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Research Article

Graphs Based on BCK/BCI-Algebras

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The associated graphs of BCK/BCI-algebras will be studied. To do so, the notions of (l-prime) quasi-ideals and zero divisors are first introduced and related properties are investigated. The concept of associative graph of a BCK/BCI-algebra is introduced, and several examples are displayed.

1. Introduction

Many authors studied the graph theory in connection with (commutative) semigroups and (commutative and noncommutative) rings as we can refer to references. For example, Beck [1] associated to any commutative ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0), with two vertices a, b joined by an edge in case $ab = 0$. Also, DeMeyer et al. [2] defined the zero-divisor graph of a commutative semigroup S with zero ($0x = 0\ \forall x \in S$).

In this paper, motivated by these works, we study the associated graphs of BCK/BCI-algebras. We first introduce the notions of (l-prime) quasi-ideals and zero divisors and investigated related properties. We introduce the concept of associative graph of a BCK/BCI-algebra and provide several examples. We give conditions for a proper (quasi-) ideal of a BCK/BCI-algebra to be l-prime. We show that the associative graph of a BCK-algebra is a connected graph in which every nonzero vertex is adjacent to 0, but the associative graph of a BCI-algebra is not connected by providing an example.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following axioms:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)$ $(x * x = 0)$,

(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCK/BCI-algebra X satisfies the following conditions:

(a1) $(\forall x \in X)$ $(x * 0 = x)$,

(a2) $(\forall x, y, z \in X)$ $(x * y = 0 \Rightarrow (x * z) * (y * z) = 0, (z * y) * (z * x) = 0)$,

(a3) $(\forall x, y, z \in X)$ $((x * y) * z = (x * z) * y)$,

(a4) $(\forall x, y, z \in X)$ $(((x * z) * (y * z)) * (x * y) = 0)$.

We can define a partial ordering $\leq$ on a BCK/BCI-algebra X by $x \leq y$ if and only if $x * y = 0$.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b1) $0 \in A$,

(b2) $(\forall x, y \in X)$ $(x * y \in A, y \in A \Rightarrow x \in A)$.

We refer the reader to the books [3, 4] for further information regarding BCK/BCI-algebras.

3. Associated Graphs

In what follows, let X denote a BCK/BCI-algebra unless otherwise specified.

For any subset A of X, we will use the notations $r(A)$ and $l(A)$ to denote the sets

$$\begin{aligned} r(A) &:= \{x \in X \mid a * x = 0, \forall a \in A\}, \\ l(A) &:= \{x \in X \mid x * a = 0, \forall a \in A\}. \end{aligned} \tag{3.1}$$

Proposition 3.1. *Let A and B be subsets of X, then*

(1) *$A \subseteq l(r(A))$ and $A \subseteq r(l(A))$,*

(2) *If $A \subseteq B$, then $l(B) \subseteq l(A)$ and $r(B) \subseteq r(A)$,*

(3) *$l(A) = l(r(l(A)))$ and $r(A) = r(l(r(A)))$.*

Proof. Let $a \in A$ and $x \in l(A)$, then $x * a = 0$, and so $a \in r(l(A))$. This says that $A \subseteq r(l(A))$. Dually, $A \subseteq l(r(A))$. Hence, (1) is valid.

Assume that $A \subseteq B$ and let $x \in l(B)$, then $x * b = 0$ for all $b \in B$, which implies from $A \subseteq B$ that $x * b = 0$ for all $b \in A$. Thus, $x \in l(A)$, which shows that $l(B) \subseteq l(A)$. Similarly, we have $r(B) \subseteq r(A)$. Thus, (2) holds.

Using (1) and (2), we have $l(r(l(A))) \subseteq l(A)$ and $r(l(r(A))) \subseteq r(A)$. If we apply (1) to $l(A)$ and $r(A)$, then $l(A) \subseteq l(r(l(A)))$ and $r(A) \subseteq r(l(r(A)))$. Hence, $l(A) = l(r(l(A)))$ and $r(A) = r(l(r(A)))$. $\square$

Table 1: $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	a	a	0	0
d	d	b	a	b	0

Table 2: $*$-operation.

$*$	0	1	a	b	c
0	0	0	c	c	a
1	1	0	c	c	a
a	a	a	0	0	c
b	b	a	1	0	c
c	c	c	a	a	0

Definition 3.2. A nonempty subset I of X is called a *quasi-ideal* of X if it satisfies

$$(\forall x \in X) \quad (\forall y \in I) \quad (x * y = 0 \Longrightarrow x \in I). \tag{3.2}$$

Example 3.3. Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 1, then $(X; *, 0)$ is a BCK-algebra (see [4]). The set $I := \{0, a, b\}$ is a quasi-ideal of X.

Obviously, every quasi-ideal I of a BCK-algebra X contains the zero element 0. The following example shows that there exists a quasi-ideal I of a BCI-algebra X such that $0 \notin I$.

Example 3.4. Let $X = \{0, 1, a, b, c\}$ be a set with the $*$-operation given by Table 2, then $(X; *, 0)$ is a BCI-algebra (see [3]). The set $I := \{0, 1, a\}$ is a quasi-ideal of X containing the zero element 0, but the set $J := \{a, b, c\}$ is a quasi-ideal of X which does not contain the zero element 0.

Obviously, every ideal of X is a quasi-ideal of X, but the converse is not true. In fact, the quasi-ideal $I := \{0, a, b\}$ in Example 3.3 is not an ideal of X. Also, quasi-ideals I and J in Example 3.4 are not ideals of X.

Definition 3.5. A (quasi-) ideal I of X is said to be *l-prime* if it satisfies

(i) I is proper, that is, $I \neq X$,

(ii) $(\forall x, y \in X)$ $(l(\{x, y\}) \subseteq I \Rightarrow x \in I$ or $y \in I)$.

Example 3.6. Consider the BCK-algebra $X = \{0, a, b, c, d\}$ with the operation $*$ which is given by the Table 3, then the set $I = \{0, a, c\}$ is an l-prime ideal of X.

Theorem 3.7. *A proper (quasi-) ideal I of X is l-prime if and only if it satisfies*

$$l(\{x_1, \ldots, x_n\}) \subseteq I \Longrightarrow (\exists i \in \{1, \ldots, n\}) \quad (x_i \in I), \tag{3.3}$$

for all $x_1, \ldots, x_n \in X$.

Table 3: $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	a	c	0	a
d	d	d	d	d	0

Table 4: $*$-operation.

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Proof. Assume that I is an l-prime (quasi-) ideal of X. We proceed by induction on n. If $n = 2$, then the result is true. Suppose that the statement holds for $n-1$. Let $x_1, \ldots, x_n \in X$ be such that $l(\{x_1, \ldots, x_{n-1}, x_n\}) \subseteq I$. If $y \in l(\{x_1, \ldots, x_{n-1}\})$, then $l(\{y, x_n\}) \subseteq l(\{x_1, \ldots, x_{n-1}, x_n\}) \subseteq I$. Assume that $x_n \notin I$, then $y \in I$ by the l-primeness of I, which shows that $l(\{x_1, \ldots, x_{n-1}\}) \subseteq I$. Using the induction hypothesis, we conclude that $x_i \in I$ for some $i \in \{1, \ldots, n-1\}$. The converse is clear. □

For any $x \in X$, we will use the notation Z_x to denote the set of all elements $y \in X$ such that $l(\{x, y\}) = \{0\}$, that is,

$$Z_x := \{y \in X \mid l(\{x, y\}) = \{0\}\}, \tag{3.4}$$

which is called the set of zero divisors of x.

Lemma 3.8. *If X is a BCK-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X$.*

Proof. Let $x \in X$ and $a \in l(\{x, 0\})$, then $a * x = 0 = a * 0 = a$, and so $l(\{x, 0\}) = \{0\}$ for all $x \in X$. □

If X is a BCI-algebra, then Lemma 3.8 does not necessarily hold. In fact, let $X = \{0, 1, 2, a, b\}$ be a set with the $*$-operation given by Table 4, then $(X; *, 0)$ is a BCI-algebra (see [4]). Note that $l(\{x, 0\}) = \{0\}$ for all $x \in \{1, 2\}$ and $l(\{x, 0\}) = \emptyset$ for all $x \in \{a, b\}$.

Corollary 3.9. *If X is a BCI-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X$ with $l(\{x, 0\}) \neq \emptyset$.*

Lemma 3.10. *If X is a BCI-algebra, then $l(\{x, 0\}) = \{0\}$ for all $x \in X_+$, where X_+ is the BCK-part of X.*

Proof. Straightforward. □

Lemma 3.11. *For any elements a and b of a BCK-algebra X, if $a * b = 0$, then $l(\{a\}) \subseteq l(\{b\})$ and $Z_b \subseteq Z_a$.*

Table 5: $*$-operation.

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

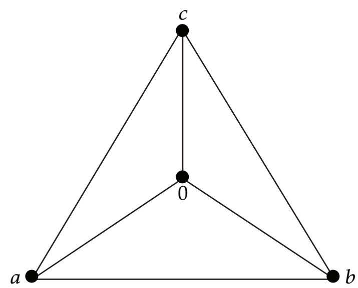

Figure 1: Associated graph $\Gamma(X)$ of X.

Proof. Assume that $a * b = 0$. Let $x \in l(\{a\})$, then $x * a = 0$, and so

$$0 = (x * b) * (x * a) = (x * b) * 0 = x * b. \tag{3.5}$$

Thus, $x \in l(\{b\})$, which shows that $l(\{a\}) \subseteq l(\{b\})$. Obviously, $Z_b \subseteq Z_a$. □

Theorem 3.12. *For any element x of a BCK-algebra X, the set of zero divisors of x is a quasi-ideal of X containing the zero element 0. Moreover, if Z_x is maximal in $\{Z_a \mid a \in X, Z_a \neq X\}$, then Z_x is l-prime.*

Proof. By Lemma 3.8, we have $0 \in Z_x$. Let $a \in X$ and $b \in Z_x$ be such that $a * b = 0$. Using Lemma 3.11, we have

$$l(\{x, a\}) = l(\{x\}) \cap l(\{a\}) \subseteq l(\{x\}) \cap l(\{b\}) = l(\{x, b\}) = \{0\}, \tag{3.6}$$

and so $l(\{x, a\}) = \{0\}$. Hence, $a \in Z_x$. Therefore, Z_x is a quasi-ideal of X. Let $a, b \in X$ be such that $l(\{a, b\}) \subseteq Z_x$ and $a \notin Z_x$, then $l(\{a, b, x\}) = \{0\}$. Let $0 \neq y \in l(\{a, x\})$ be an arbitrarily element, then $l(\{b, y\}) \subseteq l(\{a, b, x\}) = \{0\}$, and so $l(\{b, y\}) = \{0\}$, that is, $b \in Z_y$. Since $y \in l(\{a, x\})$, we have $y * x = 0$. It follows from Lemma 3.11 that $Z_x \subseteq Z_y \neq X$ so from the maximality of Z_x it follows that $Z_x = Z_y$. Hence, $b \in Z_x$, which shows that Z_x is l-prime. □

Definition 3.13. By the *associated graph* of a BCK/BCI-algebra X, denoted $\Gamma(X)$, we mean the graph whose vertices are just the elements of X, and for distinct $x, y \in \Gamma(X)$, there is an edge connecting x and y, denoted by $x - y$ if and only if $l(\{x, y\}) = \{0\}$.

Example 3.14. Let $X = \{0, a, b, c\}$ be a set with the $*$-operation given by Table 5, then X is a BCK-algebra (see [4]). The associated graph $\Gamma(X)$ of X is given by the Figure 1.

Table 6: $*$-operation.

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Table 7: $*$-operation.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	1
2	2	1	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

Example 3.15. Let $X = \{0, a, b, c, d\}$ be a set with the $*$-operation given by Table 6, then X is a BCK-algebra (see [4]). By Lemma 3.8, each nonzero point is adjacent to 0. Note that $l(\{a, b\}) = l(\{a, d\}) = l(\{b, c\}) = l(\{c, d\}) = \{0\}$, $l(\{a, c\}) = \{0, a\}$, and $l(\{b, d\}) = \{0, b\}$. Hence the associated graph $\Gamma(X)$ of X is given by the Figure 2.

Example 3.16. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the $*$-operation given by Table 7, then X is a BCK-algebra (see [4]). By Lemma 3.8, each nonzero point is adjacent to 0. Note that $l(\{1, 2\}) = \{0, 1\}$, that is, 1 is not adjacent to 2 and $l(\{1, 3\}) = l(\{1, 4\}) = l(\{2, 3\}) = l(\{2, 4\}) = l(\{3, 4\}) = \{0\}$. Hence, the associated graph $\Gamma(X)$ of X is given by Figure 3.

Example 3.17. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with the $*$-operation given by Table 4, then

$$l(\{1, a\}) = l(\{1, b\}) = l(\{2, a\}) = l(\{2, b\}) = \emptyset, \tag{3.7}$$

$l(\{a, b\}) = \{a\}$, and $l(\{1, 2\}) = \{0\}$. Since $X_+ = \{0, 1, 2\}$, we know from Lemma 3.10 that two points 1 and 2 are adjacent to 0. The associated graph $\Gamma(X)$ of X is given by Figure 4.

Theorem 3.18. *Let $\Gamma(X)$ be the associated graph of a BCK-algebra X. For any $x, y \in \Gamma(X)$, if Z_x and Z_y are distinct l-prime quasi-ideals of X, then there is an edge connecting x and y.*

Proof. It is sufficient to show that $l(\{x, y\}) = \{0\}$. If $l(\{x, y\}) \neq \{0\}$, then $x \notin Z_y$ and $y \notin Z_x$. For any $a \in Z_x$, we have $l(\{x, a\}) = \{0\} \subseteq Z_y$. Since Z_y is l-prime, it follows that $a \in Z_y$ so that $Z_x \subseteq Z_y$. Similarly, $Z_y \subseteq Z_x$. Hence, $Z_x = Z_y$, which is a contradiction. Therefore, x is adjacent to y. □

Theorem 3.19. *The associated graph of a BCK-algebra is connected in which every nonzero vertex is adjacent to 0.*

Proof. It follows from Lemma 3.8. □

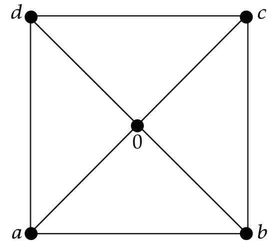

Figure 2: Associated graph $\Gamma(X)$ of X.

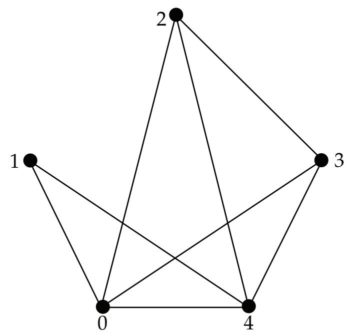

Figure 3: Associated graph $\Gamma(X)$ of X.

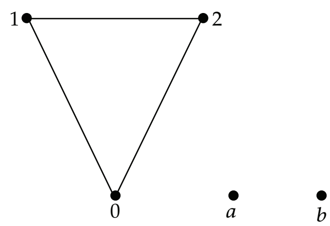

Figure 4: Associated graph $\Gamma(X)$ of X.

Example 3.17 shows that the associated graph of a proper BCI-algebra may not be connected.

4. Conclusions

We have introduced the associative graph of a BCK/BCI-algebra with several examples. We have shown that the associative graph of a BCK-algebra is connected, but the associative graph of a BCI-algebra is not connected.

Our future work is to study how to induce BCK/BCI-algebras from the given graph (with some additional conditions).

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