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Research Article

On Geometry of Submanifolds of $(\mathrm{LCS})_n$-Manifolds

Mehmet Atceken

Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, 60250 Tokat, Turkey

Correspondence should be addressed to Mehmet Atceken, mehmet.atceken@gop.edu.tr

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We show geometrical properties of a submanifold of a $(\mathrm{LCS})_n$-manifold. The properties of the induced structures on such a submanifold are also studied.

1. Introduction

The geometry of manifolds endowed with geometrical structures has been intensively studied, and several important results have been published. In this paper, we deal with manifolds having a Lorentzian concircular structure ($(\mathrm{LCS})_n$-manifold) [1–3] (see Section 2 for detail).

The study of the Lorentzian almost paracontact manifold was initiated by Matsumoto in [4]. Later on, several authors studied the Lorentzian almost paracontact manifolds and their different classes including [1, 4, 5]. Recently, the notion of the Lorentzian concircular structure manifolds was introduced in (briefly (LCS)-manifolds) with an example, which generalizes the notion of the LP-Sasakian manifolds introduced by Matsumoto in [4].

Papers related to this issue are very few in the literature so far. But the geometry of submanifolds of a (LCS)-manifold is rich and interesting. So, in the present paper we introduce the concept of submanifolds of a (LCS)-manifold and investigate the fundamental properties of such submanifolds. We obtain the necessary and sufficient conditions for a submanifold of (LCS)-manifold to be invariant. In this case, the induced structures on submanifold by the structure on ambient space are classified. I think that the results will contribute to geometry.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type $(0,2)$ such that, for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a nondegenerate inner product of signature $(-,+,\dots,+)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A nonzero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, and spacelike) if it satisfies $g_p(v,v) < 0$ (resp., ≤ 0, $= 0$, and > 0) [6].

Definition 2.1. In a Lorentzian manifold $(\overline{M}, \overline{g})$, a vector field P defined by

$$\overline{g}(X, P) = A(X), \tag{2.1}$$

for any $X \in \Gamma(T\overline{M})$, is said to be a concircular vector field if

$$\left(\overline{\nabla}_X A\right)(Y) = \alpha\{\overline{g}(X,Y) + \omega(X)A(Y)\}, \tag{2.2}$$

where α is a nonzero scalar and ω is a closed 1-form and $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Let M be an n-dimensional Lorentzian manifold admitting a unit time-like concircular vector field ξ, called the characteristic vector field of the manifold. Then, we have

$$\overline{g}(\xi, \xi) = -1. \tag{2.3}$$

Since ξ is a unit concircular vector field, it follows that there exists a nonzero 1-form η such that, for

$$\overline{g}(X, \xi) = \eta(X), \tag{2.4}$$

the equation of the following form holds:

$$\left(\overline{\nabla}_X \eta\right)(Y) = \alpha\{\overline{g}(X,Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0) \tag{2.5}$$

for all vector fields X, Y, where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a nonzero scalar function satisfying

$$\overline{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.6}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\overline{\nabla}_X \xi, \tag{2.7}$$

then from (2.5) and (2.7) we have

$$\phi X = X + \eta(X)\xi, \tag{2.8}$$

from which, it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus, the Lorentzian manifold M together with the unit time-like concircular vector field ξ, its associated 1-form η, and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(\mathrm{LCS})_n$-manifold). Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [4]. In a $(\mathrm{LCS})_n$-manifold ($n > 2$), the following relations hold:

$$\eta(\xi) = -1, \qquad \phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) + \eta(X)\eta(Y), \tag{2.9}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.10}$$

$$S(X, \xi) = (n-1)\left(\alpha^2 - \rho\right)\eta(X), \tag{2.11}$$

$$R(X, Y)\xi = \left(\alpha^2 - \rho\right)[\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$R(\xi, Y)Z = \left(\alpha^2 - \rho\right)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.13}$$

$$\left(\overline{\nabla}_X \phi\right)Y = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.15}$$

$$R(X, Y)Z = \phi R(X, Y)Z + \left(\alpha^2 - \rho\right)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi, \tag{2.16}$$

for all $X, Y, Z \in \Gamma(TM)$.

3. Submanifolds of a (LCS)-Manifold

Let M be an isometrically immersed submanifold of a $(\mathrm{LCS})_n$-manifold $\overline{M}$ with induced metric $\overline{g}$; we define the isometric immersion by $i : M \to \overline{M}$ and denote by B the differential of i. The induced Riemannian metric g on M by $\overline{g}$ satisfies $g(X, Y) = \overline{g}(BX, BY)$, for all $X, Y \in \Gamma(TM)$.

We denote the tangent and normal spaces of M at point $p \in M$ by $T_M(p)$ and $T_M^{\perp}(p)$, respectively. Let $\{N_1, N_2, , \dots, N_s\}$ be an orthonormal basis of the normal space $T_M^{\perp}(p)$, where $s = \dim(\overline{M}) - \dim(M)$, that is, $s = \operatorname{codim}(M)$.

For any $X \in \Gamma(TM)$, we can write

$$\phi BX = B\varphi X + \sum_{i=1}^{s} \upsilon_i(X) N_i, \tag{3.1}$$

$$\phi N_i = BU_i + \sum_{j=1}^{s} \lambda_{ij} N_j, \quad 1 \le i \le s, \tag{3.2}$$

where ψ, υ_i, U_i, and λ_{ij} denote induced (1-1)-tensor, 1-forms, vector fields and functions on M, respectively. The vector field ξ on (LCS)-manifold $\overline{M}$ can be written as follows:

$$\xi = BV + \sum_{i=1}^{s} \alpha_i N_i, \tag{3.3}$$

where V and α_i are vector field and functions on M and $\overline{M}$, respectively. From (3.1) and (3.2), we can derive

$$\begin{gathered} \upsilon_k(X) = \overline{g}(\phi BX, N_k) = \overline{g}(BX, \phi N_k) = \overline{g}(BX, BU_k) = g(U_k, X), \\ \lambda_{ik} = \overline{g}(\phi N_i, N_k) = \overline{g}(N_i, \phi N_k) = \lambda_{ki}, \end{gathered} \tag{3.4}$$

that is, λ_{ik} is symmetric and

$$\alpha_k = \overline{g}(\xi, N_k) = \eta(N_k), \quad 1 \le i, k \le s. \tag{3.5}$$

Here, we note that the induced (1-1)-tensor field ψ is also symmetric because ϕ is symmetric.

Next, we will the following Lemmas for later use.

Lemma 3.1. *Let M be an isometrically immersed submanifold of a (LCS)-manifold $\overline{M}$. Then, the following assertions are true:*

$$\psi^2 = I + \mu \otimes V - \sum_{i=1}^{s} \upsilon_i \otimes U_i, \tag{3.6}$$

$$\alpha_j V = \psi U_j + \sum_{i=1}^{s} \lambda_{ji} U_i, \quad 1 \le j \le s, \tag{3.7}$$

$$\sum_{j=1}^{s} \lambda_{kj} \lambda_{jp} = \delta_{kp} + \alpha_k \alpha_p - \upsilon_p(U_k), \quad 1 \le k, p \le s, \tag{3.8}$$

where μ denotes the induced 1-form on M by η on $\overline{M}$ and given by $\mu(X) = g(X, V) = \eta(BX)$.

Proof. For any $X \in \Gamma(TM)$, by using (2.10), (3.1), and (3.2), we have

$$\begin{aligned} \phi^2 BX &= \phi B\psi X + \sum_{i=1}^{s} \upsilon_i(X) \phi N_i \\ &= B\psi^2 X + \sum_{j=1}^{s} \upsilon_j(\psi X) N_j + \sum_{i=1}^{s} \upsilon_i(X) \left\{ BU_i + \sum_{j=1}^{s} \lambda_{ij} N_j \right\}, \\ BX + \eta(BX)\xi &= B\psi^2 X + \sum_{j=1}^{s} \upsilon_j(\psi X) N_j + \sum_{i=1}^{s} \upsilon_i(X) BU_i + \sum_{i=1}^{s} \upsilon_i(X) \sum_{j=1}^{s} \lambda_{ij} N_j. \end{aligned} \tag{3.9}$$

Also considering (3.3), we arrive at

$$BX + \mu(X)BV + \mu(X)\sum_{i=1}^{s}\alpha_i N_i = B\psi^2 X + \sum_{j=1}^{s}\upsilon_j(\psi X)N_j + \sum_{i=1}^{s}\upsilon_i(X)BU_i + \sum_{i=1}^{s}\upsilon_i(X)\sum_{j=1}^{s}\lambda_{ij}N_j. \tag{3.10}$$

From the tangential components of (3.10), we conclude that

$$X + \mu(X)V = \psi^2 X + \sum_{i=1}^{s}\upsilon_i(X)U_i, \tag{3.11}$$

which is equivalent to (3.6). On the other hand, with the normal components of (3.10), we have

$$\mu(X)\sum_{i=1}^{s}\alpha_i N_i = \sum_{j=1}^{s}\upsilon_j(\psi X)N_j + \sum_{i=1}^{s}\upsilon_i(X)\sum_{j=1}^{s}\lambda_{ij}N_j, \tag{3.12}$$

which implies that

$$\mu(X)\alpha_k = \upsilon_k(\psi X) + \sum_{i=1}^{s}\upsilon_i(X)\lambda_{ik}, \tag{3.13}$$

that is,

$$g(X,V)\alpha_k = g(\psi X, U_k) + \sum_{i=1}^{s}g(X,U_i)\lambda_{ik}. \tag{3.14}$$

This proves (3.7). In order to prove (3.8), taking (2.10) and (3.2), into account we have

$$\begin{aligned}
\phi^2 N_k &= \phi BU_k + \sum_{j=1}^{s}\lambda_{jk}\phi N_j,\\
N_k + \eta(N_k)\xi &= B\psi U_k + \sum_{i=1}^{s}\upsilon_i(U_k)N_i + \sum_{j=1}^{s}\lambda_{jk}\left\{BU_j + \sum_{t=1}^{s}\lambda_{jt}N_t\right\},\\
N_k + \alpha_k BV + \alpha_k\sum_{i=1}^{s}\alpha_i N_i &= B\psi U_k + \sum_{i=1}^{s}\upsilon_i(U_k)N_i + \sum_{j=1}^{s}\lambda_{jk}BU_j\\
&\quad + \sum_{j=1}^{s}\lambda_{jk}\sum_{t=1}^{s}\lambda_{jt}N_t.
\end{aligned} \tag{3.15}$$

Taking the product of (3.15) with N_p, $1 \leq p \leq s$, we reach

$$\sum_{j=1}^{s} \lambda_{jk}\lambda_{jp} = \delta_{kp} + \alpha_k\alpha_p - \upsilon_p(U_k), \tag{3.16}$$

which gives us (3.8). □

Lemma 3.2. *Let M be an isometrically immersed submanifold of a (LCS)-manifold $\overline{M}$. Then, the following assertions are true:*

$$\psi V + \sum_{i=1}^{s} \alpha_i U_i = 0, \qquad \upsilon_p(V) + \sum_{i=1}^{s} \alpha_i \lambda_{ip} = 0, \quad 1 \leq p \leq s, \tag{3.17}$$

$$\mu(V) = -1 - \sum_{i=1}^{s} \alpha_i^2, \tag{3.18}$$

$$g(\psi X, \psi Y) = g(X, Y) + \mu(X)\mu(Y) + \sum_{i=1}^{s} \upsilon_i(X)\upsilon_i(Y), \tag{3.19}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Making use of $\phi\xi = 0$ and (3.3), we have

$$\begin{aligned} \phi BV + \sum_{i=1}^{s} \alpha_i \phi N_i &= B\psi V + \sum_{i=1}^{s} \upsilon_i(V) N_i + \sum_{i=1}^{s} \alpha_i \left\{ BU_i + \sum_{j=1}^{s} \lambda_{ij} N_j \right\}, \\ 0 &= B\psi V + \sum_{i=1}^{s} \upsilon_i(V) N_i + B\sum_{i=1}^{s} \alpha_i U_i + \sum_{i=1}^{s} \alpha_i \left(\sum_{j=1}^{s} \lambda_{ij} N_j \right). \end{aligned} \tag{3.20}$$

From the tangential and normal components of this last equation, respectively, we get

$$\psi V + \sum_{i=1}^{s} \alpha_i U_i = 0, \qquad \upsilon_p(V) + \sum_{i=1}^{s} \alpha_i \lambda_{ip} = 0. \tag{3.21}$$

Again, taking into account that ξ is time-like vector and (3.3), we reach

$$\begin{aligned} \overline{g}\left(BV + \sum_{i=1}^{s} \alpha_i N_i, BV + \sum_{j=1}^{s} \alpha_j N_j \right) &= g(V, V) - \sum_{i,j=1}^{s} \alpha_j \alpha_i \overline{g}(N_i, N_j), \\ -1 &= \mu(V) + \sum_{i=1}^{s} \alpha_i^2. \end{aligned} \tag{3.22}$$

Finally, we conclude that

$$\begin{aligned} g(\psi X, \psi Y) &= \overline{g}(B\psi X, B\psi Y) = \overline{g}\left(\phi BX - \sum_{i=1}^{s} \upsilon_i(X)N_i, \phi BY - \sum_{j=1}^{s} \upsilon_j(Y)N_j\right) \\ &= \overline{g}(BX, BY) + \eta(BX)\eta(BY) + \sum_{i=1}^{s} \upsilon_i(X)\upsilon_i(Y) \\ &= g(X, Y) + \mu(X)\mu(Y) + \sum_{i=1}^{s} \upsilon_i(X)\upsilon_i(Y). \end{aligned} \tag{3.23}$$

This proves our assertions. □

Now, we suppose that $\{N_1, N_2, , \ldots, N_s\}$ and $\{\overline{N}_1, \overline{N}_2, \ldots, \overline{N}_s\}$ are two orthonormal bases of $T_M^{\perp}(p)$ at $p \in M$ and set

$$\overline{N}_i = \sum_{j=1}^{s} k_{ji} N_i, \quad 1 \leq i \leq s, \tag{3.24}$$

by means of $g(\overline{N}_i, \overline{N}_j) = \sum_{p=1}^{s} k_{ip} k_{pj} = \delta_{ij}$. So, we mean that the basis with another basis transition matrix (k_{ij}) is an orthogonal matrix. From (3.24) we have

$$N_j = \sum_{p=1}^{s} k_{jp} \overline{N}_p. \tag{3.25}$$

Taking (3.24) into account, (3.1), (3.2), and (3.3) are, respectively, written in the following way:

$$\phi BX = B\psi X + \sum_{k=1}^{s} \overline{\upsilon}_k(X) \overline{N}_k, \tag{3.26}$$

$$\phi \overline{N}_p = B\overline{U}_p + \sum_{t=1}^{s} \overline{\lambda}_{pt} \overline{N}_t, \quad 1 \leq p \leq s, \tag{3.27}$$

$$\xi = BV + \sum_{\ell=1}^{s} \overline{\alpha}_\ell \overline{N}_\ell, \tag{3.28}$$

where

$$\overline{\upsilon}_p(X) = \sum_{i=1}^{s} k_{ip} \upsilon_i(X), \qquad \overline{U}_\ell = \sum_{i=1}^{s} k_{i\ell} U_i, \tag{3.29}$$

$$\overline{\lambda}_{pt} = \sum_{ij}^{s} k_{ip} \lambda_{ij} \lambda_{it}, \qquad \overline{\lambda}_{pt} = \overline{\lambda}_{tp}, \qquad \overline{\alpha}_\ell = \sum_{i=1}^{s} k_{i\ell} \alpha_i. \tag{3.30}$$

Furthermore, because λ_{ij} is symmetric, from (3.30), we can derive that under the suitable transformation (3.24) λ_{ij} reduce to $\overline{\lambda}_{ij} = \lambda_i \delta_{ij}$, where λ_i are eigenvalues of matrix (λ_{ij}). So, again (3.27) and (3.8) can be, respectively, written in the following way:

$$\begin{aligned} \phi \overline{N}_\ell &= B\overline{U}_\ell + \lambda_\ell \overline{N}_\ell, \\ \overline{v}_p\left(\overline{U}_k\right) &= \delta_{kp} + \overline{\alpha}_k \overline{\alpha}_p - \overline{\lambda}_p \overline{\lambda}_k \delta_{kj}, \end{aligned} \tag{3.31}$$

which implies that $\overline{v_k}(\overline{U_k}) = 1 - \overline{\alpha}_k^2 - \overline{\lambda}_k^2$ and $\overline{v}_k(\overline{U}_j) = -\overline{\alpha}_k \overline{\alpha}_j$ for $k \neq j$.

Now, let M be an isometrically immersed submanifold of a (LCS)-manifold $\overline{M}$. If $\phi(BT_M(p)) \subset T_M(p)$ for any point $p \in M$, then M is said to be an invariant submanifold of $\overline{M}$. In this case, (3.1), (3.2), and (3.3) become, respectively,

$$\phi BX = B\psi X, \tag{3.32}$$

$$\phi N_i = \sum_{j=1}^{s} \lambda_{ij} N_j, \tag{3.33}$$

$$\xi = BV + \sum_{i=1}^{s} \alpha_i N_i \tag{3.34}$$

for any $X \in \Gamma(TM)$.

Lemma 3.3. *Let M be an invariant submanifold of a (LCS)-manifold $\overline{M}$. Then, the following assertions are true:*

$$\psi^2 = I + \mu \otimes V, \qquad \alpha_i V = 0, \tag{3.35}$$

$$\delta_{kj} + \alpha_k \alpha_j - \sum_{i=1}^{s} \lambda_{ki} \lambda_{ij} = 0, \qquad \psi V = 0, \qquad \sum_{i=2}^{s} \alpha_i \lambda_{ij} = 0, \tag{3.36}$$

$$-v(V) = 1 + \sum_{i=1}^{s} \alpha_i^2, \qquad g(\psi X, \psi Y) = g(X, Y) + \mu(X)\mu(Y), \tag{3.37}$$

for any $X, Y \in \Gamma(TM)$.

Proof. The proof is obvious. Therefore, we omit it. □

Theorem 3.4. *Let M be an invariant submanifold of a (LCS)-manifold $\overline{M}$. One of the following cases occurs.*

(1) *If ξ is normal to M, then the induced structure (ψ, g) on M is an almost product Riemannian structure whenever ψ is nontrivial.*

(2) *If ξ is tangent to M, then the induced structure (ψ, V, μ, g) on M is a Lorentzian concircular structure.*

Proof. (1) If ξ is normal to the submanifold, then the vector field $V = 0$. From (3.35) and (3.37), we have $\psi^2 = I$, $g(\psi X, \psi Y) = g(X, Y)$, that is, (ψ, g) is an almost product Riemannian structure whenever ψ is nontrivial.

(2) If ξ is tangent to the submanifold (i.e., $V \neq 0$, $\alpha_i = 0$), then we have $\mu(X) = g(X, V)$, $\psi^2 = I + \mu \otimes V$, $\psi V = 0$, $\mu(V) = -1$, that is, (ψ, V, μ, g) is a Lorentzian concircular structure. □

Theorem 3.5. *Let M be a submanifold of a (LCS)-manifold $\overline{M}$. The submanifold M of a (LCS)-manifold $\overline{M}$ is invariant if and only if the induced structure (ψ, g) on M is an almost product Riemannian structure whenever ψ is nontrivial or the induced structure (ψ, V, μ, g) on M is a Lorentzian concircular structure.*

Proof. From Theorem 3.4 we know that the necessary is obvious.

Conversely, we suppose that the induced structure (ψ, g) is an almost product Riemannian structure. Then, from (3.19), we have

$$\mu^2(X) + \sum_{i=1}^{s} v_i^2(X) = 0, \tag{3.38}$$

that is, $\mu(X) = v_i(X) = 0$, $1 \leq i \leq s$. So from (3.1) and (3.3) we can derive that the submanifold M is invariant and ξ is normal to M.

Now, we suppose that the induced structure (ψ, V, μ, g) is a Lorentzian concircular structure. Then, from (3.6), we get

$$\sum_{i=1}^{s} v_i(X) U_i = 0, \tag{3.39}$$

which implies that $v_i(X) = 0$, $1 \leq i \leq s$. From (3.7), by a direct calculation, we derive $\alpha_i = 0$, $1 \leq i \leq s$. So from (3.1) and (3.3), we conclude that M is invariant submanifold and ξ is tangent to M. □

Theorem 3.6. *Let M be an isometrically immersed submanifold of (LCS)-manifold $\overline{M}$. Then, M is invariant submanifold if and only if the normal space $T_M^{\perp}(p)$, at every point $p \in M$, admits an orthonormal basis consisting of the eigenvectors of the matrix (ϕ).*

Proof. Let us suppose that M is invariant.

(1) When ξ is normal to M, at $p \in M$ we consider an s-dimensional vector space W and investigate the eigenvalues of the matrix $(\lambda_{ij})_{s\times s}$. From (3.36) and (3.37), it is easy to see that the vector $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of the vector space W is a unit eigenvector of the matrix $(\lambda_{ij})_{s\times s}$ and its eigenvalue is equal to 0.

Now, we suppose that a vector $(\omega_1, \omega_2, \ldots, \omega_s)$ satisfying $\sum_{i=2}^{s} \alpha_i \omega_i = 0$ is an eigenvector and its eigenvalue is λ. Then, we have

$$\sum_{j=1}^{s} \lambda_{ij} \omega_j = \lambda \omega_i, \quad 1 \leq i \leq s, \tag{3.40}$$

which implies that

$$\sum_{i,j=1}^{s}\lambda_{ki}\lambda_{ji}\omega_j = \lambda\sum_{i=1}^{s}\lambda_{ki}\omega_i, \quad 1 \le k \le s, \tag{3.41}$$

from which

$$\sum_{j=1}^{s}\left(\sum_{i=1}^{s}\lambda_{ki}\lambda_{ji}\right)\omega_j = \lambda^2\omega_k. \tag{3.42}$$

On the other hand, from (3.36) we get

$$\sum_{j=1}^{s}(\delta_{kj} - \alpha_k\alpha_j)\omega_j = \sum_{i=1}^{s}\left(\sum_{j=1}^{s}\lambda_{kj}\lambda_{ij}\right)\omega_j = \lambda^2\omega_k, \tag{3.43}$$

that is, $\omega_k = \lambda^2\omega_k$, which is equivalent to $\lambda^2 = 1$.

Consequently, if by a suitable transformation of the orthonormal basis $\{N_1, N_2, \dots, N_s\}$ of $T_M^{\perp}(p)$, the matrix λ_{ij} becomes a diagonal matrix, then the diagonal components $\lambda_1, \lambda_2, \dots, \lambda_s$ satisfy relations

$$\lambda_1^s = \lambda_2^2 = \cdots = \lambda_{s-1}^2, \qquad \lambda_s = 0. \tag{3.44}$$

In this case, if we denote by $\{\overline{N}_1, \overline{N}_2, \dots, \overline{N}_s\}$ another orthonormal basis of $T_M^{\perp}(p)$, then, from (3.31), we have $\phi\overline{N}_\ell = \lambda_\ell\overline{N}_\ell$, $1 \le \ell \le s$. So, $\overline{N}_\ell$, $1 \le \ell \le s$, are eigenvectors of the matrix-(ϕ) and $\overline{N}_s = \xi$.

(2) When ξ is tangent to M, since $\alpha_i = 0$, $1 \le i \le s$, from (3.36), we have

$$\delta_{kj} = \sum_{i=1}^{s}\lambda_{ki}\lambda_{ij}. \tag{3.45}$$

If we denote by $\{\omega_1, \omega_2, \dots, \omega_s\}$ an eigenvector of matrix (λ_{ij}) and by λ its eigenvalue, then we have

$$\sum_{j=1}^{s}\lambda_{ji}\omega_j = \lambda\omega_i, \quad 1 \le i \le s. \tag{3.46}$$

So, we obtain

$$\sum_{i,j=1}^{s}\lambda_{ki}\lambda_{ji}\omega_j = \lambda\sum_{i=1}^{s}\lambda_{ki}\omega_i, \quad 1 \le k \le s, \tag{3.47}$$

that is, $\omega_k = \lambda^2 \omega_k$, which implies that $\lambda^2 = 1$. Since the eigenvalues of (λ_{ij}) are ± 1, by a suitable transformation of the orthonormal basis of $T_M^{\perp}(p)$, $\{N_1, N_2, \dots, N_s\}$ to become $\{\overline{N}_1, \overline{N}_2, \dots, \overline{N}_s\}$, then $\overline{N}_1, \overline{N}_2, \dots, \overline{N}_s$ are eigenvectors of matrix-(ϕ).

Conversely, if the orthonormal basis $\{\overline{N}_1, \overline{N}_2, \dots, \overline{N}_s\}$ of $T_M^{\perp}(p)$ consists of eigenvectors of matrix-(ϕ) and these eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$ satisfy $\lambda_1^2 = \lambda_2^2, \dots, \lambda_{s-1}^2 = 1$ and $\lambda_s^2 = \pm 1$ or 0, then we have $\phi \overline{N}_\ell = \lambda \overline{N}_\ell$ and we conclude that $\overline{U}_\ell = 0$, $1 \leq \ell \leq s$, and so M is invariant. □

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