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Research Article

Some Generalizations of Jungck's Fixed Point Theorem

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We are going to generalize the Jungck's fixed point theorem for commuting mappings by mean of the concepts of altering distance functions and compatible pair of mappings, as well as, by using contractive inequalities of integral type and contractive inequalities depending on another function.

1. Introduction and Preliminary Facts

In 1922, Banach introduced his famous result in the metric fixed point theory, the Banach contraction Principle (BCP), as follows.

Theorem 1.1 (see [1]). *Let (M, d) be a complete metric space and let $S : M \to M$ be a self-map that satisfies the following condition: there is $a \in [0, 1)$ such that*

$$d(Sx, Sy) \leq ad(x, y) \tag{BC}$$

for all $x, y \in M$. Then, S has a unique fixed point $z_0 \in M$ such that for each $x \in M, \lim_{n\to\infty} S^n x = z_0$. One says that a mapping S belongs to the class BC if it satisfies the condition (BC).

Since then, several generalizations of the BCP have been appeared, some of them can be found, for instance, in [2–7] and into the references therein. In this paper we will focus our attention on an extension of the BCP introduced in 1976 by Jungck. More precisely, we are going to improve and generalize the following extension of the BCP.

Theorem 1.2 (see [8]). *Let S_1, S_2 be two self-maps on a complete metric space, (M, d) such that*

$(J_1)(S_1, S_2)$ *is a commuting pair,*

$(J_2)S_2$ *is continuous,*

$(J_3)S_1(M) \subset S_2(M)$,

(J_4) *there is $a \in [0,1)$ such that*

$$d(S_1x, S_1y) \leq ad(S_2x, S_2y) \tag{JC}$$

for all $x, y \in M$.

Then, S_1 and S_2 have a unique common fixed point $z_0 \in M$.

The class of all pair of mappings satisfying condition(JC) will be denoted by JC.

Notice that by taking $S_2x = x$, for all $x \in M$, in (JC) we obtain the condition (BC). That is, the Theorem 1.1 is obtained from Theorem 1.2.

In order to attain our goals, we are going to use the notions of altering distance functions [9] and compatible pair of mappings [10], also we will use some contractive inequalities of integral type [11] and contractive inequalities depending on another function [12].

We would like to start recalling that in 1984 Khan et al. [9] introduced the concept of altering distance function as follows. A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called an altering distance function if the following conditions are satisfied:

(ψ_1) $\psi(0) = 0$,

(ψ_2) ψ is monotonically nondecreasing,

(ψ_3) ψ is continuous.

We will denote by Ψ the set of all altering distance functions. Using this notion the same authors introduce the following class of mappings and studied the conditions for the existence of fixed points.

Definition 1.3 (see [9]). Let (M, d) be a metric space, $\psi \in \Psi$ and $S : M \to M$ be a self-map. The map S is called a ψ-Banach contraction, $(\psi\text{-}BC)$, if for each $x, y \in M$ there exists $a \in [0,1)$ such that

$$\psi[d(Sx, Sy)] \leq a\psi[d(x, y)]. \tag{ψ-BC}$$

The class of all mappings satisfying condition $(\psi\text{-}BC)$ will be called the ψ-BC class. Notice that letting $\psi(x) = i_M x = x$ in the inequality $(\psi\text{-}BC)$ we obtain (BC), so the ψ-BC class generalizes the BC ones.

In 2002, Branciari [12] introduced a new generalization of Banach contraction mappings, which satisfies a general contractive condition of integral type. For that generalization first let us consider the set $\Phi = \{\varphi/\varphi : \mathbb{R}_+ \to \mathbb{R}_+\}$ where φ satisfies the following conditions:

(φ_1) φ is nonnegative,

(φ_2) φ is Lebesgue integrable mapping which is summable on each compact subset of $\mathbb{R}_+$,

(φ_3) for each $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)dt > 0$.

Definition 1.4 (see [12]). Let (M, d) be a metric space. A mapping $S_1 : M \to M$ is said to be a $\int \varphi$-Banach contraction, ($\int \varphi$-BC), if there is $a \in [0, 1)$ such that for all $x, y \in M$, one has

$$\int_0^{d(S_1x,S_1y)} \varphi(t)dt \leq a \int_0^{d(x,y)} \varphi(t)dt, \qquad \left(\int \varphi\text{-}BC\right)$$

where $\varphi \in \Phi$.

The class of all the mappings satisfying the definition above will be denoted by$\int \varphi$-BC.

Notice that the Definition 1.4 is a generalization of the BCP. In fact, letting $\varphi(t) = 1$ for each $t > 0$ in ($\int \varphi$-BC), we have

$$\int_0^{d(S_1x,yS_1)} \varphi(t)dt = d(S_1x, S_1y) \leq ad(x, y) = a \int_0^{d(x,y)} \varphi(t)dt. \tag{1.1}$$

Thus, a Banach contraction mapping also satisfies the condition ($\int \varphi$-BC). The converse is not true as we will see in [12, Example 3.6].

Theorem 1.5 (see [12]). *Let (M, d) be a complete metric space and let $S_1 : M \to M$ be a $\int \varphi$-BC. Then S_1 has a unique fixed point $z_0 \in M$ such that for each $x \in M$, $\lim_{n\to\infty} S_1^n x = z_0$.*

General contractive inequalities of integral type are becoming popular for extend classes of contractive mappings with fixed points. See for example, [13–20] and references therein.

In 2009, Beiranvand et al. [11] introduced a new class of contraction mappings, by generalizing the contraction condition (BC) in terms of another function.

Definition 1.6 (see [11]). Let (M, d) be a metric space and $T, S_1 : M \to M$ be two mappings. The mapping S_1 is said to be a T-Banach contraction, (or belongs to the T-BC class). If there is $a \in [0, 1)$ such that

$$d(TS_1x, TS_1y) \leq ad(Tx, Ty) \qquad (T\text{-}BC)$$

for all $x, y \in M$.

By taking $Tx = x$, for all $x \in M$, we get that the (T-BC) and (BC) are equivalent conditions.

Contraction conditions depending on another function have been used in order to generalize other well-known contraction type maps as the Contractive, Chatterjea, Ćirić, Hardy-Rogers, Kannan, Reich and Rhoades mappings [11, 21–25].

Definition 1.7 (see [11]). Let (M, d) be a metric space. A mapping $T : M \to M$ is said sequentially convergent if one has, for every sequence (y_n), if $T(y_n)$ is convergent, then (y_n) also is convergent. T is said subsequentially convergent if one has, for every sequence (y_n), if $(T(y_n))$ is convergent, then (y_n) has a convergent subsequence.

The conditions for the existence of a unique fixed point for mappings in the class T-BC are given in the following result.

Theorem 1.8 (see [11]). *Let (M, d) be a complete metric space and $T : M \to M$ be a one to one, continuous, and subsequentially convergent mapping. Then, every T-BC continuous mapping $S_1 : M \to M$, has a unique fixed point $z_0 \in M$. Moreover, if T is sequentially convergent, then for each $x \in M$, $\lim_{n\to\infty} S^n x = z_0$.*

2. A Version of the Jungck's Fixed Point Theorem Using Altering Distance Functions

In this section we are going to generalize the Jungck's fixed point Theorem 1.2 by using the altering distance function and the JC class. More precisely, we will introduce the class of ψ-J-contraction mappings which generalize the JC class, and therefore the BC class of mappings.

Definition 2.1. Let (M, d) be a metric space, $S, T : M \to M$ be self-mappings and $\psi \in \Psi$. The pair (S, T) is called a ψ-Jungck contraction, (ψ-J-Contraction) if for each $x, y \in M$ there exists $a \in [0, 1)$ such that

$$\psi[d(Sx, Sy)] \le a\psi[d(Tx, Ty)]. \qquad (\psi\text{-}JC)$$

It is clear that letting $Tx = x$ in $(\psi$-$JC)$ we obtain $(\psi$-$BC)$.

Example 2.2. Let $M = [0, 1] \subset \mathbb{R}$ endowed with the Euclidean metric. We define $Sx = x^2$, $Tx = (2/\sqrt{a})x$, with $0 < a < 1$ and $\psi(x) = x^2$. Then,

$$\begin{aligned}\psi(d(Sx, Sy)) &= \left(x^2 - y^2\right)^2, \\ \psi(d(Tx, Ty)) &= \frac{4}{a}(x - y)^2.\end{aligned} \qquad (2.1)$$

From here we get that the pair (S, T) belongs to the class of mappings ψ-JC. On the other hand,

$$\begin{aligned}\psi(d(Sx, Sy)) &= \left(x^2 - y^2\right)^2, \\ \psi(d(x, y)) &= (x - y)^2.\end{aligned} \qquad (2.2)$$

Hence, the pair (S, T) does not belong to the class ψ-BC.

In order to prove the fixed point theorem for pair of mappings belonging to the class ψ-JC we will need the following notions.

Definition 2.3 (see [26]). Let (M, d) be a metric space and let $S_1, S_2 : M \to M$ be two mappings. Suppose that $S_1(M) \subset S_2(M)$ and for every $x_0 \in M$ one considers the sequence $(x_n) \subset M$ defined by, $S_1(x_{n-1}) = S_2(x_n)$, for all $n \in \mathbb{N}$, one says that $(S_2(x_n))$ is a (S_1, S_2)-sequence of initial point x_0.

On the other hand, recall that a pair of mappings (T, S) is said to be compatible [10] if and only if $d(TSx_n, STx_n) = 0$, whenever $(x_n) \subset M$ is such that

$$\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t \tag{2.3}$$

for some $t \in M$.

Theorem 2.4. *Let (M, d) be a complete metric space and let $S, T : M \to M$ be self-mappings such that*

(a) *T is a continuous function,*

(b) *$S(M) \subset T(M)$,*

(c) *(S, T) is compatible,*

(d) *the pair (S, T) belongs to the class of mappings ψ-JC.*

Then, S and T have a unique common fixed point $z_0 \in M$.

Proof. Let $x_0 \in M$ be an arbitrary point, we will prove that the (S, T)-sequence $(T(x_n))$ of initial point x_0 is a Cauchy sequence in M. For each $n \in \mathbb{N}$ from inequality contraction $(\psi$-$JC)$ we have

$$\begin{aligned} \psi[d(Tx_{n+1}, Tx_n)] &= \psi[d(Sx_n, Sx_{n-1})] \\ &\le a\psi[d(Tx_n, Tx_{n-1})] \le \cdots \le a^n\psi[d(Tx_1, Tx_0)] \end{aligned} \tag{2.4}$$

since $0 \le a < 1$, it follows that

$$\lim_{n\to\infty} \psi[d(Tx_{n+1}, Tx_n)] = 0. \tag{2.5}$$

From conditions (ψ_1) and (ψ_3) of the altering distance function ψ we obtain

$$\begin{aligned} &\lim_{n\to\infty} d(Tx_{n+1}, Tx_n) = 0, \\ &\lim_{n\to\infty} d(Sx_n, Sx_{n-1}) = 0. \end{aligned} \tag{2.6}$$

Now, we want to prove that $(Tx_n) \subset M$ is a Cauchy sequence in M. Suppose that (Tx_n) is not a Cauchy sequence in M. So, there exists $\varepsilon_0 > 0$ for which we can find subsequences $(Tx_{m(k)})$ and $(Tx_{n(k)})$ of (Tx_n) with $k < m(\mathrm{k}) < n(k)$ such that

$$d\left(Tx_{m(k)}, Tx_{n(k)}\right) \ge \varepsilon_0 \tag{2.7}$$

and for this $m(k)$ we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ satisfying (2.7). Then

$$d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) < \varepsilon_0. \tag{2.8}$$

Thus, we have

$$\begin{aligned}\varepsilon_0 &\leq d\left(Tx_{m(k)}, Tx_{n(k)}\right)\\ &\leq d\left(Tx_{m(k)}, Tx_{n(k)-1}\right) + d\left(Tx_{n(k)-1}, Tx_{n(k)}\right)\\ &< \varepsilon_0 + d\left(Tx_{n(k)-1}, Tx_{n(k)}\right).\end{aligned} \tag{2.9}$$

From (2.6), (2.7), and (2.9), it follows that

$$\lim_{k\to\infty} d\left(Tx_{m(k)}, TSx_{n(k)}\right) = \varepsilon_0. \tag{2.10}$$

In a similar way we obtain

$$\lim_{k\to\infty} d\left(Tx_{n(k)-1}, Tx_{m(k)-1}\right) = \varepsilon_0. \tag{2.11}$$

Now, using the definition of ψ-JC mappings for $x = Tx_{m(k)}$ and $y = Tx_{n(k)}$ we have

$$\begin{aligned}\psi(\varepsilon_0) &\leq \psi\left[d\left(Tx_{m(k)}, Tx_{n(k)}\right)\right]\\ &= \psi\left[d\left(Sx_{m(k)-1}, Sx_{n(k)-1}\right)\right]\\ &\leq a\psi\left[d\left(Tx_{m(k)-1}, Tx_{n(k)-1}\right)\right].\end{aligned} \tag{2.12}$$

Letting $k \to \infty$ and using (2.11) we have

$$\psi(\varepsilon_0) \leq a\psi(\varepsilon_0), \tag{2.13}$$

which is a contraction if $\varepsilon_0 > 0$. This shows that (Tx_n) is a Cauchy sequence in M and hence it is convergent in the complete metric space M. So there is $z_0 \in M$ such that

$$\lim_{n\to\infty} T(x_{n+1}) = \lim_{n\to\infty} S(x_n) = z_0. \tag{2.14}$$

Since T is a continuous function

$$\lim_{n\to\infty} T^2 x_{n+1} = \lim_{n\to\infty} TSx_n = Tz_0, \tag{2.15}$$

which proves that the pair (S, T) is compatible. Thus

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0. \tag{2.16}$$

Now

$$\psi[d(TSx_n, Sx_n)] \leq a\psi\left[d\left(T^2x_n, \mathrm{T}x_n\right)\right] \tag{2.17}$$

letting $n \to \infty$ we have

$$\psi[d(Tz_0, z_0)] \leq a\psi[d(Tz_0, z_0)] \tag{2.18}$$

then, it follows that

$$\psi[d(Tz_0, z_0)] = 0, \tag{2.19}$$

which implies that $d(Tz_0, z_0) = 0$, so $Tz_0 = z_0$. Again,

$$\psi[d(STx_n, Sz_0)] \leq a\psi\left[d\left(T^2x_n, Tz_0\right)\right] \tag{2.20}$$

letting $n \to \infty$

$$\psi[d(Tz_0, Sz_0)] \leq a\psi[d(Tz_0, Tz_0)] = 0 \tag{2.21}$$

we have that $\psi[d(Tz_0, Sz_0)] = 0$, so it follows that $d(Tz_0, Sz_0) = 0$. Then $z_0 = Tz_0 = Sz_0$.

Therefore, z_0 is a common fixed point of S and T. Now, we are going to suppose that z_1 is another common fixed point of S and T. Then

$$\psi[d(z_0, z_1)] = \psi[d(Sz_0, Sz_1)] \leq a\psi[d(z_0, z_1)] \tag{2.22}$$

thus we conclude that $\psi[d(z_0, z_1)] = 0$, therefore $d(z_0, z_1) = 0$, or equivalently, $z_0 = z_1$. □

Notice that letting $\psi = i_M$, in the definition of the ψ-JC class, we obtain a natural generalization of Jungck's fixed point Theorem 1.2 to compatible pair of mappings, by replacing condition (J_1) for the following:

(J_1') (S_1, S_2) is compatible.

Also, notice that the fixed point criterion for the class of mappings introduced by Khan et al. in [9] is obtained now as a consequence of Theorem 2.4 by letting $Tx = x$.

3. The Jungck's Fixed Point Theorem with Contractive Inequality Depending on Another Function

In this section our main goal is to generalize the Theorem 1.2 by considering its contractive condition depending on another function. Now, using the ideas of Beiranvand et al. given in [11], we introduce the following class of pair of mappings.

Definition 3.1. Let (M, d) be a metric space and let $T, S_1, S_2 : M \to M$ be mappings. The pair (S_1, S_2) is said to be a T-J-contraction, $(T\text{-}JC)$ if there is $a \in [0, 1)$ such that

$$d(TS_1x, TS_1y) \leq ad(TS_2x, TS_2y) \tag{T-JC}$$

for all $x, y \in M$.

Notice that taking $Tx = x$ in the inequality above, we get that $(T\text{-}JC)$ and (JC) are equivalent conditions. The next example shows that the class $T\text{-}JC$ is more general than the JC ones.

Example 3.2. Let $M = [1, +\infty) \subset \mathbb{R}$ endowed with the Euclidean metric. We define $S_1x = 4\sqrt{x}$, $S_2x = x$, and $Tx = 1 + \ln x$ for all $x, y \in M$. Then,

$$\begin{aligned} d(S_1x, S_1y) &= |S_1x - S_1y| = |4\sqrt{x} - 4\sqrt{y}| \\ &\leq 2|x - y| = 2d(S_2x, S_2y). \end{aligned} \tag{3.1}$$

Thus, the pair (S_1, S_2) does not belong to JC. However,

$$\begin{aligned} d(TS_1x, TS_1y) &= |TS_1x - TS_1y| = |T(4\sqrt{x}) - T(4\sqrt{y})| \\ &= \frac{1}{2}|\ln x - \ln y| = \frac{1}{2}d(TS_2x, TS_2y). \end{aligned} \tag{3.2}$$

Hence, the pair (S_1, S_2) belongs to the class $T\text{-}JC$.

Theorem 3.3. *Let (M, d) be a complete metric space and let $T, S_1, S_2 : M \to M$ be mappings such that*

(a) *T is one to one, continuous, and sequentially convergent,*

(b) *S_2 is continuous,*

(c) *$S_1(M) \subset S_2(M)$,*

(d) *S_1 and S_2 are commuting mappings,*

(e) *the pair (S_1, S_2) is a member of the class $T\text{-}JC$.*

Then, S_1 and S_2 have a unique common fixed point $z_0 \in M$.

Proof. Let $x_0 \in M$ be an arbitrary point. We will prove that the (S_1, S_2)-sequence $(S_2(x_n))$ of initial point x_0 is a Cauchy sequence in M. For each $n \in \mathbb{N}$ from the $(T\text{-}JC)$ condition we get

$$\begin{aligned} d(TS_2x_{n+1}, TS_2x_n) &= d(TS_1x_n, TS_1x_{n-1}) \\ &\leq ad(TS_2x_n, TS_2x_{n-1}) \end{aligned} \tag{3.3}$$

consequently, by repeating the argument above we can conclude that

$$d(TS_2x_{n+1}, TS_2x_n) \leq a^n d(TS_2x_1, TS_2x_0). \tag{3.4}$$

Taking limits in the last inequality we have that

$$\lim_{n\to\infty} d(TS_2x_{n+1}, TS_2x_n) = \lim_{n\to\infty} d(TS_1x_n, TS_1x_{n-1}) = 0. \tag{3.5}$$

From (3.3), we have that (TS_2x_n) is a Cauchy sequence in M. Since M is a complete metric space, then (TS_2x_n) is convergent in M, and due to the fact that T is sequentially convergent, then (S_2x_n) is convergent in M. So, there exists $z_0 \in M$ such that

$$\begin{aligned} &\lim_{n\to\infty} S_2x_n = z_0, \\ &\lim_{n\to\infty} S_1x_n = z_0. \end{aligned} \tag{3.6}$$

Since T and S_2 are continuous maps, then TS_2 is a continuous mapping, and using the contractive inequality $(T\text{-}JC)$ we conclude then that TS_1 is continuous. Thus

$$\begin{aligned} &\lim_{n\to\infty} TS_1S_2x_n = TS_1z_0, \\ &\lim_{n\to\infty} TS_1S_1x_n = TS_1z_0. \end{aligned} \tag{3.7}$$

Using that S_1 and S_2 are commuting mappings we obtain

$$\begin{aligned} &\lim_{n\to\infty} TS_1S_2x_n = TS_1z_0, \\ &\lim_{n\to\infty} TS_2S_1x_n = TS_2z_0. \end{aligned} \tag{3.8}$$

Therefore

$$\begin{aligned} TS_1z_0 &= TS_2z_0, \\ TS_1^2z_0 &= TS_2S_1z_0. \end{aligned} \tag{3.9}$$

Now

$$\begin{aligned} d\left(TS_1^2z_0, TS_1z_0\right) &\le ad(TS_2(S_1z_0), TS_2z_0) \\ &= ad\left(TS_1^2z_0, TS_1z_0\right). \end{aligned} \tag{3.10}$$

Then, it follows that

$$d\left(TS_1^2z_0, TS_1z_0\right) = 0 \tag{3.11}$$

so, $TS_1^2z_0 = TS_1z_0$. Since T is one to one, we obtain that $S_1(S_1z_0) = S_1z_0$, which implies that S_1z_0 is a fixed point of S_1 and also that S_1z_0 is a fixed point of S_2. Hence, S_1z_0 is the common fixed point of S_1 and S_2.

Now, we will prove that $z_0 \in M$ is a common fixed point of S_1 and S_2. Using the $(T\text{-}JC)$ condition, we have

$$\begin{aligned} d(Tz_0, TS_1z_0) &\le d(Tz_0, TS_1x_n) + d(TS_1x_n, TS_1z_0) \\ &\le d(Tz_0, TS_1x_n) + ad(TS_2x_n, TS_2z_0) \end{aligned} \tag{3.12}$$

taking the limit as $n \to \infty$ we get

$$d(Tz_0, TS_1z_0) \le d(Tz_0, Tz_0) + ad(Tz_0, TS_2z_0). \tag{3.13}$$

That is,

$$d(Tz_0, TS_1z_0) \le ad(Tz_0, TS_1z_0). \tag{3.14}$$

Therefore, $d(Tz_0, TS_1z_0) = 0$, then $Tz_0 = TS_1z_0$. Since T is one to one, we conclude that $z_0 = S_1z_0 = S_2z_0$. Finally, we suppose that $y_0 = S_1y_0 = S_2y_0$ and $z_0 \ne y_0$. Then

$$\begin{aligned} d(Tz_0, Ty_0) &= d(TS_1z_0, TS_1z_0) \le ad(TS_2z_0, TS_2y_0) \\ &= ad(Tz_0, Ty_0), \end{aligned} \tag{3.15}$$

which implies that $Tz_0 = Ty_0$, or equivalently, $z_0 = y_0$. Therefore, the common fixed point of S_1 and S_2 is unique. □

Example 3.4. As in Example 3.2, let $M = [1, +\infty) \subset \mathbb{R}$ be endowed with the euclidean metric and we define $S_1(x) = 4\sqrt{x}$, $S_2x = x$, and $Tx = 1 + \ln x$, for all $x \in M$. Then,

(a) T is one to one, continuous and sequentially convergent,

(b) S_2 is continuous,

(c) $S_1(M) \subset S_2(M)$,

(d) S_1 and S_2 are commuting mappings,

(e) In the Example 3.2, we proved that the pair (S_1, S_2) does not belong to JC, thus we cannot apply the Theorem 1.2.

(f) In the Example 3.2 we have proved that the pair (S_1, S_2) is a $T\text{-}JC$, so we apply the Theorem 3.3,

(g) It is clear that $z_0 = 16$ is the unique common fixed point of S_1 and S_2.

Notice that if in the Theorem 3.3 we take $Tx = x$, for all $x \in M$, then we obtain Theorem 1.2.

4. Contractive Conditions of Integral Type for the Jungck's Fixed Point Theorem

This section is devoted to generalize the Theorem 1.2 using the ideas of Branciari [12]. More precisely, for a pair of mappings we introduce contractive conditions of integral type depending on another function. Afterwards, we will give (common) fixed point results for this new class of mappings.

Definition 4.1. Let (M, d) be a metric space and let $T, S_1, S_2 : M \to M$ be mappings. The pair (S_1, S_2) is called T-$\int \varphi$-JC if there is $a \in [0, 1)$ such that for all $x, y \in M$, we have

$$\int_0^{d(TS_1x, TS_1y)} \varphi(t)dt \le a \int_0^{d(TS_2x, TS_2y)} \varphi(t)dt, \qquad (T\text{-}\textstyle\int \varphi\text{-}JC)$$

where $\varphi \in \Phi$.

The class of pair of mappings satisfying the definition above will be denoted by T-$\int \varphi$-JC.

Example 4.2. Let us consider T, S_1, S_2 and $(M, \|\cdot\|)$ as in Example 3.2 and consider the function $\varphi \in \Phi$ defined by $\varphi(t) = t$. So, the pair (S_1, S_2) belongs to the class T-$\int \varphi$-JC if we have that

$$\int_0^{d(TS_1x, TS_1y)} tdt \le a \int_0^{d(TS_2x, TS_2y)} tdt, \quad 0 \le a < 1. \tag{4.1}$$

Or, equivalently

$$\frac{1}{2}\left(d(TS_1x, TS_1y)\right)^2 \le \frac{a}{2}\left(d(TS_2x, TS_2y)\right)^2, \tag{4.2}$$

which is hold if and only if

$$d(TS_1x, TS_1y) \le \sqrt{a} d(TS_2x, TS_2y). \tag{4.3}$$

On the other hand, from Example 3.2 we have that the inequality above is valid if we take $a = 1/4$. Therefore, the pair (S_1, S_2) belongs to T-$\int \varphi$-JC with contractive constant $a = 1/4$.

Theorem 4.3. *Let (M, d) be a complete metric space and let $T, S_1, S_2 : M \to M$ be mappings such that*

(a) *T is one to one, continuous, and sequentially convergent,*

(b) *S_1 and S_2 are continuous mappings,*

(c) *$S_1(M) \subset S_2(M)$,*

(d) *S_1 and S_2 are commuting mappings,*

(e) *The pair (S_1, S_2) is a T-$\int \varphi$-JC.*

Then, S_1 and S_2 have a unique common fixed point.

Proof. Let $x_0 \in M$ be an arbitrary point. We must prove that the (S_1, S_2)-sequence $(S_1 x_{n-1}) = (S_2 x_n)$ of initial point x_0 is a Cauchy sequence in M. For each $n \in \mathbb{N}$ from the condition $(T\text{-}\int\varphi\text{-}JC)$ we have

$$\begin{aligned}\int_0^{d(TS_1x_{n-1},TS_1x_n)} \varphi(t)dt &\le a\int_0^{d(TS_2x_{n-1},TS_2x_n)} \varphi(t)dt \\ &= a\int_0^{d(TS_1x_{n-2},TS_1x_{n-1})} \varphi(t)dt.\end{aligned} \tag{4.4}$$

Consequently

$$\int_0^{d(TS_1x_{n-1},TS_1x_n)} \varphi(t)dt \le a^n \int_0^{d(TS_2x_0,TS_2x_1)} \varphi(t)dt \tag{4.5}$$

since $0 \le a < 1$, it follows from (4.5) that

$$\lim_{n\to\infty}\int_0^{d(TS_1x_{n-1},TS_1x_n)} \varphi(t)dt \le 0 \tag{4.6}$$

using that $\int_0^{\varepsilon}\varphi(t)dt > 0$ for each $\varepsilon > 0$, we conclude that

$$\lim_{n\to\infty} d(TS_1x_{n-1},TS_1x_n) = 0. \tag{4.7}$$

Now, we can prove that (TS_1x_{n-1}) is a Cauchy sequence in M. We are going to follow the proof of the Theorem 2.1 of [27].

In first place, we prove that for all $\varepsilon > 0$, $d(TS_1x_{m(p)}, TS_1x_{n(p)-1}) \le \varepsilon$. Suppose that it is not true. Then, there exist $\varepsilon > 0$ and subsequences $(m_{p(k)})$ and $(n_{p(k)})$ such that for each positive integer k, $n_{p(k)}$ is minimal in the sense that

$$\begin{aligned}d\Big(TS_1x_{m_{p(k)}}, TS_1x_{n_{p(k)}}\Big) &\ge \varepsilon, \\ d\Big(TS_1x_{m_{p(k)}}, TS_1x_{n_{p(k)-1}}\Big) &< \varepsilon.\end{aligned} \tag{4.8}$$

Now,

$$\begin{aligned}\varepsilon < d\Big(TS_1x_{m_{p(k)-1}}, TS_1x_{n_{p(k)-1}}\Big) &\le d\Big(TS_1x_{m_{p(k)-1}}, TS_1x_{m_{p(k)}}\Big) + d\Big(TS_1x_{m_{p(k)}}, TS_1x_{n_{p(k)-1}}\Big) \\ &< \varepsilon + d\Big(TS_1x_{m_{p(k)-1}}, TS_1x_{m_{p(k)}}\Big).\end{aligned} \tag{4.9}$$

From (4.7) and (4.8), letting $k \to \infty$, we have

$$\begin{aligned}
&\lim_{k\to\infty} d\left(TS_1x_{m_{p(k)-1}}, TS_1x_{n_{p(k)-1}}\right) = \varepsilon, \\
\int_0^{\varepsilon}\varphi(t)dt &\le \int_0^{d(TS_1x_{m_{p(k)}},TS_1x_{n_{p(k)}})}\varphi(t)dt \\
&\le a\int_0^{d(TS_2x_{m_{p(k)}},TS_2x_{n_{p(k)}})}\varphi(t)dt \\
&= a\int_0^{d(TS_1x_{m_{p(k)-1}},TS_1x_{n_{p(k)-1}})}\varphi(t)dt
\end{aligned} \tag{4.10}$$

letting $k \to \infty$, we have the contradiction

$$\int_0^{\varepsilon}\varphi(t)dt \le a\int_0^{\varepsilon}\varphi(t)dt. \tag{4.11}$$

Thus

$$d\left(TS_1x_{m_{p(k)-1}}, TS_1x_{n_{p(k)-1}}\right) \le \varepsilon. \tag{4.12}$$

Now, we prove that (TS_1x_n) is a Cauchy sequence in M. From (4.7)–(4.12),

$$\begin{aligned}
\int_0^{\varepsilon}\varphi(t)dt &\le \int_0^{d(TS_1x_{m_{p(k)}},TS_1x_{n_{p(k)}})}\varphi(t)dt \le a\int_a^{d(TS_2x_{m_{p(k)}},TS_2x_{n_{p(k)}})} \\
&= a\int_0^{d(TS_1x_{m_{p(k)-1}},TS_1x_{n_{p(k)-1}})}\varphi(t)dt \le a\int_0^{\varepsilon}\varphi(t)dt,
\end{aligned} \tag{4.13}$$

which is a contradiction. Therefore (TS_1x_n) is a Cauchy sequence in M, and since M is a complete metric space, then (TS_1x_n) is a convergent sequence in M. Now, from the fact that T is sequentially convergent, we get that (S_1x_n) is a convergent sequence in M. Thus, there exists $z_0 \in M$ satisfying

$$\begin{aligned}
&\lim_{n\to\infty} S_1x_n = z_0, \\
&\lim_{n\to\infty} S_2x_n = z_0
\end{aligned} \tag{4.14}$$

using that T, S_1 and S_2 are continuous mappings, we conclude that TS_1 and TS_2 are continuous, hence from (4.14) we get

$$\begin{aligned}
&\lim_{n\to\infty} TS_2S_1x_n = TS_2z_0, \\
&\lim_{n\to\infty} TS_1S_2x_n = TS_1z_0.
\end{aligned} \tag{4.15}$$

On the other hand, by the commutative property of S_1 and S_2, it follows that

$$\lim_{n\to\infty} TS_1S_2x_n = TS_2z_0,$$
$$\lim_{n\to\infty} TS_1S_2x_n = TS_1z_0 \tag{4.16}$$

therefore, $TS_1z_0 = TS_2z_0$ and because T is injective, we have

$$S_1z_0 = S_2z_0. \tag{4.17}$$

It follows for the commuting of the pair (S_1, S_2) that

$$TS_1S_2z_0 = TS_2S_1z_0 = TS_2S_2z_0. \tag{4.18}$$

Now, suppose that $TS_1z_0 \neq TS_1(S_1z_0)$. Using the condition $(T\text{-}\int\varphi\text{-}JC)$ we have

$$\begin{aligned}\int_0^{d(TS_1z_0,TS_1S_1z_0)} \varphi(t)dt &\leq a\int_0^{d(TS_2z_0,TS_2S_1z_0)} \varphi(t)dt \\ &= a\int_0^{d(TS_1z_0,TS_1S_1z_0)} \varphi(t)dt.\end{aligned} \tag{4.19}$$

That is,

$$(1-a)\int_0^{d(TS_1z_0,TS_1S_1z_0)} \varphi(t)dt \leq 0, \tag{4.20}$$

or equivalently

$$\int_0^{d(TS_1z_0,TS_1S_1z_0)} \varphi(t)dt \leq 0, \tag{4.21}$$

which is a contradiction. Therefore,

$$\int_0^{d(TS_1z_0,TS_1S_1z_0)} \varphi(t)dt = 0. \tag{4.22}$$

Since $\int_0^{\varepsilon} \varphi(t)dt > 0$, for each $\varepsilon > 0$, then $d(TS_1z_0, TS_1S_1z_0) = 0$ which implies that

$$TS_1z_0 = TS_1S_1z_0 \tag{4.23}$$

again, since T is one to one, we have that $S_1 z_0 = S_1(S_1 z_0)$. Thus, $S_1 z_0$ is a fixed point of S_1. Now, due to the equalities $TS_2S_1z_0 = TS_1S_1z_0 = TS_1z_0$ and because T is one to one, we get that

$$S_2(S_1 z_0) = S_1 z_0 \tag{4.24}$$

therefore, $S_1 z_0$ is a fixed point of S_2, hence $S_1 z_0$ is a common fixed point of S_1 and S_2.

Now we are going to prove that $z_0 \in M$ is a common fixed point of S_1 and S_2. Using the inequality condition $(T\text{-}\int \varphi\text{-}JC)$, we have

$$\begin{aligned} \int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt &\leq \int_0^{d(Tz_0,TS_1x_n)} \varphi(t)dt + \int_0^{d(TS_1x_n,TS_1z_0)} \varphi(t)dt \\ &\leq \int_0^{d(Tz_0,TS_1x_n)} \varphi(t)dt + a\int_0^{d(TS_2x_n,TS_2z_0)} \varphi(t)dt. \end{aligned} \tag{4.25}$$

Taking the limit as $n \to \infty$ we obtain

$$\int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt \leq \int_0^{d(Tz_0,Tz_0)} \varphi(t)dt + a\int_0^{d(Tz_0,TS_2z_0)} \varphi(t)dt = a\int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt \tag{4.26}$$

thus we have

$$(1-a)\int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt \leq 0 \tag{4.27}$$

since $0 \leq a < 1$, then we get that

$$\int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt \leq 0, \tag{4.28}$$

which is a contradiction, therefore

$$\int_0^{d(Tz_0,TS_1z_0)} \varphi(t)dt = 0. \tag{4.29}$$

From here we conclude that

$$d(Tz_0, TS_1z_0) = 0 \tag{4.30}$$

therefore, $Tz_0 = TS_1z_0$, or equivalently, $z_0 = S_1z_0 = S_2z_0$. Now, we will prove the uniqueness of the common fixed point. Let us suppose that $y_0 = S_1y_0 = S_2y_0$ with $y_0 \neq z_0$. Then

$$\begin{aligned}\int_0^{d(Tz_0,Ty_0)} \varphi(t)dt &= \int_0^{d(TS_1z_0,TS_1y_0)} \varphi(t)dt \leq a\int_0^{d(TS_2z_0,TS_2y_0)} \varphi(t)dt \\ &= a\int_0^{d(Tz_0,Ty_0)} \varphi(t)dt\end{aligned} \tag{4.31}$$

then

$$(1-a)\int_0^{d(Tz_0,Ty_0)} \varphi(t)dt \leq 0, \tag{4.32}$$

which implies that

$$\int_0^{d(Tz_0,Ty_0)} \varphi(t)dt = 0 \tag{4.33}$$

thus, $d(Tz_0, Ty_0) = 0$. That is, $Tz_0 = Ty_0$ and since T is one to one, then $z_0 = y_0$. Therefore, we have proved that z_0 is the unique common fixed point of S_1 and S_2. □

5. Further Generalizations

Using contractive conditions of integral type and altering distance functions we can introduce new classes of mapping that generalize the JC class.

Definition 5.1. Let (M, d) be a metric space, $\psi \in \Psi$ and $S, T : M \to M$ be self-mappings. The pair (S, T) is called a $\int\varphi$-ψ-Jungck contraction, $(\int\varphi\text{-}\psi\text{-}JC)$ if for each $x, y \in M$ there exists $a \in [0, 1)$ such that

$$\int_0^{\psi[d(Sx,Sy)]} \varphi(t)dt \leq a\int_0^{\psi[d(Tx,Ty)]} \varphi(t)dt, \tag{$\int\varphi$-ψ-JC}$$

where $\varphi \in \Phi$.

The class of pairs of mappings fulfilling inequality above will be denoted by $\int\varphi$-ψ-JC. Similarly, we introduce the following class of mappings.

Definition 5.2. Let (M, d) be a metric space, $\psi \in \Psi$ and $S, T : M \to M$ be self-mappings. The pair (S, T) is called a ψ-$\int\varphi$-Jungck contraction, $(\psi\text{-}\int\varphi\text{-}JC)$ if for each $x, y \in M$ there exists $a \in [0, 1)$ such that

$$\psi\left[\int_0^{d(Sx,Sy)} \varphi(t)dt\right] \leq a\psi\left[\int_0^{d(Tx,Ty)} \varphi(t)dt\right], \tag{ψ-$\int\varphi$-JC}$$

where $\varphi \in \Phi$.

By ψ-$\int\varphi$-JC we mean the class of mappings given by definition above.

The existence and uniqueness of the common fixed point of pair of mappings in these new classes of self-maps is a consequence of the Theorem 2.4.

Proposition 5.3. *Let (M,d) be a complete metric space and $S,T : M \to M$ be self-mappings satisfying the following conditions:*

(a) *T is a continuous function,*

(b) *$S(M) \subset T(M)$,*

(c) *The pair (S,T) is compatible,*

(d) *The pair (S,T) belongs to $\int\varphi$-ψ-JC.*

Then, S and T have a unique common fixed point $z_0 \in M$.

Proof. Let $\psi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be a function defined by

$$\psi_0(S) = \int_0^S \varphi(t)dt, \tag{5.1}$$

where $\varphi \in \Phi$. It is clear that $\psi_0 \in \Psi$. Moreover,

$$\begin{aligned}\psi_0[\psi(d(Sx,Sy))] &= \int_0^{\psi(d(Sx,Sy))} \varphi(t)dt \\ &\leq a\int_0^{\psi(d(Tx,Ty))} \varphi(t)dt = a\psi_0(\psi(d(Tx,Ty)))\end{aligned} \tag{5.2}$$

it follows that

$$\psi_0 \circ \psi(d(Sx,Sy)) \leq a\psi_0 \circ \psi(d(Tx,Ty)) \tag{5.3}$$

for all $x,y \in M$ and $\psi_0 \circ \psi \in \Psi$. Hence by the Theorem 2.4, the mappings S and T have a unique common fixed point $z_0 \in M$. □

In similar form we can prove the following result.

Proposition 5.4. *Let (M,d) be a complete metric space and $S,T : M \to M$ be self-mappings satisfying the conditions:*

(a) *T is a continuous mapping,*

(b) *$S(M) \subset T(M)$,*

(c) *The pair* (S, T) *is compatible,*

(d) *The pair* (T, S) *belongs to* ψ-$\int \varphi$-JC.

Then, S and T have a unique common fixed point $z_0 \in M$.

Notice that taking $\psi = i_M$ in $(\int \varphi$-ψ-$JC)$ or $(\psi$-$\int \varphi$-$JC)$ we obtain the results given by Kumar et al. in [28].

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