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Research Article

On Solvable Groups of Arbitrary Derived Length and Small Commutator Length

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Wreath product constructions has been used to obtain for any positive integer n, solvable groups of derived length n, and commutator length at most equal to 2.

1. Introduction

Let G be a group and G' its commutator subgroup. Denote by $c(G)$ the minimal number such that every element of G' can be expressed as a product of at most $c(G)$ commutators. A group G is called a c-group if $c(G)$ is finite. For any positive integer n, denote by c_n the class of groups with commutator length, $c(G) = n$.

Let $F_{n,t} = \langle x_1, \ldots, x_n \rangle$ and $M_{n,t} = \langle x_1, \ldots, x_n \rangle$ be, respectively, the free nilpotent group of rank n and nilpotency class t and the free metabelian nilpotent group of rank n and nilpotency class t. Stroud, in his Ph.D. thesis [1] in 1966, proved that for all t, every element of the commutator subgroup $F'_{n,t}$ can be expressed as a product of n commutators. In 1985, Allambergenov and Roman'kov [2] proved that $c(M_{n,t})$ is precisely n, provided that $n \geq 2$, $t \geq 4$, or $n \geq 3$, $t \geq 3$. In [3], Bavard and Meigniez considered the same problem for the n-generator free metabelian group M_n. They showed that the minimum number $c(M_n)$ of commutators required to express an arbitrary element of the derived subgroup M'_n satisfies

$$\left[\frac{n}{2}\right] \leq c(M_n) \leq n, \tag{1.1}$$

where $[n/2]$ is the greatest integer part of $n/2$.

Since $F_{n,3}$ groups are metabelian, the result of Allambergenov and Roman'kov [2] shows that $c(M_n) \geq n$ for $n \geq 3$, and in [4], we considered the remaining case $n = 2$. We have $c(M_n) = n$, for all $n \geq 2$. These results were extended in [4] to the larger class of abelian by nilpotent groups, and it was shown that $c(G) = n$ if G is a (non-abelian) free abelian by nilpotent group of rank n.

In [5], we proved that $2 \leq c(W) \leq 3$, where $W = G \wr C_\infty$ is the wreath product of a nontrivial group G with the infinite cyclic group. Recently in [6], we have generalized this result. Let $W = G \wr H$ be the wreath product of G by a n-generator abelian group H. We have proved that every element of W' is a product of at most $n+2$ commutators, and every element of W^2 is a product of at most $3n + 4$ squares in W.

In the case of a finite d-generator solvable group G of solvability length r, Hartley [7] proved that $c(G) \leq d + (2d - 1)(r - 1)$. And in a recent paper, Segal [8] has proved that in a finite d-generator solvable group G, every element of G' can be expressed as a product of $72d^2 + 46d$ commutators.

The problem remains open for the d-generator solvable group in general. In the section of open problems in the site of Magnus project (http://www.grouptheory.org), Kargapolov asks the question ($S4$) as follows:

> "Is there a number $N = N(k, d)$ so that every element of the commutator subgroup of a free solvable group of rank k and solvability length d, is a product of N commutators?"

The answer is "yes" for free metabelian groups; see [2], and for free solvable groups of solvability length 3, see [9].

In [10], we found lower and upper bound for the commutator length of a finitely generated nilpotent by abelian group. We also considered an n-generator solvable group G such that G has a nilpotent by abelian normal subgroup K of finite index. If K is an s-generator group, then $c(G) \leq s(s+1)/2+72n^2+47n$. We considered the class of solvable group of finite Prüfer rank s, and we proved that every element of its commutator subgroup is equal to a product of at most $s(s + 1)/2 + 72s^2 + 47s$. And as a consequence of the above results, we proved that if A is a normal subgroup of a solvable group G such that G/A is a d-generator finite group and A has finite Prüfer rank s, then $c(G) \leq s(s + 1)/2 + 72(s^2 + n^2) + 47(s + n)$. These bounds depend only on the number of generators of the groups.

In [11], we considered a solvable group satisfying the maximal condition for normal subgroups. We found an upper bound for the commutator length of this class of groups. The bound depends on the number of generators of the group G, the solvability length of the group, and the number of generators of the group $G^{(k)}$ as a G-subgroup. In particular, if in a finitely generated solvable group G, each term of the derived series is finitely generated as a G-subgroup, then G is a c-group. We also gave the precise formulas for expressing every element of the derived group to the product of commutators.

In the present paper, we use wreath product constructions to obtain for any positive integer n, solvable groups of derived length n and commutator length equal to 1 or 2.

2. Main Results

Notation. Let N be a subgroup of a group G, and $x, y \in G$. Then, $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$ and $[N, x] = \{[n, x] : n \in N\}$.

The main results of this paper are as follows.

First, we need the following generalization of Lemma 9.22 in [12].

Lemma 2.1. *Let G be any solvable group, say of derived length n, and let T be a cyclic group. Then, $W = G \wr T$ is a solvable group of derived length $n+1$.*

Theorem 2.2. *For any positive integer n, there are solvable groups of derived length n, in which every element of G' is a commutator.*

Theorem 2.3. *The commutator length of the wreath product of a c_1-group by the infinite cyclic group is at most equal to 2.*

In particular, we have the following consequences of these results.

Corollary 2.4. *For any positive integer n, there are solvable groups of derived length n, with commutator length at most equal to 2.*

Corollary 2.5. *For any positive integer n, there are n-generator solvable groups of derived length n, in which every element of G' is a commutator.*

3. Proofs

The proof of Lemma 2.1 is similar to the proof of Lemma 9.22 in [12].

Proof of Lemma 2.1. Let $T = \langle t \rangle$ and $W = G \wr T$. Let $B = \mathrm{Dr}_{1 \le i \le |T|} G_i$ be the base group of W. Then, $W = B \rtimes T$ is the semidirect product of B by T, where the action of T on B is given by $g_i^t = g_{i+1} \in G_{i+1}$. Since G is solvable of derived length n, B is also solvable of derived length n.

Since W/B is abelian, W is solvable and $W' \le B$. It is clear that for any i,

$$g_{i-1}^{-1} g_i = g_{i-1}^{-1} g_{i-1}^{t} = \left[g_{i-1}, t\right] \in W'. \tag{3.1}$$

Now, assume that π denotes the projection of B on to G_i and let $\pi' = \pi|_{W'}$. In view of (3.1), it is clear that π' is surjective. And W' is a solvable group of derived length at least n. Since $W' \le B$ and B is solvable, of derived length n, W' is a solvable group of derived length n. Therefore, W is of derived length equal to $n+1$. □

The proof of Theorem 2.2 requires the following theorem proved in [9].

Theorem 3.1 (Rhemtulla [9]). *The commutator length of the wreath product of a c_1-group by a finite cyclic group is again a c_1-group.*

Now, we turn to the proof of Theorem 2.2.

Proof of Theorem 2.2. Let A be any nontrivial abelian group, and let $T = \langle t \rangle$ be a finite cyclic group. Define $G_1 = A, \quad G_n = G_{n-1} \wr T$. Repeated application of Lemma 2.1 shows that for every positive integer n, G_n is a solvable group of derived length n. By our assumption, $G_2 = A \wr T$. Let B be the base group of G_2. Then, $G_2 = B \rtimes T$ and

$$G_2' = [BT, BT] = [B, T] = \{[b, t]; b \in B\}. \tag{3.2}$$

This easily follows from the relations $[b, t^{-1}] = [(b^{-1})^{t^{-1}}, t]$ and $[b, t][b', t] = [bb', t]$ for all $b, b' \in B$ which hold when B is a normal abelian subgroup. Hence, every elements of G_2' is a commutator. Now, for every positive integer $n \geq 3$, since $G_n = G_{n-1} \wr T$ and every elements of G'_{n-1} is a commutator, repeated application of Lemma 2.1 and Rhemtulla's result shows that the group G_n, obtained by taking successive wreath product of finite cyclic groups satisfies the desired property and the proof is complete. □

The proof of Theorem 2.3 requires the following lemma proved in [5].

Lemma 3.2. *Let A be a free abelian group and $W = A\mathrm{wr}C_\infty$, where C_∞ is the infinite cyclic group, then W is a c-group and furthermore the commutator length of W is equal to 1.*

Proof of Theorem 2.3. Let $W = G \wr T$, where G is a c_1-group and $T = \langle t \rangle \simeq \mathbb{Z}$. Then, $W = B \rtimes T$, where $B = \mathrm{Dr}_{i\in Z} G_i$, where $G_i \simeq G$. Modulo $B', W \simeq A \wr \mathbb{Z}$, where $A \simeq G/G'$. Since A is isomorphic to F/K for some free group F and $A \wr \mathbb{Z}$ is a quotient of $F \wr \mathbb{Z}$, it is clear that $c(A \wr \mathbb{Z}) \leq c(F \wr \mathbb{Z})$; hence, by Lemma 3.2, every element of W'/B' is a commutator. Now, $B' = \mathrm{Dr}_{i\in Z} G'_i$, and since $G \in c_1$, every element of W' is a product of two commutators. □

Remark 3.3. Rhemtulla had introduced in [9] a group which is the wreath product of a c_1-group by the infinite cyclic group, and it is no longer a c_1-group.

Now, we prove Corollary 2.4.

Proof of Corollary 2.4. Let G_{n-1} be the group defined in Theorem 2.2, and let $H = G_{n-1} \wr \mathbb{Z}$. Since G_{n-1} is a c_1-group of derived length $n-1$, it follows from Lemma 2.1 that H is a solvable group of derived length n. Now, to complete the proof, it is enough to apply Theorem 2.3 to H. □

Finally, we prove Corollary 2.5.

Proof of Corollary 2.5. Let A be any non trivial cyclic group, T any nontrivial finite cyclic group, and G_n the group defined in Theorem 2.2. Then, by Theorem 2.3 [13], G_n is a n generator group. □

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