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Research Article

Euler Basis, Identities, and Their Applications

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Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. We show that $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ is a good basis for the space V_n, for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n, then $p(x) = \sum_{l=0}^{n} b_l E_l(x)$ for some uniquely determined $b_l \in \mathbb{Q}$. In this paper we develop method for computing b_l from the information of $p(x)$.

1. Introduction

The Euler polynomials, $E_n(x)$, are given by

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.1}$$

(see [1–20]) with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0, E_n(0) = E_n$ are called the nth Euler numbers. The Bernoulli numbers are also defined by

$$\frac{t}{e^t-1} = e^{Bt} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \tag{1.2}$$

(see [1–20]) with the usual convention about replacing B^n by B_n. As is well known, the Bernoulli polynomials are given by

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l} = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \tag{1.3}$$

(see [9–15]) From (1.1), (1.2), and (1.3), we note that

$$B_n(1) - B_n = \delta_{1,n}, \qquad E_n(1) + E_n = 2\delta_{0,n}, \tag{1.4}$$

where $\delta_{k,n}$ is the kronecker symbol.

Let $m, n \in \mathbb{Z}_+$ with $m + n \geq 2$. The formula

$$B_m(x)B_n(x) = \sum_r \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \frac{m!n!B_{m+n}}{(m+n)!}, \tag{1.5}$$

is proved in [4–6]. Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Probably, $\{1, x, \ldots, x^n\}$ is the most natural basis for this space. But $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ is also a good basis for the space V_n, for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n, then

$$p(x) = \sum_{l=0}^{n} b_l E_l(x), \tag{1.6}$$

for some uniquely determined $b_l \in \mathbb{Q}$. Further,

$$b_k = \frac{1}{2k!} \left\{ p^{(k)}(1) + p^{(k)}(0) \right\} \quad (k = 0, 1, 2, \ldots, n), \tag{1.7}$$

where $p^{(k)}(x) = d^k p(x)/dx^k$. In this paper we develop methods for computing b_l from the information of $p(x)$. Apply these results to arithmetically and combinatorially interesting identities involving $E_0(x), E_1(x), \ldots, E_n(x), B_0(x), \ldots, B_n(x)$. Finally, we give some applications of those obtained identities.

2. Euler Basis, Identities, and Their Applications

Let us take $p(x)$ the polynomial of degree n as follows:

$$p(x) = \sum_{k=0}^{n} B_k(x) B_{n-k}(x). \tag{2.1}$$

From (2.1), we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!} \sum_{l=k}^{n} B_{l-k}(x) B_{n-l}(x). \tag{2.2}$$

By (1.7) and (2.2), we get

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} \\ &= \frac{1}{2}\binom{n+1}{k}\sum_{l=k}^{n}\{(B_{l-k} + \delta_{1,l-k})(B_{n-l} + \delta_{1,n-l}) + B_{l-k}B_{n-l}\}, \end{aligned} \tag{2.3}$$

Thus, we have

$$b_k = \binom{n+1}{k}\left(\sum_{l=k}^{n} B_{l-k}B_{n-l} + B_{n-k-1}\right), \quad (0 \le k \le n-3), \tag{2.4}$$

$$b_{n-2} = \frac{7}{72}n\left(n^2 - 1\right), \qquad b_n = n+1, \qquad b_{n-1} = 0. \tag{2.5}$$

By (1.6), (2.1), (2.3), and (2.4), we get

$$\begin{aligned} &\sum_{k=0}^{n} B_k(x)B_{n-k}(x) \\ &\quad = \sum_{k=0}^{n-3}\binom{n+1}{k}\left(\sum_{l=k}^{n} B_{l-k}B_{n-l} + B_{n-k-1}\right)E_k(x) + \frac{7}{72}n\left(n^2-1\right)E_{n-2}(x) + (n+1)E_n(x). \end{aligned} \tag{2.6}$$

Let us consider the following triple identities:

$$p(x) = \sum_{r+s+t=n} B_r(x)B_s(x)B_t(x) = \sum_{k=0}^{n} b_k E_k(x), \tag{2.7}$$

where the sum runs over all $r, s, t \in \mathbb{Z}_+$ with $r + s + t = n$. Thus, by (2.7), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k} B_r(x)B_s(x)B_t(x). \tag{2.8}$$

From (1.7) and (2.8), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} \\ &= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{B_r(1)B_s(1)B_t(1) + B_rB_sB_t\} \\ &= \frac{\binom{n+2}{k}}{2}\left\{2\sum_{r+s+t=n-k} B_rB_sB_t + \sum_{r+s+t=n-k}\delta_{1,r}B_sB_t + \sum_{r+s+t=n-k} B_r\delta_{1,s}B_t\right. \end{aligned}$$

$$+\sum_{r+s+t=n-k} B_r B_s \delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s}B_t + \sum_{r+s+t=n-k} \delta_{1,r}B_s\delta_{1,t}$$
$$\left. + \sum_{r+s+t=n-k} B_r\delta_{1,s}\delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s}\delta_{1,t}\right\}. \tag{2.9}$$

Therefore, by (2.7) and (2.9), we obtain the following theorem.

Theorem 2.1. *For $r, s, t \in \mathbb{Z}_+$, and $n \in \mathbb{N}$ with $n \geq 3$, one has*

$$\begin{aligned}
&\sum_{r+s+t=n} B_r(x)B_s(x)B_t(x) \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s+t=n-k} B_rB_sB_t + 3\sum_{r+s=n-k-1} B_rB_s + 3B_{n-k-2} + \delta_{k,n-3}\right\}E_k(x) \\
&\qquad + \binom{n+2}{2}E_n(x).
\end{aligned} \tag{2.10}$$

Let us take the polynomial $p(x)$ as follows:

$$p(x) = \sum_{r+s+t=n} B_r(x)B_s(x)E_t(x). \tag{2.11}$$

Then, by (2.11), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k} B_r(x)B_s(x)E_t(x). \tag{2.12}$$

From (1.6), (1.7), and (2.12), we have

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} = \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{B_r(1)B_s(1)E_t(1) + B_rB_sE_t\} \\
&= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{(B_r+\delta_{1,r})(B_s+\delta_{1,s})(-E_t+2\delta_{0,t}) + B_rB_sE_t\} \\
&= \frac{\binom{n+2}{k}}{2}\left\{-\sum_{r+s+t=n-k}\delta_{1,r}B_sE_t - \sum_{r+s+t=n-k}B_r\delta_{1,s}E_t + 2\sum_{r+s+t=n-k}B_rB_s\delta_{0,t}\right. \\
&\qquad - \sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}E_t + 2\sum_{r+s+t=n-k}\delta_{1,r}B_s\delta_{0,t} + 2\sum_{r+s+t=n-k}B_r\delta_{1,s}\delta_{0,t} \\
&\qquad \left. +2\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}\delta_{0,t}\right\}.
\end{aligned} \tag{2.13}$$

Note that

$$
\begin{aligned}
b_{n-1} &= \binom{n+2}{n-1}\left\{-\sum_{s+t=0} B_s E_t - \sum_{r+t=0} B_r E_t + 2\sum_{r+s=1} B_r B_s - 0 + 2B_0 + 2B_0 + 2\cdot 0\right\} \\
&= \frac{1}{2}\binom{n+2}{n-1}\{-1-1+2(B_1+B_1)+2+2\} = 0.
\end{aligned}
\tag{2.14}
$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$
\begin{aligned}
&\sum_{r+s+t=n} B_r(x) B_s(x) E_t(x) \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s=n-k} B_r B_s - 2\sum_{r+t=n-k-1} B_r E_t - E_{n-k-2} + 4B_{n-k-1} + 2\delta_{k,n-2}\right\} E_k(x) \\
&\quad + \binom{n+2}{2} E_n(x).
\end{aligned}
\tag{2.15}
$$

Remark 2.3. By the same method, we obtain the following identities.

(I)

$$
\begin{aligned}
&\sum_{r+s+t=n} B_r(x) E_s(x) E_t(x) \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s+t=n-k} B_r E_s E_t + \sum_{s+t=n-k-1} E_s E_t - 4\sum_{r+s=n-k} B_r E_s + 4B_{n-k} - 4E_{n-k-1}\right\} E_k(x) \\
&\quad + \binom{n+2}{2} E_n(x).
\end{aligned}
\tag{2.16}
$$

(II)

$$
\begin{aligned}
&\sum_{r+s+t=n} E_r(x) E_s(x) E_t(x) \\
&\quad = 3\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{r+s=n-k} E_r E_s - 2E_{n-k}\right\} E_k(x) + \binom{n+2}{2} E_n(x).
\end{aligned}
\tag{2.17}
$$

Let us consider the polynomial $p(x)$ as follows:

$$
p(x) = \sum_{r+s+t=n} B_r(x) B_s(x) x^t.
\tag{2.18}
$$

Thus, by (2.18), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k} B_r(x)B_s(x)x^t. \tag{2.19}$$

From (1.6), (1.7), (2.18), and (2.19), we have

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} = \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\left\{B_r(1)B_s(1)+B_rB_s0^t\right\}\\
&= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\left\{(B_r+\delta_{1,r})(B_s+\delta_{1,s})+B_rB_s0^t\right\}\\
&= \frac{\binom{n+2}{k}}{2}\left\{\sum_{r+s+t=n-k}B_rB_s+\sum_{r+s+t=n-k}B_r\delta_{1,s}+\sum_{r+s+t=n-k}\delta_{1,r}B_s+\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}+\sum_{r+s+t=n-k}B_rB_s0^t\right\}.
\end{aligned} \tag{2.20}$$

Here we note that

$$\begin{aligned}
&\sum_{r+s+t=n-k}B_rB_s = \sum_{t=0}^{n-k}\sum_{r+s=n-k-t}B_rB_s = \sum_{t=0}^{n-k}\sum_{r+s=t}B_rB_s\\
&\sum_{r+s+t=n-k}B_r\delta_{1,s} = \begin{cases}\sum_{r=0}^{n-k-1}B_r, & \text{if } k\le n-1,\\ 0, & \text{if } k=n,\end{cases}\\
&\sum_{r+s+t=n-k}B_s\delta_{1,r} = \begin{cases}\sum_{r=0}^{n-k-1}B_r, & \text{if } k\le n-1,\\ 0, & \text{if } k=n,\end{cases}\\
&\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s} = \begin{cases}1, & \text{if } k\le n-2,\\ 0, & \text{if } k=n-1 \text{ or } n,\end{cases}\\
&\sum_{r+s+t=n-k}B_rB_s0^t = \sum_{r+s=n-k}B_rB_s, \quad \forall k.
\end{aligned} \tag{2.21}$$

It is easy to show that

$$\begin{aligned}
b_{n-1} &= \frac{1}{2}\binom{n+2}{n-1}\left\{\sum_{r+s=0}B_rB_s+2\sum_{r+s=1}B_rB_s+2B_0\right\}\\
&= \frac{1}{2}\binom{n+2}{n-1}\{1+2(B_1+B_2)+2\} = \frac{1}{2}\binom{n+2}{n-1}.
\end{aligned} \tag{2.22}$$

Therefore, by (1.6), (2.18), (2.20), and (2.22), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned}
&\sum_{r+s+t=n} B_r(x)B_s(x)x^t \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{t=0}^{n-k-1}\sum_{r+s=t} B_rB_s + 2\sum_{r+s=n-k} B_rB_s + 2\sum_{r=0}^{n-k-1} B_r + 1\right\}E_k(x) \\
&\qquad + \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{n}E_n(x).
\end{aligned} \tag{2.23}$$

Remark 2.5. By the same method, we can derive the following identities.

(I)

$$\begin{aligned}
&\sum_{r+s+t=n} B_r(x)E_s(x)x^t \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{-\sum_{t=0}^{n-k-1}\sum_{r+s=t} B_rE_s - \sum_{s=0}^{n-k-1} E_s + 2\sum_{r=0}^{n-k} B_r + 2\right\}E_k(x) \\
&\qquad + \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{n}E_n(x).
\end{aligned} \tag{2.24}$$

(II)

$$\begin{aligned}
&\sum_{r+s+t=n} E_r(x)E_s(x)x^t \\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{t=0}^{n-k-1}\sum_{r+s=t} E_rE_s + 2\sum_{r+s=n-k} E_rE_s - 4\sum_{r=0}^{n-k} E_r + 4\right\}E_k(x) \\
&\qquad + \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{2}E_n(x).
\end{aligned} \tag{2.25}$$

Now we generalize the above consideration to the completely arbitrary case. Let

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x), \tag{2.26}$$

where the sum runs over all nonnegative integers $i_1, i_2, \ldots, i_r, j_1, \ldots, j_s$ satisfying $i_1 + i_2 + \cdots + i_r + j_1 + \cdots + j_s = n$. From (2.26), we note that

$$p^{(k)}(x) = (n+r+s-1)\cdots(n+r+s-k)\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} B_{i_1}(x)\cdots B_{i_r}(x) \times E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.27}$$

By (1.6), (1.7), (2.18), and (2.27), we get

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} \\
&= \frac{1}{2}\binom{n+r+s-1}{k}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\
&= \frac{1}{2}\binom{n+r+s-1}{k} \\
&\quad\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\left\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\right. \\
&\qquad\qquad \left.\times(-E_{j_1}+2\delta_{0,j_1})\cdots(-E_{j_s}+2\delta_{0,j_s})+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\
&= \frac{1}{2}\binom{n+r+s-1}{k}\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\right. \\
&\qquad\qquad \left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.
\end{aligned} \tag{2.28}$$

Note that

$$\begin{aligned}
b_n &= \frac{1}{2}\binom{n+r+s-1}{n}\left\{\sum_{0\le c\le s}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c}\right. \\
&\qquad\qquad \left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\
&= \frac{1}{2}\binom{n+r+s-1}{n}\left((2-1)^s+1\right)=\binom{n+r+s-1}{n}, \\
b_{n-1} &= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{\sum_{\substack{r-1\le a\le r\\ 0\le c\le s}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right. \\
&\qquad\qquad \times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=1+a-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\
&\qquad\qquad \left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}
\end{aligned}$$

$$= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{r(2-1)^s + \sum_{0\le c\le s}\binom{s}{c}(-1)^c 2^{s-c}\left[-\frac{1}{2}(r+c)\right] - \frac{1}{2}(r+s)\right\}$$
$$= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{r - \frac{1}{2}r + \frac{1}{2}s - \frac{1}{2}r - \frac{1}{2}s\right\} = 0. \tag{2.29}$$

Therefore, by (1.6), (2.28), and (2.29), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$ with $n \ge 2$, one has*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+r+s-1}{k}\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}E_k(x)$$
$$+ \binom{n+r+s-1}{n}E_n(x). \tag{2.30}$$

Let us consider the polynomial $p(x)$ of degree n as

$$p(x) = \sum_{t+i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t. \tag{2.31}$$

Then, from (2.31), we have

$$p^{(k)}(x) = (n+r+s)(n+r+s-1)\cdots(n+r+s-k+1) \times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t. \tag{2.32}$$

By (1.7) and (2.32), we get

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} \\ &= \frac{1}{2}\binom{n+r+s}{k}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\ &= \frac{1}{2}\binom{n+r+s}{k} \\ &\quad\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k}\left\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\right. \\ &\qquad\left.\times\left(-E_{j_0}+2\delta_{0,j_1}\right)\cdots\left(-E_{j_s}+2\delta_{1,j_s}\right)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \end{aligned} \tag{2.33}$$

From (2.33), we can derive the following equation:

$$\begin{aligned} b_k = \frac{1}{2}\binom{n+r+s}{k}&\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{t=0}^{n+a-k-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\right. \\ &\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}. \end{aligned} \tag{2.34}$$

Observe now that

$$\begin{aligned} b_n &= \frac{1}{2}\binom{n+r+s}{n}\left\{\sum_{c=0}^{s}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c}\right. \\ &\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{1}{2}\binom{n+r+s}{n}\left[(2-1)^s+1\right]=\binom{n+r+s}{n}, \end{aligned} \tag{2.35}$$

$$b_{n-1}=\frac{1}{2}\binom{n+r+s}{n-1}\left\{\sum_{\substack{r-1\le a\le r\\0\le c\le s}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{t=0}^{1+a-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\right.$$
$$\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}$$
$$=\frac{1}{2}\binom{n+r+s}{n-1}\left\{r+1-\frac{1}{2}r+\frac{1}{2}s-\frac{1}{2}r-\frac{1}{2}s\right\}=\frac{1}{2}\binom{n+r+s}{n-1}. \tag{2.36}$$

Therefore, by (1.6), (2.31), (2.34), (2.35), and (2.36), we obtain the following theorem.

Theorem 2.7. *For $n\in\mathbb{N}$ with $n\ge 2$, one has*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t$$
$$=\frac{1}{2}\sum_{k=0}^{n-2}\binom{n+r+s}{k}\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\right.$$
$$\times\sum_{t=0}^{n+a-k-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \tag{2.37}$$
$$\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}E_k(x)$$
$$+\frac{1}{2}\binom{n+r+s}{n-1}E_{n-1}(x)+\binom{n+r+s}{n}E_n(x).$$

Let us consider the following polynomial of degree n.

$$p(x)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.38}$$

Thus, by (2.38), we get

$$p^{(k)}(x)=(r+s)^k\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\times B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.39}$$

From (1.7), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} \\ &= \frac{(r+s)^k}{2k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!} \\ &\quad \times\left\{B_{i_1}(1)\cdots B_{i_r}(1)\times E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{(r+s)^k}{2k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!} \\ &\quad \times\left\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\times\left(-E_{j_1}+2\delta_{0,j_1}\right)\cdots\left(-E_{j_s}+2\delta_{0,j_s}\right)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}. \end{aligned} \tag{2.40}$$

Thus, by (2.40), we get

$$\begin{aligned} b_k = \frac{(r+s)^k}{2k!}&\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}\right. \\ &\quad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\}. \end{aligned} \tag{2.41}$$

Now, we note that

$$\begin{aligned} b_n = \frac{(r+s)^n}{2n!}&\left\{\sum_{c=0}^{s}\binom{s}{c}(-1)^c2^{s-c}\right. \\ &\quad\left.\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_r!j_1!\cdots j_c!}+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\} \\ &= \frac{(r+s)^n}{2n!}\left[(2-1)^s+1\right]=\frac{(r+s)^n}{n!}, \end{aligned}$$

$$b_{n-1} = \frac{(r+s)^{n-1}}{2(n-1)!}\left\{ \sum_{\substack{r-1\le a\le r\\ 0\le c\le s}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=1+a-r} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!} + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!} \right\}$$

$$= \frac{(r+s)^{n-1}}{2(n-1)!}\left\{ r(2-1)^s + \sum_{c=0}^{s}\binom{s}{c}(-1)^c 2^{s-c}\left[-\frac{1}{2}(r+c)\right] - \frac{1}{2}(r+s)\right\} = 0. \tag{2.42}$$

Therefore, by (1.6), (2.38), (2.41), and (2.42), we obtain the following theorem.

Theorem 2.8. *For $n \in \mathbb{N}$ with $n \ge 2$, one has*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} \frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)}{i_1!\cdots i_r! j_1!\cdots j_s!}$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\frac{(r+s)^k}{k!}\left\{ \sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!} + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!} \right\} E_k(x)$$
$$+ \frac{(r+s)^n}{n!}E_n(x). \tag{2.43}$$

By the same method, we can obtain the following identity:

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t}{i_1!\cdots i_r! j_1!\cdots j_s! t!}$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\frac{(r+s+1)^k}{k!}\left\{ \sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \right.$$

$$\times \sum_{t=0}^{n+a-k-r} \frac{1}{(n+a-k-r-t)!} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} \frac{B_{i_1}\cdots B_{i_a} E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!}$$

$$\left. + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} \frac{B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!} \right\} E_k(x)$$

$$+ \frac{(r+s+1)^{n-1}}{2(n-1)!} E_{n-1}(x) + \frac{(r+s+1)^n}{n!} E_n(x). \tag{2.44}$$

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