\circ = x^\circ \xi$, for every $x \in S$;*

(4) *$xyy^\circ x^\circ xy(xy)^\circ = xy(xy)^\circ,\ xy(xy)^\circ xyy^\circ x^\circ = xyy^\circ x^\circ$, for all $x, y \in S$;*

(5) *$xyy^\circ x^\circ = xy(xy)^\circ,\ y^\circ x^\circ xy = (xy)^\circ xy$, for all $x, y \in S$;*

(6) *$(xy)^\circ = y^\circ x^\circ$, for all $x, y \in S$.*

Proof. By Lemma 2.1, it is known that ξ is the greatest idempotent of S. Since S is a naturally ordered regular semigroup with the greatest idempotent ξ, by a result of Blyth and McFadden in [7], we know immediately that (1), (2), and (3) hold. Since S is orthodox, $y^\circ x^\circ \in V(xy)$ and hence (4) holds. By (4), the idempotents $xyy^\circ x^\circ$ and $xy(xy)^\circ$ are clearly $\mathcal{R}$-related. But if $e\mathcal{R}f$ for the idempotents e, f in S, then $e = fe \leq f\xi$ and so $e\xi \leq f\xi$. Similarly, $f\xi \leq e\xi$. Thus, from (3), we deduce the equality $xyy^\circ x^\circ = xy(xy)^\circ$. Similarly, we can also prove $y^\circ x^\circ xy = (xy)^\circ xy$, and hence (5) holds. From (5), we have that $(xy)^\circ = (xy)^\circ xy(xy)^\circ = y^\circ x^\circ xyy^\circ x^\circ = y^\circ x^\circ$. This implies (6) holds. □

In order to establish a structure theorem for an epiorthodox semigroup, we restate here the notion of strong semilattice of ordered semigroups.

Suppose that Y is a semilattice and $(S_\alpha)_{\alpha\in Y}$ is a family of pairwise disjoint semigroups. For $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\varphi_{\beta,\alpha} : S_\alpha \rightarrow S_\beta$ be a morphism satisfying the following conditions:

(a) $(\forall\, \alpha \in Y)\ \varphi_{\alpha,\alpha} = \mathrm{id}_{S_\alpha}$;

(b) if $\alpha \geq \beta \geq \gamma$, then $\varphi_{\gamma,\beta}\varphi_{\beta,\alpha} = \varphi_{\gamma,\alpha}$.

Then, it is known that the set $\bigcup_{\alpha\in Y} S_\alpha$ under the following multiplication:

$$(\forall x \in S_\alpha)\ (\forall y \in S_\beta) \quad xy = \varphi_{\alpha\beta,\alpha}(x)\varphi_{\alpha\beta,\beta}(y) \tag{2.3}$$

forms a semigroup which is called the only *strong semilattice of semigroups.*

By using the strong semilattices of semigroups, Blyth and Almeida Santos [8] established the following result.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a strong semilattice of semigroups. Suppose that each S_α is an ordered semigroup and that each of the structure maps $\varphi_{\beta,\alpha}$ is isotone. Then the relation "$\sqsubseteq$" defined on S by

$$(\forall x \in S_\alpha)\ (\forall y \in S_\beta) \quad x \sqsubseteq y \Longleftrightarrow \alpha \leq \beta, \quad x \leq \varphi_{\alpha,\beta}(y) \tag{2.4}$$

is a partial order on S, and so $S = \bigcup_{\alpha\in Y} S_\alpha$ forms an ordered semigroup.

We now call an ordered semigroup constructed in the above manner a *strong semilattice of ordered semigroups.*

The following definition of *"pointed semilattice of pointed semigroups"* was given by Blyth and Almeida Santos [8].

Definition 2.3. An ordered semigroup S is said to be a pointed semilattice of pointed semigroups if the following conditions are satisfied:

(1) $S = \bigcup_{\alpha\in Y} S_\alpha$ is a strong semilattice of ordered semigroups;

(2) the semilattice Y has the greatest element;

(3) every ordered semigroup S_α has the greatest element.

By the above definition and the notion of strong semilattice of ordered semigroups, we have the following lemma.

Lemma 2.4. *Let S be an epiorthodox semigroup. Then the band E of idempotents of S is a pointed semilattice of pointed rectangular bands on which the order "$\sqsubseteq$" coincides with the order "$\leq$" on S.*

Proof. By Lemma 2.1, the set of idempotents E of the semigroup S is normal. By applying a theorem of Yamada and Kimura [21], E is known to be a strong semilattice of rectangular bands which can be regarded as the $\mathfrak{D}$-class of E. Clearly, the $\mathfrak{D}$-classes of E are the same classes as the $\mathcal{Y}$-classes, where $\mathcal{Y}$ is the finest inverse semigroup congruence given by

$$(e,f) \in \mathcal{Y} \Longleftrightarrow V(e) = V(f). \tag{2.5}$$

By Lemma 2.2(1), we know that, $e^\circ = \xi e \xi$. Thus, if $(e,f) \in \mathfrak{D}$ then $e^\circ = \xi e \xi = \xi f \xi = f^\circ$. Conversely, if $\xi e \xi = \xi f \xi$ then

$$e = e^3 = e\xi e \xi e = e \xi f \xi e = efe. \tag{2.6}$$

Similarly, we can prove that $f = fef$. Consequently, $e \in V(f)$. This leads to $V(e) = V(f)$, whence $(e,f) \in \mathfrak{D}$. Thus, $\mathfrak{D}$ is defined on E by

$$(e,f) \in \mathfrak{D} \Longleftrightarrow e^\circ = f^\circ. \tag{2.7}$$

Hence, each $\mathfrak{D}$-class has the greatest element and so e° is the greatest element of D_e. Thus, the structure semilattice of E is the set $Y = \xi E \xi = \{e^\circ : e \in E\}$ which has the greatest element, that is, ξ. Now if $e^\circ, f^\circ \in Y$ with $e^\circ \geq f^\circ$, then the structure map $\varphi_{f^\circ, e^\circ} : D_{e^\circ} \to D_{f^\circ}$ is given by

$$(\forall x \in D_{e^\circ}) \quad \varphi_{f^\circ, e^\circ}(x) = x f^\circ x. \tag{2.8}$$

These maps are clearly isotone. Observe that the order "$\sqsubseteq$" coincides with the order "$\leq$" on the semigroup S. In fact, if $x \in D_{e^\circ}$ and $y \in D_{f^\circ}$ satisfy the relation $x \sqsubseteq y$, then $e^\circ \leq f^\circ$ and $x \leq \varphi_{e^\circ, f^\circ}(y) = y e^\circ y \leq y f^\circ y = y$ because D_{f° is a rectangular band. Conversely, if $x \leq y$ then

$$x = x^3 = x \cdot \xi x \xi \cdot x \leq y \cdot \xi x \xi \cdot y = \varphi_{\xi x \xi, \xi y \xi}(y). \tag{2.9}$$

This shows that E is a pointed semilattice of pointed rectangular bands. □

Remark 2.5. It can be easily seen that the structure map $\varphi_{f^\circ, e^\circ} : D_{e^\circ} \to D_{f^\circ}$ preserves the greatest element in order. In fact, we have that

$$\varphi_{f^\circ, e^\circ}(e^\circ) = e^\circ f^\circ e^\circ \geq f^\circ f^\circ f^\circ = f^\circ, \tag{2.10}$$

and whence, $\varphi_{f^\circ, e^\circ}(e^\circ) = f^\circ$ since $f^\circ = f^{\circ\circ} = \max D_{f^\circ}$.

We have already proved in Lemma 2.1 that if S is an epiorthodox semigroup then the band E of S is normal. Now, if we simply ignore the order on S, then S is isomorphic to the quasidirect product of a left normal band, an inverse semigroup, and a right normal band. In studying the regular semigroups whose idempotents satisfy some permutation identities, Yamada established an important result in [22]. To be more precise, we state the following lemma.

Lemma 2.6 (see [22]). *Let S be an inverse semigroup with a semilattice E of idempotents of S. Let L and R be, respectively, a left normal band and a right normal band with a structural decomposition $L = \bigcup_{\alpha \in E} L_\alpha$ and $R = \bigcup_{\beta \in E} R_\beta$.*

Then on the set

$$L \otimes S \otimes R = \{(e, x, f) : x \in S,\ e \in L_{xx^{-1}},\ f \in R_{x^{-1}x}\} \tag{2.11}$$

the multiplication given by

$$(e, x, f)(g, y, h) = (eu, xy, vh), \tag{2.12}$$

where $u \in L_{xy(xy)^{-1}}$ and $v \in R_{(xy)^{-1}xy}$, is well defined and $L \otimes S \otimes R$ forms an orthodox semigroup with a normal band of idempotents. Conversely, every such semigroup can be constructed in the above manner.

In order to establish a structure theorem for the epiorthodox semigroups, we need to find some suitable conditions satisfying the requirements of the construction method given by Yamada in [2] so that an epiorthodox semigroup can be so constructed.

We formulate the following Lemma.

Lemma 2.7. *Let S be a naturally ordered inverse semigroup satisfying the Dubreil-Jacotin condition. Suppose that Green's relations $\mathcal{R}, \mathcal{L}$ on S are, respectively, the left and right regular relations on S. Let L be an ordered left normal band with the greatest element 1_L which is a right identity, and let R be an ordered right normal band with the greatest element 1_R which is a left identity. Then the following statements hold.*

(i) *$L = \bigcup_{\alpha \in E} L_\alpha$ is a pointed semilattice of pointed left zero semigroups, and $R = \bigcup_{\beta \in E} R_\beta$ is also a pointed semilattice of pointed right zero semigroups.*

(ii) *Let $(L \otimes S \otimes R)_c$ denote the set*

$$L \otimes S \otimes R = \{(e, x, f) : x \in S,\ e \in L_{xx^{-1}},\ f \in R_{x^{-1}x}\} \tag{2.13}$$

equipped with the Cartesian order and the multiplication

$$(e, x, f)(g, y, h) = \left(eL^*_{xy(xy)^{-1}}, xy, R^*_{(xy)^{-1}xy}h\right), \tag{2.14}$$

where L^*_α is the greatest element of L_α and R^*_α is the greatest element of R_α. Then $(L \otimes S \otimes R)_c$ forms an epiorthodox semigroup on which Green's relations $\mathcal{L}, \mathcal{R}$ are, respectively, the right and left regular relations.

Proof. (i) Since 1_L is a right identity for L, for $e, f \in L$,

$$e \preceq f \Longrightarrow e = fe \leq f1_L = f, \tag{2.15}$$

and so L is naturally ordered. Since 1_L is the greatest element of L, L satisfies the Dubreil-Jacotin condition. Hence, by applying Lemma 2.4 with $S = E = L$, L is a pointed semilattice of pointed rectangular bands. These $\mathfrak{D}$-class rectangular bands are left zero semigroups, for if $(e, f) \in \mathfrak{D}$, then $1_L e 1_L = e^\circ = f^\circ = 1_L f 1_L$, and so $1_L e = 1_L f$. Consequently, we can deduce that $e = e1_Le = e1_Lf = ef$. Similarly, R is also a pointed semilattice of pointed right zero semigroups. Recall from Lemma 2.4 that the order "$\sqsubseteq$" coincides with the order "$\leq$" in both L and R.

(ii) Suppose that L and R admit the structure decompositions $L = \bigcup_{\alpha\in E} L_\alpha$ and $R = \bigcup_{\beta\in E} R_\beta$, respectively. Observe that, for $\alpha,\beta \in E$, we have that

$$\beta \leq \alpha \Longrightarrow L^*_\beta \leq L^*_\alpha, \quad R^*_\beta \leq R^*_\alpha. \tag{2.16}$$

If $\beta \leq \alpha$, then since the structure mapping in L maps the greatest element to the greatest element, $\phi_{\beta,\alpha}(L^*_\alpha) = L^*_\beta$. Consequently, $L^*_\beta \sqsubseteq L^*_\alpha$ and by Lemma 2.4, $L^*_\beta \leq L^*_\alpha$. Similarly, we have that $R^*_\beta \leq R^*_\alpha$. By applying the Yamada construction in Lemma 2.6, now, we can see that $(L\otimes S\otimes R)_c$ is an orthodox semigroup. Thus, under the Cartesian order and the left regularity of $\mathcal{R}$ and the right regular regularity of $\mathcal{L}$ on S, we can easily see that $(L\otimes S\otimes R)_c$ forms an ordered semigroup. At first, we let $(e,x,f) \leq (e_1,x_1,f_1)$. Then, $x \leq x_1$ and so $xy \leq x_1y$ for every $y \in S$. Since $\mathcal{R}$ is a left regular relation on S, $xy(xy)^{-1} \leq x_1y(x_1y)^{-1}$ and so by the above observation, $L^*_{xy(xy)^{-1}} \leq L^*_{x_1y(x_1y)^{-1}}$. By applying the right regularity of $\mathcal{L}$, we can similarly show that $R^*_{(xy)^{-1}xy} \leq R^*_{(x_1y)^{-1}x_1y}$. Thus, we obtain the following:

$$\begin{aligned}(e,x,f)(g,y,h) &= \left(eL^*_{xy(xy)^{-1}}, xy, R^*_{(xy)^{-1}xy}h\right)\\ &\leq \left(e_1L^*_{x_1y(x_1y)^{-1}}, x_1y, R^*_{(x_1y)^{-1}x_1y}h\right)\\ &= (e_1,x_1,f_1)(g,y,h).\end{aligned} \tag{2.17}$$

By using similar arguments, we can show that

$$(g,y,h)(e,x,f) \leq (g,y,h)(e_1,x_1,f_1), \tag{2.18}$$

and so $(L\otimes S\otimes R)_c$ forms an ordered semigroup.

Since each L_α is a left zero semigroup and each R_α is a right zero semigroup, we can easily verify that the idempotents of $(L\otimes S\otimes R)_c$ are the elements of the form (e,x,f), where $x \in E$. Suppose that $(e,x,f),(g,y,h)$ are idempotents in $(L\otimes S\otimes R)_c$ with $(e,x,f) \preceq (g,y,h)$. Then $(e,x,f) = (e,x,f)(g,y,h) = (g,y,h)(e,x,f)$. This leads to $x = xy = yx$ and so $x \preceq y$ in E. Since S is naturally ordered, we have that $x \leq y$. Also, we have $e = gL^*_{yx(yx)^{-1}} \leq g1_L = g$ and similarly, $f \leq h$. This shows that $(L\otimes S\otimes R)_c$ is naturally ordered.

Since S satisfies the Dubreil-Jacotin condition, there exists an ordered group G and an isotone surjective homomorphism $\theta : S \to G$ such that $\{x \in S : \theta(x^2) \leq \theta(x)\}$ has the greatest element ξ. Define the mapping $\psi : (L\otimes S\otimes R)_c \to G$ by $\psi(e,x,f) = \theta(x)$. Then, ψ is an isotone surjective homomorphism.

We now proceed to show that

$$\max\left\{(e,x,f) \in (L\otimes S\otimes R)_c : \psi\left[(e,x,f)^2\right] \leq \psi(e,x,f)\right\} = (1_L,\xi,1_R). \tag{2.19}$$

Since $(1_L, \xi, 1_R)$ is an idempotent,

$$(1_L, \xi, 1_R) \in \left\{ (e, x, f) \in (L \otimes S \otimes R)_c : \psi\left[(e, x, f)^2 \right] \le \psi(e, x, f) \right\}. \tag{2.20}$$

On the other hand, we have that

$$\theta\left(x^2\right) = \psi\left[(e, x, f)^2\right] \le \psi(e, x, f) = \theta(x) \Longrightarrow x \le \xi, \tag{2.21}$$

and $e \le 1_L, f \le 1_R$ are clear. Hence, $(e, x, f) \le (1_L, \xi, 1_R)$, and so

$$\max\left\{ (e, x, f) \in (L \otimes S \otimes R)_c : \psi\left[(e, x, f)^2 \right] \le \psi(e, x, f) \right\} = (1_L, \xi, 1_R). \tag{2.22}$$

Consequently, we can see that $(1_L, \xi, 1_R)$ is the greatest idempotent of $(L \otimes S \otimes R)_c$. This shows that $(L \otimes S \otimes R)_c$ is indeed a semigroup satisfying the Dubreil-Jacotin condition. □

Now, we have proved that $(L \otimes S \otimes R)_c$ is an epiorthodox semigroup. Finally, we consider Green's relation $\mathcal{R}$ on the semigroup $(L \otimes S \otimes R)_c$. We need to show that the relation $\mathcal{R}$ is a left regular relation on $(L \otimes S \otimes R)_c$. For this purpose, we need to identify $(e, x, f)^\circ$. By Lemma 2.2, we have that $(e, x, f)^\circ = (1_L, \xi, 1_R)(e, x, f)'(1_L, \xi, 1_R)$, where $(e, x, f)' \in V(e, x, f)$. We now show that $(eL^*_{x^{-1}x}, x^{-1}, g) \in V(e, x, f)$. In fact, we can deduce the following equalities:

$$\begin{aligned}
(e, x, f)\left(eL^*_{x^{-1}x}, x^{-1}, g\right)(e, x, f) &= \left(eL^*_{xx^{-1}}, xx^{-1}, R^*_{xx^{-1}}g\right)(e, x, f) \\
&= \left(e, xx^{-1}, R^*_{xx^{-1}}g\right)(e, x, f) \\
&= \left(eL^*_{xx^{-1}}, x, R^*_{x^{-1}x}f\right) \\
&= (e, x, f), \\
\left(eL^*_{x^{-1}x}, x^{-1}, g\right)(e, x, f)\left(eL^*_{x^{-1}x}, x^{-1}, g\right) &= \left(eL^*_{x^{-1}x}, x^{-1}x, R^*_{x^{-1}x}f\right)\left(e, x^{-1}, g\right) \\
&= \left(eL^*_{x^{-1}x}L^*_{x^{-1}x}, x^{-1}, R^*_{xx^{-1}}g\right) \\
&= \left(eL^*_{x^{-1}x}, x^{-1}, g\right).
\end{aligned} \tag{2.23}$$

So $(eL^*_{x^{-1}x}, x^{-1}, g) \in V(e, x, f)$. Now, we have that

$$(e, x, f)^\circ = (1_L, \xi, 1_R)\left(eL^*_{x^{-1}x}, x^{-1}, g\right)(1_L, \xi, 1_R). \tag{2.24}$$

Hence

$$\begin{aligned}(e,x,f)(e,x,f)^{\circ} &= (e,x,f)(1_L,\xi,1_R)\left(eL^{*}_{x^{-1}x},x^{-1},g\right)(1_L,\xi,1_R)\\ &= (eL^{*}_{xx^{-1}},x,R^{*}_{x^{-1}x}1_R)\left(eL^{*}_{x^{-1}x}L^{*}_{x^{-1}x},x^{-1},R^{*}_{xx^{-1}}1_R\right)\\ &= (e,x,R^{*}_{x^{-1}x}1_R)\left(eL^{*}_{x^{-1}x},x^{-1},R^{*}_{xx^{-1}}1_R\right)\\ &= \left(eL^{*}_{xx^{-1}},xx^{-1},R^{*}_{xx^{-1}}R^{*}_{xx^{-1}}1_R\right)\\ &= \left(e,xx^{-1},R^{*}_{xx^{-1}}1_R\right).\end{aligned} \tag{2.25}$$

Consequently, we can deduce that $\mathcal{R}$ is a left regular relation on $(L\otimes S\otimes R)_c$. Similarly, $\mathcal{L}$ is a right regular relation on $(R\otimes S\otimes L)_c$.

We formulate the following lemma.

Lemma 2.8. *If $Y = (L\otimes S\otimes R)_c$ has a band of idempotents $E(Y)$ and containing the greatest idempotent ξ, then there are ordered semigroup isomorphisms*

$$\xi Y\xi \simeq S, \qquad E(Y)\xi \simeq L, \qquad \xi E(Y) \simeq R. \tag{2.26}$$

Proof. Since $\xi = (1_L,1,1_R) = (L^{*}_1,1,R^{*}_1)$, it can be readily seen that

$$\xi(e,x,f)\xi = (L^{*}_{xx^{-1}},x,R^{*}_{x^{-1}x}). \tag{2.27}$$

The mapping

$$\theta : \xi Y\xi \longrightarrow S, \quad \theta(L^{*}_{xx^{-1}},x,R^{*}_{x^{-1}x}) = x \tag{2.28}$$

is a semigroup isomorphism. Since Green's relations $\mathcal{R},\mathcal{L}$ are, respectively, the left and right regular relations on S, θ is an order isomorphism.

For $\alpha,\beta\in E$, since the structure maps preserve the greatest elements, we have that

$$L^{*}_{\alpha}L^{*}_{\beta} = \varphi_{\alpha\beta,\alpha}(L^{*}_{\alpha})\varphi_{\alpha\beta,\beta}\left(L^{*}_{\beta}\right) = L^{*}_{\alpha\beta}L^{*}_{\alpha\beta} = L^{*}_{\alpha\beta}. \tag{2.29}$$

Now $E(Y) = \{(e,\alpha,f);\alpha\in E\}$, and

$$(e,x,f)\xi = (e,x,f)(L^{*}_1,1,R^{*}_1) = (eL^{*}_{\alpha},\alpha,R^{*}_{\alpha}R^{*}_1) = (e,\alpha,R^{*}_{\alpha}). \tag{2.30}$$

Consider the mapping $\psi : E(Y)\xi \rightarrow L$, $\psi(e,\alpha,R^{*}_{\alpha}) = e$. Because

$$(e,\alpha,R^{*}_{\alpha})\left(g,\beta,R^{*}_{\beta}\right) = \left(eL^{*}_{\alpha\beta},\alpha\beta,R^{*}_{\alpha\beta}R^{*}_{\beta}\right) = \left(eL^{*}_{\alpha\beta},\alpha\beta,R^{*}_{\alpha\beta}\right), \tag{2.31}$$

and since $L_{\alpha\beta}$ is a left zero semigroup, we can deduce that

$$eL^*_{\alpha\beta} = \varphi_{\alpha\beta,\alpha}(e)\varphi_{\alpha\beta,\alpha\beta}\left(L^*_{\alpha\beta}\right) = \varphi_{\alpha\beta,\alpha}(e)L^*_{\alpha\beta} = \varphi_{\alpha\beta,\alpha}(e),$$
$$eg = \varphi_{\alpha\beta,\alpha}(e)\varphi_{\alpha\beta,\beta}(g) = \varphi_{\alpha\beta,\alpha}(e). \tag{2.32}$$

Thus, ψ is a semigroup homomorphism. Obviously, ψ is surjective. Since for arbitrary $e \in L_\alpha$ and $g \in L_\beta$ the equality $e = g$ implies that $\alpha = \beta$, ψ is injective. ψ is clearly isotone. Finally, if $e \leq g$ with $e \in L_\alpha, g \in L_\beta$, then $e \sqsubseteq g$ and therefore $\alpha \leq \beta$ by Lemma 2.4. This shows that ψ is an order isomorphism.

Similarly, we can prove that $\xi E(Y) \simeq R$. □

3. Main Theorem

In this section, we will give a structure theorem for the epiorthodox semigroups. We establish the converse of Lemma 2.7 by showing every epiorthodox semigroup S on which Green's relations $\mathcal{R}, \mathcal{L}$ are, respectively, the left and right regular relations which arise in the way as stated in Lemma 2.7. Our Lemma 2.8 indicates how this goal can be achieved.

Theorem 3.1 (main theorem). *Let ξ be the greatest idempotent of an epiorthodox semigroup T and E the band of idempotents of T. Then $E\xi$ is an ordered left normal band with the greatest element which is a right identity, and ξE is an ordered right normal band with the greatest element which is a left identity. Moreover, $\xi T\xi$ is a naturally ordered inverse semigroup satisfying the Dubreil-Jacotin condition, and its semilattice of idempotents $\xi E\xi$ is the structure semilattice of $E\xi$ and of ξE. If Green's relations $\mathcal{R}, \mathcal{L}$ are, respectively, the left and right regular relations on T, then T and $(E\xi \otimes \xi T\xi \otimes \xi E)_c$ are order isomorphic, that is,*

$$T \simeq (E\xi \otimes \xi T\xi \otimes \xi E)_c. \tag{3.1}$$

Proof. Clearly, ξ is the greatest element of $E\xi$ and ξ is a right identity for $E\xi$. By Lemma 2.1, E is a normal band and hence $efgh = egfh$ for all $e, f, g, h \in E$. Take $e, f, g \in E\xi$ and $h = \xi$. Then $efg = egf$ for all $e, f, g \in E\xi$ because ξ is a right identity of $E\xi$. Thus, $E\xi$ is a band since ξ is a middle unit and so $E\xi$ is a left normal band. Similarly, ξE is a right normal band with the greatest element ξ which is a left identity. As for $\xi T\xi$, it is clear that it is a subsemigroup of T and is regular because T itself is regular and ξ is a middle unit. If $x \in E$, then it is clear that $\xi x\xi \in E(\xi T\xi)$. Conversely, if $\xi x\xi \in E(\xi T\xi)$, then $\xi x\xi = \xi x\xi \cdot \xi x\xi = \xi x^2\xi$. Let $x' \in V(x)$. Then, we have that $x'\xi x\xi x' = x'\xi x^2\xi x'$ and so $x' = x'x^2x'$. This leads to $x = xx'x = xx'x^2x'x = x^2$ and so $x \in E$. Consequently, $E(\xi T\xi) = \xi E\xi$. Clearly, $E(\xi T\xi)$ is a semilattice since E is a normal band. Thus,

$$\xi e\xi \cdot \xi f\xi = \xi \cdot \xi e\xi \cdot \xi f\xi \cdot \xi = \xi \cdot \xi f\xi \cdot \xi e\xi \cdot \xi = \xi f\xi \cdot \xi e\xi. \tag{3.2}$$

Hence, we have proved that $\xi T\xi$ is an inverse subsemigroup of T.

To show that the Dubreil-Jacotin condition is satisfied by $\xi T\xi$, we let $\varphi : T \to G$ be an isotone epimorphism from T onto an ordered group G such that $\{x \in T : \varphi(x^2) \leq \varphi(x)\}$ has the greatest element. Then $\varphi(\xi) = 1_G$ and so $\varphi|_{\xi T\xi} : \xi T\xi \rightarrow G$ is also an isotone epimorphism. It can be easily verified that

$$\max\left\{ x \in \xi T\xi : \varphi|_{\xi T\xi}\left(x^2\right) \leq \varphi\Big|_{\xi T\xi}(x) \right\} = \xi, \tag{3.3}$$

and hence the Dubreil-Jacotin condition is satisfied on $\xi T\xi$. Moreover, since T is naturally ordered, so is $\xi T\xi$. The structure semilattice of $E\xi$ is $\xi E\xi$. This follows from the proof of Lemma 2.4. Similarly, the structure semilattice of ξE is $\xi E\xi$.

By Lemma 2.7, we can construct an epiorthodox semigroup $(E\xi \otimes \xi T\xi \otimes \xi E)_c$. Now, suppose that Green's relations $\mathcal{R}, \mathcal{L}$ are, respectively, the left and right regular relations on T. Then we consider the following mapping:

$$\chi : T \longrightarrow (E\xi \otimes \xi T\xi \otimes \xi E)_c \tag{3.4}$$

defined by

$$\chi(x) = (xx^{\circ}, \xi x\xi, x^{\circ}x). \tag{3.5}$$

Note that χ is well defined. On the one hand, by Lemma 2.2, we can easily deduce that $xx^{\circ} = xx^{\circ}\xi \in E\xi$ and, similarly, $x^{\circ}x \in \xi E$; on the other hand, we have that

$$xx^{\circ} \in L_{\xi xx^{\circ}\xi} = L_{\xi x\xi(\xi x\xi)^{\circ}} = L_{\xi x\xi(\xi x\xi)^{-1}}. \tag{3.6}$$

Because ξ is a middle unit, $xx^{\circ} \cdot \xi x\xi \cdot x^{\circ}x = x$ and so χ is injective.

To show that χ is also surjective, we first let $(e\xi, \xi x\xi, \xi f) \in (E\xi \otimes \xi T\xi \otimes \xi E)_c$. Then we have that

$$e\xi \in L_{\xi x\xi(\xi x\xi)^{-1}} = L_{\xi x\xi(\xi x\xi)^{\circ}} = L_{\xi xx^{\circ}}, \tag{3.7}$$

and hence $(e\xi, \xi xx^{\circ}) \in \mathfrak{D}$. Therefore, by Lemma 2.2, we have that

$$\xi e\xi = (e\xi)^{\circ} = (\xi xx^{\circ})^{\circ} = \xi xx^{\circ}. \tag{3.8}$$

Similarly, $\xi f \in R_{x^{\circ}x\xi}$ implies that $\xi f\xi = x^{\circ}x\xi$. Now, consider

$$\chi(exf) = (exf(exf)^{\circ}, \xi exf\xi, (exf)^{\circ}exf). \tag{3.9}$$

By the above observation, we have that

$$\begin{aligned} exf(exf)^{\circ} &= exff^{\circ}x^{\circ}e^{\circ} \\ &= ex\xi f\xi x^{\circ}e^{\circ} \\ &= exx^{\circ}x\xi x^{\circ}e^{\circ} \\ &= exx^{\circ}e^{\circ} \\ &= e\xi xx^{\circ}e^{\circ} \\ &= e\xi e\xi\xi e\xi \\ &= e\xi. \end{aligned} \tag{3.10}$$

Similarly, $(exf)^{\circ}exf = \xi f$ and

$$\xi exf\xi = \xi e\xi x\xi f\xi = \xi xx^{\circ}xx^{\circ}x\xi = \xi x\xi. \tag{3.11}$$

Thus

$$\chi(exf) = (e\xi, \xi x\xi, \xi f), \tag{3.12}$$

and so we have proved that χ is a surjective mapping.

We now deduce that

$$\begin{aligned} \chi(x)\chi(y) &= (xx^{\circ}, \xi x\xi, x^{\circ}x)(yy^{\circ}, \xi y\xi, y^{\circ}y) \\ &= \left(xx^{\circ}L^{*}_{\xi xy\xi(\xi xy\xi)^{-1}}, \xi xy\xi, R^{*}_{(\xi xy\xi)^{-1}\xi xy\xi}y^{\circ}y\right) \\ &= \left(xx^{\circ}L^{*}_{\xi xyy^{\circ}x^{\circ}}, \xi xy\xi, R^{*}_{y^{\circ}x^{\circ}xy\xi}y^{\circ}y\right) \\ &= (xx^{\circ}\xi xyy^{\circ}x^{\circ}\xi, \xi xy\xi, \xi y^{\circ}x^{\circ}xy\xi y^{\circ}y) \\ &= (xyy^{\circ}x^{\circ}, \xi xy\xi, y^{\circ}x^{\circ}xy) \\ &= (xy(xy)^{\circ}, \xi xy\xi, (xy)^{\circ}xy) \\ &= \chi(xy). \end{aligned} \tag{3.13}$$

Thus, χ is a semigroup isomorphism.

Since Green's relations $\mathcal{R}, \mathcal{L}$ are, respectively, the left and the right regular relations on T, χ is isotone; and since

$$\begin{aligned} \chi(x) \leq \chi(y) &\Longrightarrow xx^{\circ} \leq yy^{\circ}, \quad \xi x\xi \leq \xi y\xi, \quad x^{\circ}x \leq y^{\circ}y \\ &\Longrightarrow x = xx^{\circ}\xi x\xi x^{\circ}x \leq yy^{\circ}\xi y\xi y^{\circ}y = y, \end{aligned} \tag{3.14}$$

it follows that χ is an order isomorphism between the ordered semigroups T and $(E\xi \otimes \xi T\xi \otimes \xi E)_c$, that is,

$$T \simeq (E\xi \otimes \xi T\xi \otimes \xi E)_c. \tag{3.15}$$

The proof is completed. □

We conclude the above results by the following remark.

Remark 3.2. It is noted that the following statements hold on the semigroup $T = (E\xi \otimes \xi T\xi \otimes \xi E)_c$:

(1) $(e, x, f)\mathcal{R}^T(g, y, h) \Leftrightarrow x\mathcal{R}^{\xi T\xi}y,\ e = g$;

(2) $(e, x, f)\mathcal{L}^T(g, y, h) \Leftrightarrow x\mathcal{L}^{\xi T\xi}y,\ f = h$;

(3) $(e, x, f)\mathfrak{D}^T(g, y, h) \Leftrightarrow x\mathfrak{D}^{\xi T\xi}y$.

Proof. We first deduce the following implications:

$$\begin{aligned}
(e, x, f)\mathcal{R}^T(g, y, h) &\Longleftrightarrow (e, x, f)(e, x, f)^\circ = (g, y, h)(g, y, h)^\circ \\
&\Longleftrightarrow \left(e, xx^{-1}, R^*_{xx^{-1}}1_R\right) = \left(g, yy^{-1}, R^*_{yy^{-1}}1_R\right) \\
&\Longleftrightarrow e = g, \quad xx^{-1} = yy^{-1} \\
&\Longleftrightarrow e = g, \quad x\mathcal{R}^{\xi T\xi}y.
\end{aligned} \tag{3.16}$$

Hence (1) holds. As (2) is the dual of (1), (2) holds. It follows from (1) and (2) that (3) holds. □

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