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Research Article

Complex Hessian Equations on Some Compact Kähler Manifolds

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On a compact connected $2m$-dimensional Kähler manifold with Kähler form ω, given a smooth function $f : M \to \mathbb{R}$ and an integer $1 < k < m$, we want to solve uniquely in $[\omega]$ the equation $\tilde{\omega}^k \wedge \omega^{m-k} = e^f \omega^m$, relying on the notion of k-positivity for $\tilde{\omega} \in [\omega]$ (the extreme cases are solved: $k = m$ by (Yau in 1978), and $k = 1$ trivially). We solve by the continuity method the corresponding complex elliptic kth Hessian equation, more difficult to solve than the Calabi-Yau equation ($k = m$), under the assumption that the holomorphic bisectional curvature of the manifold is nonnegative, required here only to derive an a priori eigenvalues pinching.

1. The Theorem

All manifolds considered in this paper are *connected*.

Let (M, J, g, ω) be a compact connected Kähler manifold of complex dimension $m \geq 3$. Fix an integer $2 \leq k \leq m - 1$. Let $\varphi : M \to \mathbb{R}$ be a smooth function, and let us consider the $(1, 1)$-form $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$ and the associated 2-tensor $\tilde{g}$ defined by $\tilde{g}(X, Y) = \tilde{\omega}(X, JY)$. Consider the sesquilinear forms h and $\tilde{h}$ on $T^{1,0}$ defined by $h(U, V) = g(U, \overline{V})$ and $\tilde{h}(U, V) = \tilde{g}(U, \overline{V})$. We denote by $\lambda(g^{-1}\tilde{g})$ the eigenvalues of $\tilde{h}$ with respect to the Hermitian form h. By definition, these are the eigenvalues of the unique endomorphism A of $T^{1,0}$ satisfying

$$\tilde{h}(U, V) = h(U, AV) \quad \forall U, V \in T^{1,0}. \tag{1.1}$$

Calculations infer that the endomorphism A writes

$$\begin{aligned} A : T^{1,0} &\longrightarrow T^{1,0}, \\ U^i \partial_i &\longmapsto A_i^j U^i \partial_j = g^{j\bar{\ell}} \tilde{g}_{i\bar{\ell}} U^i \partial_j. \end{aligned} \tag{1.2}$$

A is a self-adjoint/Hermitian endomorphism of the Hermitian space $(T^{1,0}, h)$, therefore $\lambda(g^{-1}\tilde{g}) \in \mathbb{R}^m$. Let us consider the following cone: $\Gamma_k = \{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k, \sigma_j(\lambda) > 0\}$, where σ_j denotes the jth elementary symmetric function.

Definition 1.1. φ is said to be k-admissible if and only if $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$.

In this paper, we prove the following theorem.

Theorem 1.2 (the σ_k equation). *Let (M, J, g, ω) be a compact connected Kähler manifold of complex dimension $m \geq 3$ with nonnegative holomorphic bisectional curvature, and let $f : M \to \mathbb{R}$ be a function of class C^∞ satisfying $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$. There exists a unique function $\varphi : M \to \mathbb{R}$ of class C^∞ such that*

$$(1) \int_M \varphi \, \omega^m = 0, \tag{1.3}$$

$$(2)\ \tilde{\omega}^k \wedge \omega^{m-k} = \left(\frac{e^f}{\binom{m}{k}} \right) \omega^m. \tag{E_k}$$

Moreover the solution φ is k-admissible.

This result was announced in a note in the Comptes Rendus de l'Acadé-mie des Sciences de Paris published online in December 2009 [1]. The curvature assumption is used, in Section 6.2 only, for an a priori estimate on $\lambda(g^{-1}\tilde{g})$ as in [2, page 408], and it should be removed (as did Aubin for the case $k = m$ in [3], see also [4] for this case). For the analogue of (E_k) on $\mathbb{C}^m$, the Dirichlet problem is solved in [5, 6], and a Bedford-Taylor type theory, for weak solutions of the corresponding degenerate equations, is addressed in [7]. Thanks to Julien Keller, we learned of an independent work [8] aiming at the same result as ours, with a different gradient estimate and a similar method to estimate $\lambda(g^{-1}\tilde{g})$, but no proofs given for the C^0 and the C^2 estimates.

Let us notice that the function f appearing in the second member of (E_k) satisfies necessarily the normalisation condition $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$. Indeed, this results from the following lemma.

Lemma 1.3. *Consider $\int_M \tilde{\omega}^k \wedge \omega^{m-k} = \int_M \omega^m$.*

Proof. See [9, page 44]. □

Let us write (E_k) differently.

Lemma 1.4. *Consider $\tilde{\omega}^k \wedge \omega^{m-k} = (\sigma_k(\lambda(g^{-1}\tilde{g})) / \binom{m}{k}) \omega^m$.*

Proof. Let $P \in M$. It suffices to prove the equality at P in a g-normal $\tilde{g}$-adapted chart z centered at P. In such a chart $g_{i\bar{j}}(0) = \delta_{ij}$ and $\tilde{g}_{i\bar{j}}(0) = \delta_{ij}\lambda_i(0)$, so at $z = 0$, $\omega = idz^a \wedge dz^{\bar{a}}$ and $\tilde{\omega} = i\lambda_a(0)dz^a \wedge dz^{\bar{a}}$. Thus

$$\begin{aligned}\tilde{\omega}^k \wedge \omega^{m-k} &= \left(\sum_a i\lambda_a(0)dz^a \wedge dz^{\bar{a}}\right)^k \wedge \left(\sum_b idz^b \wedge dz^{\bar{b}}\right)^{m-k} \\ &= \sum_{\substack{(a_1,\dots,a_k)\in\{1,\dots,m\} \\ \text{distinct integers} \\ (b_1,\dots,b_{m-k})\in\{1,\dots,m\}\setminus\{a_1,\dots,a_k\} \\ \text{distinct integers}}} i^m \lambda_{a_1}(0)\cdots\lambda_{a_k}(0) \\ &\left(dz^{a_1} \wedge dz^{\bar{a}_1}\right) \wedge \cdots \wedge \left(dz^{a_k} \wedge dz^{\bar{a}_k}\right) \wedge \left(dz^{b_1} \wedge dz^{\bar{b}_1}\right) \wedge \cdots \wedge \left(dz^{b_{m-k}} \wedge dz^{\bar{b}_{m-k}}\right).\end{aligned} \tag{1.4}$$

Now $a_1, \dots, a_k, b_1, \dots, b_{m-k}$ are m distinct integers of $\{1, \dots, m\}$ and 2-forms commute therefore,

$$\begin{aligned}\tilde{\omega}^k \wedge \omega^{m-k} &= \left(\sum_{\substack{(a_1,\dots,a_k)\in\{1,\dots,m\} \\ \text{distinct integers} \\ (b_1,\dots,b_{m-k})\in\{1,\dots,m\}\setminus\{a_1,\dots,a_k\} \\ \text{distinct integers}}} \lambda_{a_1}(0)\cdots\lambda_{a_k}(0)\right) \\ &\underbrace{i^m\left(dz^1 \wedge dz^{\bar{1}}\right) \wedge \cdots \wedge \left(dz^m \wedge dz^{\bar{m}}\right)}_{=\frac{\omega^m}{m!}} \\ &= \left(\sum_{\substack{(a_1,\dots,a_k)\in\{1,\dots,m\} \\ \text{distinct integers}}} (m-k)! \quad \lambda_{a_1}(0)\cdots\lambda_{a_k}(0)\right)\frac{\omega^m}{m!} \\ \tilde{\omega}^k \wedge \omega^{m-k} &= \frac{(m-k)!}{m!}k!\sigma_k(\lambda_1(0),\dots,\lambda_m(0))\ \omega^m = \frac{\sigma_k\left(\lambda\left(g^{-1}\tilde{g}\right)\right)}{\binom{m}{k}}\omega^m.\end{aligned} \tag{1.5}$$

□

Consequently, (E_k) writes:

$$\sigma_k\left(\lambda\left(g^{-1}\tilde{g}\right)\right) = e^f. \tag{E'_k}$$

Let us remark that E_m corresponds to the Calabi-Yau equation $\det(\tilde{g})/\det(g) = e^f$, when E_1 is just a linear equation in Laplacian form. Since the endomorphism A is Hermitian, the spectral theorem provides an h-orthonormal basis for $T^{1,0}$ of eigenvectors $e_1, \dots, e_m$: $Ae_i = \lambda_i e_i$, $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$. At $P \in M$ in a chart z, we have $\mathrm{Mat}_{\partial_1,\dots,\partial_m} A_P = [A^i_j(z)]_{1\le i,j\le m}$, thus

$\sigma_k(\lambda(A_P)) = \sigma_k(\lambda([A^i_j(z)]_{1\le i,j\le m}))$. In addition, $A^j_i = g^{j\bar{\ell}}\tilde{g}_{i\bar{\ell}} = g^{j\bar{\ell}}(g_{i\bar{\ell}} + \partial_{i\bar{\ell}}\varphi) = \delta^j_i + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi$, so the equation writes locally:

$$\sigma_k\left(\lambda\left(\left[\delta^j_i + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]_{1\le i,j\le m}\right)\right) = e^f. \qquad (E''_k)$$

Let us notice that a solution of this equation (E''_k) is necessarily k-admissible [9, page 46]. Let us define $f_k(B) = \sigma_k(\lambda(B))$ and $F_k(B) = \ln\sigma_k(\lambda(B))$ where $B = [B^j_i]_{1\le i,j\le m}$ is a Hermitian matrix. The function f_k is a polynomial in the variables B^j_i, specifically $f_k(B) = \sum_{|I|=k} B_{II}$ (sum of the principal minors of order k of the matrix B). Equivalently (E''_k) writes:

$$F_k\left(\left[\delta^j_i + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]_{1\le i,j\le m}\right) = f. \qquad (E'''_k)$$

It is a nonlinear elliptic second order PDE of complex Monge-Ampère type. We prove the existence of a k-admissible solution by the continuity method.

2. Derivatives and Concavity of F_k

2.1. Calculation of the Derivatives at a Diagonal Matrix

The first derivatives of the symmetric polynomial σ_k are given by the following: for all $1 \le i \le m$, $(\partial\sigma_k/\partial\lambda_i)(\lambda) = \sigma_{k-1,i}(\lambda)$ where $\sigma_{k-1,i}(\lambda) := \sigma_{k-1}|_{\lambda_i=0}$. For $1 \le i \ne j \le m$, let us denote $\sigma_{k-2,ij}(\lambda) := \sigma_{k-2}|_{\lambda_i=\lambda_j=0}$ and $\sigma_{k-2,ii}(\lambda) = 0$. The second derivatives of the polynomial σ_k are given by $(\partial^2\sigma_k/\partial\lambda_i\partial\lambda_j)(\lambda) = \sigma_{k-2,ij}(\lambda)$. We calculate the derivatives of the function $f_k : \mathscr{H}_m(\mathbb{C}) \to \mathbb{R}$, where $\mathscr{H}_m(\mathbb{C})$ denotes the set of Hermitian matrices, at diagonal matrices using the formula:

$$\begin{aligned} f_k(B) &= \sum_{1\le i_1<\cdots<i_k\le m}\sum_{\sigma\in S_k}\varepsilon(\sigma)B^{i_{\sigma(1)}}_{i_1}\cdots B^{i_{\sigma(k)}}_{i_k} \\ &= \frac{1}{k!}\sum_{1\le i_1,\ldots,i_k,j_1,\ldots,j_k\le m}\varepsilon^{i_1\cdots i_k}_{j_1\cdots j_k}B^{j_1}_{i_1}\cdots B^{j_k}_{i_k}, \end{aligned} \qquad (2.1)$$

where

$$\varepsilon^{i_1\cdots i_k}_{j_1\cdots j_k} = \begin{cases} 1 & \text{if } i_1,\ldots,i_k \text{ distinct and } j_1,\ldots,j_k \text{ even permutation of } i_1,\ldots,i_k, \\ -1 & \text{if } i_1,\ldots,i_k \text{ distinct and } j_1,\ldots,j_k \text{ odd permutation of } i_1,\ldots,i_k, \\ 0 & \text{else.} \end{cases} \qquad (2.2)$$

These derivatives are given by [9, page 48]

$$\frac{\partial f_k}{\partial B_i^j}(\operatorname{diag}(b_1,\ldots,b_m)) = \begin{cases} 0 & \text{if } i \neq j, \\ \sigma_{k-1,i}(b_1,\ldots,b_m) & \text{if } i = j, \end{cases}$$

$$\text{if } i \neq j \quad \frac{\partial^2 f_k}{\partial B_j^j \partial B_i^i}(\operatorname{diag}(b_1,\ldots,b_m)) = \sigma_{k-2,ij}(b_1,\ldots,b_m) \tag{2.3}$$

$$\frac{\partial^2 f_k}{\partial B_j^i \partial B_i^j}(\operatorname{diag}(b_1,\ldots,b_m)) = -\sigma_{k-2,ij}(b_1,\ldots,b_m),$$

and all the other second derivatives of f_k at $\operatorname{diag}(b_1,\ldots,b_m)$ vanish.

Consequently, the derivatives of the function $F_k = \ln f_k : \lambda^{-1}(\Gamma_k) \subset \mathscr{H}_m(\mathbb{C}) \to \mathbb{R}$ at diagonal matrices $\operatorname{diag}(\lambda_1,\ldots,\lambda_m)$ with $\lambda = (\lambda_1,\ldots,\lambda_m) \in \Gamma_k$, where $\lambda^{-1}(\Gamma_k) = \{B \in \mathscr{H}_m(\mathbb{C}) / \lambda(B) \in \Gamma_k\}$, are given by

$$\frac{\partial F_k}{\partial B_i^j}(\operatorname{diag}(\lambda_1,\ldots,\lambda_m)) = \begin{cases} 0 & \text{if } i \neq j, \\ \dfrac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} & \text{if } i = j, \end{cases} \tag{2.4}$$

$$\text{if } i \neq j \quad \frac{\partial^2 F_k}{\partial B_j^i \partial B_i^j}(\operatorname{diag}(\lambda_1,\ldots,\lambda_m)) = -\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)}$$

$$\frac{\partial^2 F_k}{\partial B_j^j \partial B_i^i}(\operatorname{diag}(\lambda_1,\ldots,\lambda_m)) = \frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} - \frac{\sigma_{k-1,i}(\lambda)\sigma_{k-1,j}(\lambda)}{(\sigma_k(\lambda))^2} \tag{2.5}$$

$$\frac{\partial^2 F_k}{\partial B_i^i \partial B_i^i}(\operatorname{diag}(\lambda_1,\ldots,\lambda_m)) = -\frac{(\sigma_{k-1,i}(\lambda))^2}{(\sigma_k(\lambda))^2}$$

and all the other second derivatives of F_k at $\operatorname{diag}(\lambda_1,\ldots,\lambda_m)$ vanish.

2.2. The Invariance of F_k and of Its First and Second Differentials

The function $F_k : \lambda^{-1}(\Gamma_k) \to \mathbb{R}$ is invariant under unitary similitudes:

$$\forall B \in \lambda^{-1}(\Gamma_k),\ \forall U \in U_m(\mathbb{C}), \quad F_k(B) = F_k\left({}^t\overline{U}BU\right). \tag{2.6}$$

Differentiating the previous invariance formula (2.6), we show that the first and second differentials of F_k are also invariant under unitary similitudes:

$$\begin{gathered}\forall B \in \lambda^{-1}(\Gamma_k), \quad \forall \zeta \in \mathcal{H}_m(\mathbb{C}), \quad \forall U \in U_m(\mathbb{C}), \\ (dF_k)_B \cdot \zeta = (dF_k)_{{}^t\overline{U}BU} \cdot \left({}^t\overline{U}\zeta U\right),\end{gathered} \tag{2.7}$$

$$\begin{gathered}\forall B \in \lambda^{-1}(\Gamma_k), \quad \forall \zeta \in \mathcal{H}_m(\mathbb{C}), \quad \forall \Theta \in \mathcal{H}_m(\mathbb{C}), \quad \forall U \in U_m(\mathbb{C}), \\ \left(d^2F_k\right)_B \cdot (\zeta, \Theta) = \left(d^2F_k\right)_{{}^t\overline{U}BU} \cdot \left({}^t\overline{U}\zeta U, {}^t\overline{U}\Theta U\right).\end{gathered} \tag{2.8}$$

These invariance formulas are allowed to come down to the diagonal case, when it is useful.

2.3. *Concavity of* F_k

We prove in [9] the concavity of the functions $u \circ \lambda$ and more generally $u \circ \lambda_B$ when $u \in \Gamma_0(\mathbb{R}^m)$ and is symmetric [9, Theorem VII.4.2], which in particular gives the concavity of the functions $F_k = \ln \sigma_k \lambda$ [9, Corollary VII.4.30] and more generally $\ln \sigma_k \lambda_B$ [9, Theorem VII.4.29]. In this section, let us show by an elementary calculation the concavity of the function F_k.

Proposition 2.1. *The function* $F_k : \lambda^{-1}(\Gamma_k) \to \mathbb{R}$, $B \mapsto F_k(B) = \ln \sigma_k(\lambda(B))$ *is concave (this holds for all* $k \in \{1, \dots, m\}$*).*

Proof. The function F_k is of class C^2, so its concavity is equivalent to the following inequality:

$$\forall B \in \lambda^{-1}(\Gamma_k),\ \forall \zeta \in \mathcal{H}_m(\mathbb{C}) \qquad \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(B)\zeta_i^j \zeta_r^s \leq 0. \tag{2.9}$$

Let $B \in \lambda^{-1}(\Gamma_k)$, $\zeta \in \mathcal{H}_m(\mathbb{C})$, and $U \in U_m(\mathbb{C})$ such that ${}^t\overline{U}BU = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$. We have $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$. Let us denote $\tilde{\zeta} = {}^t\overline{U}\zeta U \in \mathcal{H}_m(\mathbb{C})$:

$$\begin{aligned} S &:= \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(B)\zeta_i^j \zeta_r^s \\ &= \left(d^2F_k\right)_B \cdot (\zeta, \zeta) \quad \text{so by the invariance formula (2.8)} \\ &= \left(d^2F_k\right)_{{}^t\overline{U}BU} \cdot \left({}^t\overline{U}\zeta U, {}^t\overline{U}\zeta U\right) \\ &= \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(\operatorname{diag}(\lambda_1, \dots, \lambda_m))\tilde{\zeta}_i^j \tilde{\zeta}_r^s \\ &= \sum_{i \neq j=1}^{m} -\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)}\tilde{\zeta}_i^j \underbrace{\tilde{\zeta}_j^i}_{=\overline{\tilde{\zeta}_i^j}} \end{aligned}$$

$$+\sum_{i\neq j=1}^{m}\underbrace{\left(\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)}-\frac{\sigma_{k-1,i}(\lambda)\sigma_{k-1,j}(\lambda)}{(\sigma_k(\lambda))^2}\right)}_{=:c_{ij}}\widetilde{\zeta}_i^i\,\widetilde{\zeta}_j^j+\sum_{i=1}^{m}-\frac{(\sigma_{k-1,i}(\lambda))^2}{(\sigma_k(\lambda))^2}\left(\widetilde{\zeta}_i^i\right)^2$$
$$=\sum_{i,j=1}^{m}-\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)}\left|\widetilde{\zeta}_i^j\right|^2+\sum_{i,j=1}^{m}c_{ij}\widetilde{\zeta}_i^i\,\widetilde{\zeta}_j^j. \tag{2.10}$$

But $c_{ij}=(\partial^2(\ln\sigma_k)/\partial\lambda_i\partial\lambda_j)(\lambda)$, and $\widetilde{\zeta}_i^i\in\mathbb{R}$, so $\sum_{i,j=1}^m c_{ij}\widetilde{\zeta}_i^i\widetilde{\zeta}_j^j\leq 0$ by concavity of $\ln\sigma_k$ at $\lambda\in\Gamma_k$ [10, page 269]. In addition, $\sigma_{k-2,ij}(\lambda)>0$ since $\lambda\in\Gamma_k$ [11], consequently $\sum_{i,j=1}^m-(\sigma_{k-2,ij}(\lambda)/\sigma_k(\lambda))|\widetilde{\zeta}_i^j|^2\leq 0$, which shows that $S\leq 0$ and achieves the proof. □

3. The Proof of Uniqueness

Let φ_0 and φ_1 be two smooth k-admissible solutions of (E_k''') such that $\int_M\varphi_0\omega^m=\int_M\varphi_1\omega^m=0$. For all $t\in[0,1]$, let us consider the function $\varphi_t=t\varphi_1+(1-t)\varphi_0=\varphi_0+t\varphi$ with $\varphi=\varphi_1-\varphi_0$. Let $P\in M$, and let us denote $h_k^P(t)=f_k([\delta_i^j+g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi_t(P)])$. We have $h_k^P(1)-h_k^P(0)=0$ which is equivalent to $\int_0^1 h_k^{P'}(t)dt=0$. But

$$h_k^{P'}(t)=\sum_{i,j=1}^{m}\underbrace{\left(\sum_{\ell=1}^{m}\frac{\partial f_k}{\partial B_i^{\ell}}\left(\left[\delta_i^j+g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi_t(P)\right]\right)g^{\ell\bar{j}}(P)\right)}_{=:\alpha_{ij}^t(P)}\partial_{i\bar{j}}\varphi(P). \tag{3.1}$$

Therefore we obtain

$$\mathcal{L}\varphi(P):=\sum_{i,j=1}^{m}a_{ij}(P)\partial_{i\bar{j}}\varphi(P)=0\quad\text{with } a_{ij}(P)=\int_0^1\alpha_{ij}^t(P)dt. \tag{3.2}$$

We show easily that the matrix $[a_{ij}(P)]_{1\leq i,j\leq m}$ is Hermitian [9, page 53]. Besides the function φ is continuous on the compact manifold M so it assumes its minimum at a point $m_0\in M$, so that the complex Hessian matrix of φ at the point m_0, namely, $[\partial_{i\bar{j}}\varphi(m_0)]_{1\leq i,j\leq 2m}$, is positive-semidefinite.

Lemma 3.1. *For all $t\in[0,1]$, $\lambda(g^{-1}\widetilde{g}_{\varphi_t})(m_0)\in\Gamma_k$; namely, the functions $(\varphi_t)_{t\in[0,1]}$ are k-admissible at m_0.*

Proof. Let us denote $\mathcal{W}:=\{t\in[0,1]/\lambda(g^{-1}\widetilde{g}_{\varphi_t})(m_0)\in\Gamma_k\}$. The set $\mathcal{W}$ is nonempty, it contains 0, and it is an open subset of $[0,1]$. Let t be the largest number of $[0,1]$ such that $[0,t[\subset\mathcal{W}$. Let us suppose that $t<1$ and show that we get a contradiction. Let $1\leq q\leq k$, we have $\sigma_q(\lambda(g^{-1}\widetilde{g}_{\varphi_t})(m_0))-\sigma_q(\lambda(g^{-1}\widetilde{g}_{\varphi_0})(m_0))=h_q^{m_0}(t)-h_q^{m_0}(0)=\int_0^t h_q^{m_0'}(s)ds$. Let us prove that

$h_q^{m_0'}(s) \geq 0$ for all $s \in [0,t[$. Fix $s \in [0,t[$; the quantity $h_q^{m_0'}(s)$ is intrinsic so it suffices to prove the assertion in a particular chart at m_0. Now at m_0 in a g-unitary $\tilde{g}_{\varphi_s}$-adapted chart at m_0

$$\begin{aligned} h_q^{m_0'}(s) &= \sum_{i,j,\ell=1}^{m} \frac{\partial f_q}{\partial B_i^j}\left(\left[\delta_i^j + g^{j\bar{\ell}}(m_0)\partial_{i\bar{\ell}}\varphi_s(m_0)\right]\right)g^{j\bar{\ell}}(m_0)\partial_{i\bar{\ell}}\varphi(m_0) \\ &= \sum_{i=1}^{m} \frac{\partial \sigma_q}{\partial \lambda_i}\left(\lambda\left(g^{-1}\tilde{g}_{\varphi_s}\right)(m_0)\right)\partial_{i\bar{i}}\varphi(m_0). \end{aligned} \tag{3.3}$$

But $\lambda(g^{-1}\tilde{g}_{\varphi_s})(m_0) \in \Gamma_k \subset \Gamma_q$ since $s \in [0,t[\subset \mathcal{W}$, then $(\partial\sigma_q/\partial\lambda_i)(\lambda(g^{-1}\tilde{g}_{\varphi_s})(m_0)) > 0$ for all $1 \leq i \leq m$. Besides, $\partial_{i\bar{i}}\varphi(m_0) \geq 0$ since the matrix $[\partial_{i\bar{j}}\varphi(m_0)]_{1\leq i,j\leq m}$ is positive-semi-definite. Therefore, we infer that $h_q^{m_0'}(s) \geq 0$. Consequently, we obtain that $\sigma_q(\lambda(g^{-1}\tilde{g}_{\varphi_t})(m_0)) \geq \sigma_q(\lambda(g^{-1}\tilde{g}_{\varphi_0})(m_0)) > 0$ (since φ_0 is k-admissible). This holds for all $1 \leq q \leq k$; we deduce then that $\lambda(g^{-1}\tilde{g}_{\varphi_t})(m_0) \in \Gamma_k$ which proves that $t \in \mathcal{W}$. This is a contradiction; we infer then that $\mathcal{W} = [0,1]$. □

We check easily that the Hermitian matrix $[a_{ij}(m_0)]_{1\leq i,j\leq m}$ is positive definite [9, page 54] and deduce then the following lemma since the map $P \mapsto a_{ij}(P) = \int_0^1 (\sum_{\ell=1}^{m}(\partial f_k/\partial B_i^{\ell})([\delta_i^j + g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi_t(P)])g^{\ell\bar{j}}(P))dt$ is continuous on a neighbourhood of m_0.

Lemma 3.2. *There exists an open ball B_{m_0} centered at m_0 such that for all $P \in B_{m_0}$ the Hermitian matrix $[a_{ij}(P)]_{1\leq i,j\leq m}$ is positive definite.*

Consequently, the operator $\mathcal{L}$ is elliptic on the open set B_{m_0}. But the map φ is C^∞, assumes its minimum at $m_0 \in B_{m_0}$, and satisfies $\mathcal{L}\varphi = 0$; then by the Hopf maximum principle [12], we deduce that $\varphi(P) = \varphi(m_0)$ for all $P \in B_{m_0}$. Let us denote $\mathcal{S} := \{P \in M/\varphi(P) = \varphi(m_0)\}$. This set is nonempty and it is a closed set. Let us prove that $\mathcal{S}$ is an open set: let m be a point of $\mathcal{S}$, so $\varphi(m) = \varphi(m_0)$, then the map φ assumes its minimum at the point m. Therefore, by the same proof as for the point m_0, we infer that there exists an open ball B_m centered at m such that for all $P \in B_m$ $\varphi(P) = \varphi(m)$ so for all $P \in B_m$ $\varphi(P) = \varphi(m_0)$ then $B_m \subset \mathcal{S}$, which proves that $\mathcal{S}$ is an open set. But the manifold M is connected; then $\mathcal{S} = M$, namely, $\varphi(P) = \varphi(m_0)$ for all $P \in M$. Besides $\int_M \varphi\omega^m = 0$, therefore we deduce that $\varphi \equiv 0$ on M namely that $\varphi_1 \equiv \varphi_0$ on M, which achieves the proof of uniqueness.

4. The Continuity Method

Let us consider the one parameter family of $(E_{k,t})$, $t \in [0,1]$

$$\mathcal{F}_k[\varphi_t] := F_k\left(\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi_t\right]_{1\leq i,j\leq m}\right) = tf + \ln\left(\underbrace{\frac{\binom{m}{k}\int_M \omega^m}{\int_M e^{tf}\omega^m}}_{A_t}\right). \tag{$E_{k,t}$}$$

The function $\varphi_0 \equiv 0$ is a k-admissible solution of $(E_{k,0})$: $\sigma_k(\lambda([\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi_0]_{1\leq i,j\leq m})) = \binom{m}{k}$ and satisfies $\int_M \varphi_0\omega^m = 0$. For $t = 1$, $A_1 = 1$ so $(E_{k,1})$ corresponds to (E_k''').

Let us fix $l \in \mathbb{N}$, $l \geq 5$ and $0 < \alpha < 1$, and let us consider the nonempty set (containing 0):

$$\mathcal{T}_{l,\alpha} := \Big\{ t \in [0,1]/(E_{k,t}) \text{ have a } k\text{-admissible solution } \varphi \in C^{l,\alpha}(M) \\ \text{such that } \int_M \varphi \omega^m = 0 \Big\}. \tag{4.1}$$

The aim is to prove that $1 \in \mathcal{T}_{l,\alpha}$. For this we prove, using the connectedness of $[0,1]$, that $\mathcal{T}_{l,\alpha} = [0,1]$.

4.1. $\mathcal{T}_{l,\alpha}$ Is an Open Set of $[0,1]$

This arises from the local inverse mapping theorem and from solving a linear problem. Let us consider the following sets:

$$\widetilde{S}_{l,\alpha} := \Big\{ \varphi \in C^{l,\alpha}(M), \int_M \varphi \omega^m = 0 \Big\}, \\ S_{l,\alpha} := \Big\{ \varphi \in \widetilde{S}_{l,\alpha},\ k\text{-admissible for } g \Big\}, \tag{4.2}$$

where $\widetilde{S}_{l,\alpha}$ is a vector space and $S_{l,\alpha}$ is an open set of $\widetilde{S}_{l,\alpha}$. Using these notations, the set $\mathcal{T}_{l,\alpha}$ writes $\mathcal{T}_{l,\alpha} := \{t \in [0,1]/\exists \varphi \in S_{l,\alpha} \text{ solution of } (E_{k,t})\}$.

Lemma 4.1. *The operator $\mathcal{F}_k : S_{l,\alpha} \to C^{l-2,\alpha}(M)$, $\varphi \mapsto \mathcal{F}_k[\varphi] = F_k([\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi]_{1\leq i,j\leq m})$, is differentiable, and its differential at a point $\varphi \in S_{l,\alpha}$, $d\mathcal{F}_{k\varphi} \in \mathcal{L}(\widetilde{S}_{l,\alpha}, C^{l-2,\alpha}(M))$ is equal to*

$$d\mathcal{F}_{k\varphi} \cdot \psi = \sum_{i,j=1}^{m} \frac{\partial F_k}{\partial B_i^j}\left(\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]\right) g^{j\bar{\ell}}\partial_{i\bar{\ell}}\psi \quad \forall \psi \in \widetilde{S}_{l,\alpha}. \tag{4.3}$$

Proof. See [9, page 60]. □

Proposition 4.2. *The nonlinear operator $\mathcal{F}_k$ is elliptic on $S_{l,\alpha}$.*

Proof. Let us fix a function $\varphi \in S_{l,\alpha}$ and check that the nonlinear operator $\mathcal{F}_k$ is elliptic for this function φ. This goes back to show that the linearization at φ of the nonlinear operator $\mathcal{F}_k$ is elliptic. By Lemma 4.1, this linearization is the following linear operator:

$$d\mathcal{F}_{k\varphi} \cdot v = \sum_{i,o=1}^{m} \left(\sum_{j=1}^{m} \frac{\partial F_k}{\partial B_i^j}\left[\delta_i^j + g^{j\bar{o}}\partial_{i\bar{o}}\varphi\right]_{1\leq i,j\leq m} \times g^{j\bar{o}} \right) \partial_{i\bar{o}} v. \tag{4.4}$$

In order to prove that this linear operator is elliptic, it suffices to check the ellipticity in a particular chart, for example, at the center of a g-normal $\widetilde{g}_\varphi$-adapted chart. At the center of such a chart,

$$d\mathcal{F}_{k\varphi}\cdot v=\sum_{i,o=1}^{m}\left(\frac{\partial F_k}{\partial B_i^o}\left(\operatorname{diag}\lambda\left(g^{-1}\widetilde{g}\right)\right)\right)\partial_{i\bar{o}}v=\sum_{i=1}^{m}\frac{\sigma_{k-1,i}\lambda\left(g^{-1}\widetilde{g}\right)}{\sigma_k\lambda\left(g^{-1}\widetilde{g}\right)}\partial_{i\bar{i}}v. \tag{4.5}$$

But for all $i\in\{1,\dots,m\}$ we have $\sigma_{k-1,i}\lambda(g^{-1}\widetilde{g})/\sigma_k\lambda(g^{-1}\widetilde{g})>0$ on M since $\lambda(g^{-1}\widetilde{g})\in\Gamma_k$ [11], which proves that the linearization is elliptic and achieves the proof. □

Let us denote $\mathfrak{F}_k$ the operator

$$\mathfrak{F}_k[\varphi]:=f_k\left(\left[\delta_i^j+g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]_{1\le i,j\le m}\right). \tag{4.6}$$

As $\mathcal{F}_k$, the operator $\mathfrak{F}_k: S_{l,\alpha}\to C^{l-2,\alpha}(M)$ is differentiable and elliptic on $S_{l,\alpha}$ of differential

$$d\mathfrak{F}_{k\varphi}\cdot\psi=\sum_{i,j=1}^{m}\frac{\partial f_k}{\partial B_i^j}\left(\left[\delta_i^j+g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]\right)g^{j\bar{\ell}}\partial_{i\bar{\ell}}\psi\quad\forall\psi\in\widetilde{S}_{l,\alpha}. \tag{4.7}$$

Let us denote a_φ the matrix $[\delta_i^j+g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi]_{1\le i,j\le m}$ and calculate this linearization in a different way, by using the expression (2.1) of f_k:

$$\mathfrak{F}_k[\varphi]=f_k(a_\varphi)=\frac{1}{k!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}(a_\varphi)_{i_1}^{j_1}\cdots(a_\varphi)_{i_k}^{j_k}. \tag{4.8}$$

Thus

$$\begin{aligned}
d\mathfrak{F}_{k\varphi}\cdot v&=\frac{d}{dt}\left(\mathfrak{F}_k[\varphi+tv]\right)_{|t=0}\\
&=\frac{d}{dt}\left(\frac{1}{k!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}(a_{\varphi+tv})_{i_1}^{j_1}\cdots(a_{\varphi+tv})_{i_k}^{j_k}\right)_{|t=0}\\
&=\frac{1}{k!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}\left(g^{j_1\bar{s}}\partial_{i_1\bar{s}}v\right)(a_\varphi)_{i_2}^{j_2}\cdots(a_\varphi)_{i_k}^{j_k}\\
&\quad+\frac{1}{k!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}(a_\varphi)_{i_1}^{j_1}\left(g^{j_2\bar{s}}\partial_{i_2\bar{s}}v\right)\cdots(a_\varphi)_{i_k}^{j_k}\\
&\quad+\cdots+\frac{1}{k!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}(a_\varphi)_{i_1}^{j_1}\cdots(a_\varphi)_{i_{k-1}}^{j_{k-1}}\left(g^{j_k\bar{s}}\partial_{i_k\bar{s}}v\right)\\
&=\frac{1}{(k-1)!}\sum_{1\le i_1,\dots,i_k,j_1,\dots,j_k\le m}\varepsilon_{j_1\cdots j_k}^{i_1\cdots i_k}(a_\varphi)_{i_1}^{j_1}\cdots(a_\varphi)_{i_{k-1}}^{j_{k-1}}\left(g^{j_k\bar{s}}\partial_{i_k\bar{s}}v\right)
\end{aligned}$$

by symmetry

$$= \sum_{i,j=1}^{m} \underbrace{\left(\frac{1}{(k-1)!} \sum_{1 \le i_1,\dots,i_{k-1},j_1,\dots,j_{k-1} \le m} \varepsilon^{i_1 \cdots i_{k-1} i}_{j_1 \cdots j_{k-1} j} \left(a_\varphi\right)^{j_1}_{i_1} \cdots \left(a_\varphi\right)^{j_{k-1}}_{i_{k-1}} \right)}_{=:\mathcal{C}^i_j(a_\varphi)} \nabla^j_i v. \tag{4.9}$$

We infer then the following proposition.

Proposition 4.3. *The linearization $d\mathfrak{F}_k$ of the operator $\mathfrak{F}_k$ is of divergence type:*

$$d\mathfrak{F}_{k\varphi} = \nabla_i \left(\mathcal{C}^i_j(a_\varphi) \nabla^j \right). \tag{4.10}$$

Proof. By (4.9) we have

$$\begin{aligned} d\mathfrak{F}_{k\varphi} \cdot v &= \sum_{i,j=1}^{m} \mathcal{C}^i_j(a_\varphi) \nabla^j_i v \\ &= \sum_{i=1}^{m} \nabla_i \left(\sum_{j=1}^{m} \mathcal{C}^i_j(a_\varphi) \nabla^j v \right) - \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \nabla_i \left(\mathcal{C}^i_j(a_\varphi) \right) \right) \nabla^j v. \end{aligned} \tag{4.11}$$

Moreover

$$\sum_{i=1}^{m} \nabla_i \left(\mathcal{C}^i_j(a_\varphi) \right) = \frac{1}{(k-2)!} \sum_{i=1}^{m} \sum_{1 \le i_1,\dots,i_{k-1},j_1,\dots,j_{k-1} \le m} \varepsilon^{i_1 \cdots i_{k-1} i}_{j_1 \cdots j_{k-1} j} \left(a_\varphi\right)^{j_1}_{i_1} \cdots \left(a_\varphi\right)^{j_{k-2}}_{i_{k-2}} \nabla_i \left(\left(a_\varphi\right)^{j_{k-1}}_{i_{k-1}} \right). \tag{4.12}$$

But $\nabla_i((a_\varphi)^{j_{k-1}}_{i_{k-1}}) = \nabla_i(\delta^{j_{k-1}}_{i_{k-1}} + \nabla^{j_{k-1}}_{i_{k-1}} \varphi) = \nabla^{j_{k-1}}_{i i_{k-1}} \varphi$, then

$$\sum_{i=1}^{m} \nabla_i \left(\mathcal{C}^i_j(a_\varphi) \right) = \frac{1}{(k-2)!} \sum_{i=1}^{m} \sum_{1 \le i_1,\dots,i_{k-1},j_1,\dots,j_{k-1} \le m} \varepsilon^{i_1 \cdots i_{k-1} i}_{j_1 \cdots j_{k-1} j} \left(a_\varphi\right)^{j_1}_{i_1} \cdots \left(a_\varphi\right)^{j_{k-2}}_{i_{k-2}} \nabla^{j_{k-1}}_{i i_{k-1}} \varphi. \tag{4.13}$$

Besides, the quantity $\nabla^{j_{k-1}}_{i i_{k-1}} \varphi$ is symmetric in i, i_{k-1} (indeed, $\nabla^{j_{k-1}}_{i i_{k-1}} \varphi - \nabla^{j_{k-1}}_{i_{k-1} i} \varphi = R^{j_{k-1}}_{s i i_{k-1}} \nabla^s \varphi$ and $R^{j_{k-1}}_{s i i_{k-1}} = 0$ since g is Kähler), and $\varepsilon^{i_1 \cdots i_{k-1} i}_{j_1 \cdots j_{k-1} j}$ is antisymmetric in i, i_{k-1}; it follows then that $\sum_{i=1}^{m} \nabla_i(\mathcal{C}^i_j(a_\varphi)) = 0$, consequently $d\mathfrak{F}_{k\varphi} \cdot v = \sum_{i=1}^{m} \nabla_i(\sum_{j=1}^{m} \mathcal{C}^i_j(a_\varphi) \nabla^j v)$. □

From Proposition 4.3, we infer easily [9, page 62] the following corollary.

Corollary 4.4. *The map $F : S_{l,\alpha} \to \widetilde{S}_{l-2,\alpha}, \varphi \mapsto F(\varphi) = \mathfrak{F}_k[\varphi] - \binom{m}{k}$ is well defined and differentiable and its differential equals $dF_\varphi = d\mathfrak{F}_{k\varphi} = \nabla_i(\mathcal{C}^i_j(a_\varphi)\nabla^j) \in \mathcal{L}(\widetilde{S}_{l,\alpha}, \widetilde{S}_{l-2,\alpha})$.*

Now, let $t_0 \in \mathcal{T}_{l,\alpha}$ and let $\varphi_0 \in S_{l,\alpha}$ be a solution of the corresponding equation (E_{k,t_0}): $F(\varphi_0) = e^{t_0 f} A_{t_0} - \binom{m}{k}$.

Lemma 4.5. *$dF_{\varphi_0} : \widetilde{S}_{l,\alpha} \to \widetilde{S}_{l-2,\alpha}$ is an isomorphism.*

Proof. Let $\varphi \in C^{l-2,\alpha}(M)$ with $\int_M \varphi v_g = 0$. Let us consider the equation

$$\nabla_i\left(\mathcal{C}_j^i(a_{\varphi_0})\nabla^j u\right) = \varphi. \tag{4.14}$$

We have $\mathcal{C}_j^i(a_{\varphi_0}) \in C^{l-2,\alpha}(M)$ and the matrix $[\mathcal{C}_j^i(a_{\varphi_0})]_{1\le i,j\le m} = [(\partial f_k/\partial B_i^j)([\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi_0])]_{1\le i,j\le m}$ is positive definite (since $\mathfrak{F}_k$ is elliptic at φ_0); then by Theorem 4.7 of [13, p. 104] on the operators of divergence type, we deduce that there exists a unique function $u \in C^{l,\alpha}(M)$ satisfying $\int_M u v_g = 0$ which is solution of (4.14) and then solution of $dF_{\varphi_0} u = \varphi$. Thus, the linear continuous map $dF_{\varphi_0} : \widetilde{S}_{l,\alpha} \to \widetilde{S}_{l-2,\alpha}$ is bijective, and its inverse is continuous by the open map theorem, which achieves the proof. □

We deduce then by the local inverse mapping theorem that there exists an open set U of $S_{l,\alpha}$ containing φ_0 and an open set V of $\widetilde{S}_{l-2,\alpha}$ containing $F(\varphi_0)$ such that $F : U \to V$ is a diffeomorphism. Now, let us consider a real number $t \in [0,1]$ very close to t_0 and let us check that it belongs also to $\mathfrak{T}_{l,\alpha}$: if $|t - t_0| \le \varepsilon$ is sufficiently small then $\|(e^{tf}A_t - \binom{m}{k}) - (e^{t_0 f}A_{t_0} - \binom{m}{k})\|_{C^{l-2,\alpha}(M)}$ is small enough so that $e^{tf}A_t - \binom{m}{k} \in V$, thus there exists $\varphi \in U \subset S_{l,\alpha}$ such that $F(\varphi) = e^{tf}A_t - \binom{m}{k}$ and consequently there exists $\varphi \in C^{l,\alpha}(M)$ of vanishing integral for g which is solution of $(E_{k,t})$. Hence $t \in \mathfrak{T}_{l,\alpha}$. We conclude therefore that $\mathfrak{T}_{l,\alpha}$ is an open set of $[0,1]$.

4.2. $\mathfrak{T}_{l,\alpha}$ *Is a Closed Set of* $[0,1]$*: The Scheme of the Proof*

This section is based on a priori estimates. Finding these estimates is the most difficult step of the proof. Let $(t_s)_{s\in\mathbb{N}}$ be a sequence of elements of $\mathfrak{T}_{l,\alpha}$ that converges to $\tau \in [0,1]$, and let $(\varphi_{t_s})_{s\in\mathbb{N}}$ be the corresponding sequence of functions: φ_{t_s} is $C^{l,\alpha}$, k-admissible, has a vanishing integral, and is a solution of

$$F_k\left(\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi_{t_s}\right]_{1\le i,j\le m}\right) = t_s f + \ln(A_{t_s}). \tag{E_{k,t_s}}$$

Let us prove that $\tau \in \mathfrak{T}_{l,\alpha}$. Here is the scheme of the proof.

(1) Reduction to a $C^{2,\beta}(M)$ estimate: if $(\varphi_{t_s})_{s\in\mathbb{N}}$ is bounded in a $C^{2,\beta}(M)$ with $0 < \beta < 1$, the inclusion $C^{2,\beta}(M) \subset C^2(M,\mathbb{R})$ being compact, we deduce that after extraction $(\varphi_{t_s})_{s\in\mathbb{N}}$ converges in $C^2(M,\mathbb{R})$ to $\varphi_\tau \in C^2(M,\mathbb{R})$. We show by tending to the limit that φ_τ is a solution of $(E_{k,\tau})$ (it is then necessarily k-admissible) and of vanishing integral for g. We check finally by a nonlinear regularity theorem [14, page 467] that $\varphi_\tau \in C^\infty(M,\mathbb{R})$, which allows us to deduce that $\tau \in \mathfrak{T}_{l,\alpha}$ (see [9, pages 64–67] for details).

(2) We show that $(\varphi_{t_s})_{s\in\mathbb{N}}$ is bounded in $C^0(M,\mathbb{R})$: first of all we prove a positivity Lemma 5.4 for $(E_{k,t})$, inspired by the ones of [15, page 843] (for $k = m$), but in a very different way, required since the k-positivity of $\widetilde{\omega}_{t_s}$ is weaker with $k < m$ (in this case, some eigenvalues can be nonpositive, which complicates the proof), using a polarization method of [7, page 1740] (cf. 5.2) and a Gårding inequality 5.3; we

infer then from this lemma a fundamental inequality 5.5 as Proposition 7.18 of [13, page 262]. We conclude the proof using the Moser's iteration technique exactly as for the equation of Calabi-Yau. We deal with this C^0 estimate in Section 5.

(3) We establish the key point of the proof, namely, a C^2 a priori estimate (Section 6).

(4) With the uniform ellipticity at hand (consequence of the previous step), we obtain the needed $C^{2,\beta}(M)$ estimate by the Evans-Trudinger theory (Section 7).

5. The C^0 A Priori Estimate

5.1. The Positivity Lemma

Our first three lemmas are based on the ideas of [7, Section 2].

Lemma 5.1. *Let π be a real $(1-1)$-form, it then writes $\pi = ip_{a\bar{b}}dz^a \wedge dz^{\bar{b}}$, with $p_{a\bar{b}} = p(\partial_a, \partial_{\bar{b}})$ where p is the symmetric tensor $p(U,V) = \pi(U, JV)$; hence*

$$\forall \ell \leq m \quad \pi^\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!}\sigma_\ell\left(\lambda\left[g^{-1}p\right]\right)\omega^m. \tag{5.1}$$

Proof. The same proof as Lemma 1.4. □

We consider for $1 \leq \ell \leq m$ the map $f_\ell = \sigma_\ell \circ \lambda : \mathcal{H}_m \to \mathbb{R}$ where $\mathcal{H}_m$ denotes the $\mathbb{R}$-vector space of Hermitian square matrices of size m. f_ℓ is a real polynomial of degree ℓ and in m^2 real variables. Moreover, it is I hyperbolic (cf. [16] for the proof) and it satisfies $f_\ell(I) = \sigma_\ell(1,\ldots,1) = \binom{m}{\ell} > 0$. Let $\tilde{f}_\ell$ be the totally polarized form of f_ℓ. This polarized form $\tilde{f}_\ell : \underbrace{\mathcal{H}_m \times \cdots \times \mathcal{H}_m}_{\ell \text{ times}} \to \mathbb{R}$ is characterized by the following properties:

(i) $\tilde{f}_\ell$ is ℓ-linear.

(ii) $\tilde{f}_\ell$ is symmetric.

(iii) For all $B \in \mathcal{H}_m$, $\tilde{f}_\ell(B,\ldots,B) = f_\ell(B)$.

Using these notations, we infer from Lemma 5.1 that at the center of a g-unitary chart (this guarantees that the matrix $g^{-1}p$ is Hermitian), we have

$$\pi^\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!} f_\ell\left(g^{-1}p\right)\omega^m. \tag{5.2}$$

By transition to the polarized form in this equality we obtain the following lemma.

Lemma 5.2. *Let $1 \leq \ell \leq m$ and $\pi_1,\ldots,\pi_\ell$ be real $(1-1)$-forms. These forms write $\pi_\alpha = i(p_\alpha)_{a\bar{b}}dz^a \wedge dz^{\bar{b}}$, with $(p_\alpha)_{a\bar{b}} = p_\alpha(\partial_a, \partial_{\bar{b}})$ where p_α is the symmetric tensor $p_\alpha(U,V) = \pi_\alpha(U, JV)$. Then, at the center of a g-unitary chart we have*

$$\pi_1 \wedge \cdots \wedge \pi_\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!}\tilde{f}_\ell\left(g^{-1}p_1,\ldots,g^{-1}p_\ell\right)\omega^m. \tag{5.3}$$

Proof. See [9, page 71]. □

Theorem 5 of Gårding [16] applies to f_ℓ with $2 \le \ell \le m$.

Lemma 5.3 (the Gårding inequality for f_ℓ). *Let $2 \le \ell \le m$, for all $y^1, \dots, y^\ell \in \Gamma(f_\ell, I)$,*

$$\tilde{f}_\ell\left(y^1, \dots, y^\ell\right) \ge f_\ell\left(y^1\right)^{1/\ell} \cdots f_\ell\left(y^\ell\right)^{1/\ell}. \tag{5.4}$$

Let us recall that $\Gamma(f_\ell, I)$ is the connected component of $\{y \in \mathcal{H}_m / f_\ell(y) > 0\}$ containing I. The same proof as [17, pages 129, 130] implies that

$$\Gamma(f_\ell, I) = \left\{y \in \mathcal{H}_m / \forall 1 \le i \le \ell \ f_i(y) > 0\right\} = \left\{y \in \mathcal{H}_m / \lambda(y) \in \Gamma_\ell\right\} = \lambda^{-1}(\Gamma_\ell). \tag{5.5}$$

Note that the Gårding inequality (Lemma 5.3) holds for $\tilde{\Gamma}(f_\ell, I) = \{y \in \mathcal{H}_m / \forall 1 \le i \le \ell \ f_i(y) \ge 0\}$.

Let us now apply the previous lemmas in order to prove the following positivity lemma inspired by the ones of [15, page 843] (for $k = m$); let us emphasize that the proof is very different since the k-positivity is weaker.

Lemma 5.4 (positivity lemma). *Let α be a real 1-form on M and $j \in \{1, \dots, k-1\}$, then the function $f : M \to \mathbb{R}$ defined by $f\omega^m = {}^tJ\alpha \wedge \alpha \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j$ is nonnegative.*

Proof. Let $1 \le j \le k-1$, then $2 \le \ell = j+1 \le k$. Let α be a real 1-form, it then writes $\alpha = \alpha_a \, dz^a + \overline{\alpha_a} \, dz^{\bar{a}}$. Let $\pi_1 = {}^tJ\alpha \wedge \alpha$, hence $\pi_1(\partial_a, \partial_{\bar{b}}) = \alpha(J\partial_a)\,\alpha(\partial_{\bar{b}}) - \alpha(J\partial_{\bar{b}})\,\alpha(\partial_a) = i\alpha_a\overline{\alpha_b} - (-i)\overline{\alpha_b}\alpha_a = 2i\alpha_a\overline{\alpha_b}$. Similarly, we prove that $\pi_1(\partial_a, \partial_b) = \pi_1(\partial_{\bar{a}}, \partial_{\bar{b}}) = 0$, consequently $\pi_1 = i\underbrace{2\alpha_a\overline{\alpha_b}}_{=:p_{a\bar{b}}} dz^a \wedge dz^{\bar{b}}$. Besides, set $\pi_2 = \cdots = \pi_{j+1} = \tilde{\omega} = i\tilde{g}_{a\bar{b}} \, dz^a \wedge dz^{\bar{b}}$. Now, let $x \in M$ and ϕ be a g-unitary chart centered at x. Using Lemma 5.2, we infer that at x in the chart ϕ:

$$\begin{aligned} {}^tJ\alpha \wedge \alpha \wedge \tilde{\omega}^j \wedge \omega^{m-(j+1)} &= \pi_1 \wedge \cdots \wedge \pi_{j+1} \wedge \omega^{m-(j+1)} \\ &= \frac{(m-j-1)!(j+1)!}{m!} \tilde{f}_{j+1}\left(g^{-1}p, g^{-1}\tilde{g}, \dots, g^{-1}\tilde{g}\right)\omega^m. \end{aligned} \tag{5.6}$$

But at x, $g^{-1}\tilde{g} = \tilde{g} \in \Gamma(f_{j+1}, I)$ and $g^{-1}p = p \in \tilde{\Gamma}(f_{j+1}, I)$. Indeed, $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$ since φ is k-admissible and $\Gamma_k \subset \Gamma_{j+1}$. Moreover, the Hermitian matrix $[2\alpha_a\overline{\alpha_b}]_{1 \le a, b \le m}$ is positive-semidefinite since for all $\xi \in \mathbb{C}^m$, we have $\sum_{a,b=1}^m 2\alpha_a\overline{\alpha_b}\,\xi_a\,\overline{\xi_b} = 2\left|\sum_{a=1}^m \alpha_a\xi_a\right|^2 \ge 0$; we then deduce that for all $1 \le i \le j+1$, we have at x, $f_i(g^{-1}p) = \sigma_i(\lambda(g^{-1}p)) \ge 0$. Finally, we infer by the Gårding inequality that at x in the chart ϕ we have

$$\tilde{f}_{j+1}\left(g^{-1}p, g^{-1}\tilde{g}, \dots, g^{-1}\tilde{g}\right) \ge f_{j+1}\left(g^{-1}p\right)^{1/(j+1)} f_{j+1}\left(g^{-1}\tilde{g}\right)^{j/(j+1)} \ge 0 \tag{5.7}$$

which proves the positivity lemma. □

5.2. The Fundamental Inequality

The C^0 a priori estimate is based on the following crucial proposition which is a generalization of the Proposition 7.18 of [13, page 262].

Proposition 5.5. *Let $h(t)$ be an increasing function of class C^1 defined on $\mathbb{R}$, and let φ be a C^2 k-admissible function defined on M, then the following inequality is satisfied:*

$$\int_M \left[\binom{m}{k} - f_k\left(g^{-1}\tilde{g}\right)\right] h(\varphi)\omega^m \geq \frac{1}{2m}\binom{m}{k}\int_M h'(\varphi)|\nabla\varphi|_g^2\omega^m. \tag{5.8}$$

Proof. We have the equality $\int_M[\binom{m}{k} - f_k(g^{-1}\tilde{g})]h(\varphi)\omega^m = \binom{m}{k}\int_M h(\varphi)(\omega^m - \tilde{\omega}^k \wedge \omega^{m-k})$. Besides, since $\Lambda^2 M$ is commutative $\omega^m - \tilde{\omega}^k \wedge \omega^{m-k} = (\omega - \tilde{\omega}) \wedge \underbrace{(\omega^{m-1} + \omega^{m-2} \wedge \tilde{\omega} + \cdots + \omega^{m-k} \wedge \tilde{\omega}^{k-1})}_{=:\Omega}$, namely, $\omega^m - \tilde{\omega}^k \wedge \omega^{m-k} = -(1/2)dd^c\varphi \wedge \Omega$, then $\int_M[\binom{m}{k} - f_k(g^{-1}\tilde{g})]h(\varphi)\omega^m = -(1/2)\binom{m}{k}\int_M dd^c\varphi \wedge (h(\varphi)\Omega)$. But $d(d^c\varphi \wedge h(\varphi)\Omega) = dd^c\varphi \wedge h(\varphi)\Omega + (-1)^1 d^c\varphi \wedge d(h(\varphi)\Omega)$, and $d(h(\varphi)\Omega) = h'(\varphi)d\varphi \wedge \Omega + (-1)^0 h(\varphi)\underbrace{d\Omega}_{=0 \text{ since } \omega \text{ and } \overline{\omega} \text{ are closed}}$ so $dd^c\varphi \wedge h(\varphi)\Omega = d^c\varphi \wedge h'(\varphi)d\varphi \wedge \Omega + d(\text{something})$. In addition by Stokes' theorem, $\int_M d(\text{something}) = 0$; therefore,

$$\begin{aligned}\int_M \left[\binom{m}{k} - f_k\left(g^{-1}\tilde{g}\right)\right] h(\varphi)\omega^m &= -\frac{1}{2}\binom{m}{k}\int_M h'(\varphi)d^c\varphi \wedge d\varphi \wedge \Omega \\ &= \frac{1}{2}\binom{m}{k}\Bigg(\underbrace{\int_M h'(\varphi)(-d^c\varphi) \wedge d\varphi \wedge \omega^{m-1}}_{T_1} \\ &\qquad + \underbrace{\sum_{j=1}^{k-1}\int_M h'(\varphi)(-d^c\varphi) \wedge d\varphi \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j}_{T_2}\Bigg).\end{aligned} \tag{5.9}$$

Let us prove that $T_2 \geq 0$ (using the positivity lemma) and that $T_1 = (1/m)\int_M h'(\varphi)|\nabla\varphi|_g^2\omega^m$. Let us apply the positivity lemma to $d\varphi$: the function $f : M \to \mathbb{R}$ defined by $f\omega^m = {}^tJd\varphi \wedge d\varphi \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j$ is nonnegative for all $1 \leq j \leq k-1$. But ${}^tJd\varphi = -d^c\varphi$ and h is an increasing function; then $T_2 \geq 0$. Let us now calculate T_1. Fix $x \in M$, and let us work in a g-unitary chart centered at x and satisfying $d\varphi/|d\varphi|_g = (dz^m + dz^{\overline{m}})/\sqrt{2}$ at x. We have then $\omega = idz^a \wedge dz^{\overline{a}}$ at x and ${}^tJd\varphi \wedge d\varphi = i\,|d\varphi|_g^2\, dz^m \wedge dz^{\overline{m}}$; therefore,

$$
\begin{aligned}
&{}^tJd\varphi \wedge d\varphi \wedge \omega^{m-1} \\
&\quad = \sum_{\substack{a_1,\dots,a_{m-1}\in\{1,\dots,m-1\}\\ 2\,\text{by}\,2\neq}} i^m \left|d\varphi\right|_g^2 \left(dz^m \wedge dz^{\overline{m}}\right) \wedge \left(dz^{a_1} \wedge dz^{\overline{a}_1}\right) \wedge \cdots \wedge \left(dz^{a_{m-1}} \wedge dz^{\overline{a}_{m-1}}\right) \\
&\quad = \left(\sum_{\substack{a_1,\dots,a_{m-1}\in\{1,\dots,m-1\}\\ 2\,\text{by}\,2\neq}} 1 \right) \left|d\varphi\right|_g^2 i^m \left(dz^1 \wedge dz^{\overline{1}}\right) \wedge \cdots \wedge \left(dz^m \wedge dz^{\overline{m}}\right) \\
&\quad = (m-1)!\, \left|d\varphi\right|_g^2 \frac{\omega^m}{m!} = \frac{1}{m} \left|\nabla\varphi\right|_g^2 \omega^m.
\end{aligned}
\tag{5.10}
$$

Thus $T_1 = (1/m)\int_M h'(\varphi)|\nabla\varphi|_g^2\omega^m$, consequently $\int_M [\binom{m}{k} - f_k(g^{-1}\tilde{g})]h(\varphi)\omega^m \geq (1/2)\binom{m}{k}T_1 = (1/2m)\binom{m}{k}\int_M h'(\varphi)|\nabla\varphi|_g^2\,\omega^m$, which achieves the proof of the proposition. □

5.3. The Moser Iteration Technique

We conclude the proof using the Moser's iteration technique exactly as for the equation of Calabi-Yau. Let us apply the proposition to φ_{t_s} in order to obtain a crucial inequality (the inequality (IN1)) from which we will infer the a priori estimate of $\|\varphi_{t_s}\|_{C^0}$. Let $p \geq 2$ be a real number. The function φ_{t_s} is C^2 admissible. Let us consider the function $h(u) := u|u|^{p-2} : \mathbb{R} \to \mathbb{R}$. This function is of class C^1 and $h'(u) = |u|^{p-2} + u(p-2)u\,|u|^{p-4} = (p-1)|u|^{p-2} \geq 0$, so h is increasing. Therefore we infer by the previous proposition that

$$
\frac{p-1}{2m}\binom{m}{k}\int_M \left|\varphi_{t_s}\right|^{p-2}\left|\nabla\varphi_{t_s}\right|^2 v_g \leq \int_M \left[\binom{m}{k} - f_k\left(g^{-1}\tilde{g}\right)\right]\varphi_{t_s}\left|\varphi_{t_s}\right|^{p-2} v_g. \tag{5.11}
$$

Besides, $|\nabla|\varphi_{t_s}|^{p/2}|^2 = 2g^{a\overline{b}}\partial_a\,|\varphi_{t_s}|^{p/2}\,\partial_{\overline{b}}\,|\varphi_{t_s}|^{p/2} = 2g^{a\overline{b}}((p/2)\varphi_{t_s}|\varphi_{t_s}|^{p/2-2})^2\partial_a\,\varphi_{t_s}\,\partial_{\overline{b}}\,\varphi_{t_s} = (p^2/4)|\varphi_{t_s}|^{p-2}|\nabla\varphi_{t_s}|^2$, so the previous inequality writes:

$$
\int_M \left|\nabla\left|\varphi_{t_s}\right|^{p/2}\right|^2 v_g \leq \frac{mp^2}{2(p-1)\binom{m}{k}}\int_M \left[\binom{m}{k} - f_k\left(g^{-1}\tilde{g}\right)\right]\varphi_{t_s}\left|\varphi_{t_s}\right|^{p-2} v_g. \tag{IN1}
$$

Let us infer from the inequality (IN1) another inequality (the inequality (IN4)) that is required for the proof. It follows from the continuous Sobolev embedding $H_1^2(M) \subset L^{2m/(m-1)}(M)$ that

$$
\left\|\left|\varphi_{t_s}\right|^p\right\|_{m/(m-1)} = \left\|\left|\varphi_{t_s}\right|^{p/2}\right\|_{2m/(m-1)}^2 \leq Cste\left(\int_M \left|\nabla\left|\varphi_{t_s}\right|^{p/2}\right|^2 + \int_M \left|\varphi_{t_s}\right|^{(p/2)\cdot 2}\right), \tag{IN2}
$$

where $Cste$ is independent of p. Besides, $f_k(g^{-1}\tilde{g})$ is uniformly bounded; indeed,

$$\left|f_k\left(g^{-1}\tilde{g}\right)\right| = e^{t_s f}\frac{\binom{m}{k}\mathrm{Vol}(M)}{\int_M e^{t_s f}v_g} \leq \binom{m}{k}e^{2t_s\|f\|_\infty} \leq \binom{m}{k}e^{2\|f\|_\infty}. \tag{IN3}$$

Using the inequalities (IN1), (IN2), (IN3), and $p^2/2(p-1) \leq p$ we obtain

$$\left\||\varphi_{t_s}|^p\right\|_{m/(m-1)} \leq Cste' \times p\left(\int_M |\varphi_{t_s}|^{p-1} + \int_M |\varphi_{t_s}|^p\right) \quad (p \geq 2), \tag{IN4}$$

where $Cste'$ is independent of p. Suppose that $Cste' \geq 1$.

Using the Green's formula and the inequalities of Sobolev-Poincaré (IN2) and of Hölder, we prove following [13] these L_q estimates.

Lemma 5.6. *There exists a constant* μ *such that for all* $1 \leq q \leq 2m/(m-1)$,

$$\|\varphi_{t_s}\|_q \leq \mu. \tag{5.12}$$

Proof. M is a compact Riemannian manifold and $\varphi_{t_s} \in C^2$, so by the Green's formula $\varphi_{t_s}(x) = (1/\mathrm{Vol}(M))\underbrace{\int_M \varphi_{t_s}dv}_{=0} + \int_M G(x,y)\Delta\varphi_{t_s}(y)dv(y)$, where $G(x,y) \geq 0$ and $\int_M G(x,y)dv(y)$ is independent of x. Here $\Delta\varphi_{t_s}$ denotes the real Laplacian. Then, we infer that $\|\varphi_{t_s}\|_1 \leq C\|\Delta\varphi_{t_s}\|_1$. But $\|\Delta\varphi_{t_s}\|_1 = \int_M \Delta\varphi_{t_s}^+ + \Delta\varphi_{t_s}^-$ and $\int_M \Delta\varphi_{t_s} = \int_M \Delta\varphi_{t_s}^+ - \Delta\varphi_{t_s}^- = 0$; then $\|\Delta\varphi_{t_s}\|_1 = 2\int_M \Delta\varphi_{t_s}^+$. Besides $\Delta\varphi_{t_s} < 2m$ since φ_{t_s} is k-admissible: indeed, at x in a g-normal $\tilde{g}$-adapted chart, namely, a chart satisfying $g_{a\bar{b}} = \delta_{ab}$, $\tilde{g}_{a\bar{b}} = \delta_{ab}\lambda_a$ and $\partial_\nu g_{a\bar{b}} = 0$ for all $1 \leq a, b \leq m$, $\nu \in \{1,\ldots,m,\bar{1},\ldots,\bar{m}\}$, we have $\lambda(g^{-1}\tilde{g}) = (\lambda_1,\ldots,\lambda_m)$ so $\lambda = (\lambda_1\ldots,\lambda_m) \in \Gamma_k$ since φ_{t_s} is k-admissible; consequently $\Delta\varphi_{t_s} = -2g^{a\bar{b}}\partial_{a\bar{b}}\varphi_{t_s} = -2\sum_a \partial_{a\bar{a}}\varphi_{t_s} = 2\sum_a(1-\lambda_a) = 2m - 2\sigma_1(\lambda)$, but $\sigma_1(\lambda) > 0$ since $\lambda \in \Gamma_k$ which proves that $\Delta\varphi_{t_s} < 2m$. Therefore $\Delta\varphi_{t_s}^+ < 2m$ and $\|\Delta\varphi_{t_s}\|_1 \leq 4m\mathrm{Vol}(M)$. We infer then that $\|\varphi_{t_s}\|_1 \leq 4mC\mathrm{Vol}(M)$. Now let us take $p = 2$ in the inequality (IN2): $\|\varphi_{t_s}\|_{2m/(m-1)}^2 \leq Cste(\int_M |\nabla|\varphi_{t_s}||^2 + \int_M |\varphi_{t_s}|^2)$. Besides, $\varphi_{t_s} \in H_1^2(M)$ and has a vanishing integral; then by the Sobolev-Poincaré inequality we infer $\|\varphi_{t_s}\|_2 \leq \mathcal{A}\|\nabla\varphi_{t_s}\|_2$. But $|\nabla|\varphi_{t_s}|| = |\nabla\varphi_{t_s}|$ almost everywhere; therefore $\|\varphi_{t_s}\|_{2m/(m-1)} \leq Cste\|\nabla|\varphi_{t_s}|\|_2$. Using the inequality (IN1) with $p = 2$ and the fact that $f_k(g^{-1}\tilde{g})$ is uniformly bounded, we obtain that $\|\nabla|\varphi_{t_s}|\|_2^2 \leq Cste\|\varphi_{t_s}\|_1 \leq Cste'$. Consequently, we infer that $\|\varphi_{t_s}\|_{2m/(m-1)} \leq Cste$.

Let $1 \leq q \leq 2m/(m-1) =: 2\delta$. By the Hölder inequality we have $\|\varphi_{t_s}\|_q^q = \int_M |\varphi_{t_s}|^q \cdot 1 \leq (\int_M |\varphi_{t_s}|^{q\cdot(2\delta/q)})^{q/2\delta}\mathrm{Vol}(M)^{1-q/2\delta}$. Therefore $\|\varphi_{t_s}\|_q \leq \mathrm{Vol}(M)^{(1/q)-(1/2\delta)}\|\varphi_{t_s}\|_{2\delta}$. But

$$\mathrm{Vol}(M)^{1/q-1/2\delta} = e^{(1/q-1/2\delta)\ln(\mathrm{Vol}(M))} \leq \begin{cases} 1 & \text{if } \mathrm{Vol}(M) \leq 1, \\ \mathrm{Vol}(M)^{1-1/2\delta} & \text{if } \mathrm{Vol}(M) \geq 1 \end{cases} \tag{5.13}$$

and $\|\varphi_{t_s}\|_{2\delta} \leq Cste$, thus $\|\varphi_{t_s}\|_q \leq \mu := Cste \times \mathrm{Max}(1, \mathrm{Vol}(M)^{1-1/2\delta})$. □

Suppose without limitation of generality that $\mu \geq 1$. Now, we deduce from the previous lemma and the inequality (IN4), by induction, these more general L_p estimates using the same method as [13].

Lemma 5.7. *There exists a constant C_0 such that for all $p \geq 2$,*

$$\|\varphi_{t_s}\|_p \leq C_0\left(\delta^{m-1}Cp\right)^{-m/p}, \tag{5.14}$$

with $\delta = m/(m-1)$ and $C = Cste'(1 + \mathrm{Max}(1, \mathrm{Vol}(M)^{1/2})) \geq 1$ where $Cste'$ is the constant of the inequality (IN4).

Proof. We prove this lemma by induction: first we check that the inequality is satisfied for $2 \leq p \leq 2\delta = 2m/(m-1)$; afterwards we show that if the inequality is true for p, then it is satisfied for δp too. Denote $C_0 = \mu\delta^{m(m-1)}C^m e^{m/e}$. For $2 \leq p \leq 2\delta$ we have $\|\varphi_{t_s}\|_p \leq \mu$, so it suffices to check that $\mu \leq C_0(\delta^{m-1}Cp)^{-m/p}$. This inequality is equivalent to $\delta^{m(m-1)}C^m e^{m/e}(\delta^{m-1}Cp)^{-m/p} \geq 1$; then $(\delta^{m(m-1)}C^m)e^{m/e} \geq (\delta^{m(m-1)}C^m)^{1/p}p^{m/p}$. But if $x \geq 1$, then $x \geq x^{1/p}$ (since $p \geq 1$), and $\delta^{m(m-1)}C^m \geq 1$ (since $C \geq 1, m \geq 1$ and $\delta \geq 1$); therefore $\delta^{m(m-1)}C^m \geq (\delta^{m(m-1)}C^m)^{1/p}$. Besides, $p^{m/p} = e^{m(\ln p/p)} \leq e^{m/e}$, which proves the inequality for $2 \leq p \leq 2\delta$. Now let us fix $p \geq 2$. Suppose that $\|\varphi_{t_s}\|_p \leq C_0(\delta^{m-1}Cp)^{-m/p}$ and prove that $\|\varphi_{t_s}\|_{\delta p} \leq C_0(\delta^{m-1}C\delta p)^{-m/\delta p}$. The inequality (IN4) proved previously writes:

$$\left\| |\varphi_{t_s}|^p \right\|_\delta \leq Cste' \times p\left(\int_M |\varphi_{t_s}|^{p-1} + \int_M |\varphi_{t_s}|^p\right) \quad (p \geq 2), \tag{IN4$'$}$$

where $Cste'$ is independent of p, namely, $\|\varphi_{t_s}\|_{\delta p}^p \leq Cste' \times p(\|\varphi_{t_s}\|_{p-1}^{p-1} + \|\varphi_{t_s}\|_p^p)$. But since $1 \leq p-1 \leq p$, we have by the Hölder inequality that $\|\varphi_{t_s}\|_{p-1} \leq \mathrm{Vol}(M)^{1/(p-1)-1/p}\|\varphi_{t_s}\|_p$; therefore $\|\varphi_{t_s}\|_{\delta p}^p \leq Cste' \times p(\mathrm{Vol}(M)^{1/p}\|\varphi_{t_s}\|_p^{p-1} + \|\varphi_{t_s}\|_p^p)$.

(i) If $\|\varphi_{t_s}\|_p \leq 1$, then $\|\varphi_{t_s}\|_{\delta p}^p \leq C \times p$; therefore $\|\varphi_{t_s}\|_{\delta p} \leq (Cp)^{1/p}$. Let us check that $(Cp)^{1/p} \leq C_0(\delta^{m-1}C\delta p)^{-m/\delta p}$. This inequality is equivalent to $p^{(1/p)(1+m/\delta)} \leq \mu\delta^{m(m-1)(1-1/p)}e^{m/e} \times C^{m-m/\delta p-1/p}$, but $1 + m/\delta = m$ so it is equivalent to $p^{m/p} \leq \mu\delta^{m(m-1)(1-1/p)}e^{m/e} \times C^{m(1-1/p)}$. Besides $p^{m/p} \leq e^{m/e}$ and $\mu\delta^{m(m-1)(1-1/p)} \geq 1$, then it suffices to have $C^{m(1-1/p)} \geq 1$, and this is satisfied since $C \geq 1$.

(ii) If $\|\varphi_{t_s}\|_p \geq 1$, we infer that $\|\varphi_{t_s}\|_{\delta p}^p \leq C \times p\|\varphi_{t_s}\|_p^p$, therefore $\|\varphi_{t_s}\|_{\delta p} \leq C^{1/p} \times p^{1/p}\|\varphi_{t_s}\|_p \leq (Cp)^{1/p}C_0(\delta^{m-1}Cp)^{-m/p}$ by the induction hypothesis. But $(1-m)/p = -m/\delta p$; then we obtain the required inequality $\|\varphi_{t_s}\|_{\delta p} \leq C_0\delta^{-m^2/\delta p}(Cp)^{-m/\delta p} = C_0(\delta^{m-1}C\delta p)^{-m/\delta p}$. □

By tending to the limit $p \to +\infty$ in the inequality of the previous lemma, we obtain the needed C^0 a priori estimate.

Corollary 5.8. *Consider*

$$\|\varphi_{t_s}\|_{C^0} \leq C_0. \tag{5.15}$$

6. The C^2 A Priori Estimate

6.1. Strategy for a C^2 Estimate

First, we will look for a uniform upper bound on the eigenvalues $\lambda([\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi_t]_{1\le i,j\le m}$. Secondly, we will infer from it the uniform ellipticity of the continuity equation $(E_{k,t})$ and a uniform gradient bound. Thirdly, with the uniform ellipticity at hand, we will derive a one-sided estimate on pure second derivatives and finally get the needed C^2 bound.

6.2. Eigenvalues Upper Bound

6.2.1. The Functional

Let $t \in \mathcal{T}_{l,\alpha}$, and let $\varphi_t : M \to \mathbb{R}$ be a $C^{l,\alpha}$ k-admissible solution of $(E_{k,t})$ satisfying $\int_M \varphi_t \omega^m = 0$. Consider the following functional:

$$\begin{aligned} B : UT^{1,0} &\longrightarrow \mathbb{R} \\ (P,\xi) &\longmapsto B(P,\xi) = \widetilde{h}_P(\xi,\xi) - \varphi_t(P), \end{aligned} \tag{6.1}$$

where $UT^{1,0}$ is the unit sphere bundle associated to $(T^{1,0},h)$ and $\widetilde{g}$ is related to g by: $\widetilde{\omega} = \omega + i\partial\bar{\partial}\varphi_t$. B is continuous on the compact set $UT^{1,0}$, so it assumes its maximum at a point $(P_0,\xi_0) \in UT^{1,0}$. In addition, for fixed $P \in M$, $\xi \in UT_P^{1,0} \mapsto \widetilde{h}_P(\xi,\xi)$ is continuous on the compact subset $UT_P^{1,0}$ (the fiber); therefore it attains its maximum at a unit vector $\xi_P \in UT_P^{1,0}$, and by the min-max principle we can choose ξ_P as the direction of the largest eigenvalue of A_P, $\lambda_{\max}(A_P)$. Specifically, we have the following.

Lemma 6.1 (min-max principle). *Consider*

$$\widetilde{h}_P(\xi_P,\xi_P) = \max_{\xi\in T_P^{1,0},\ h_P(\xi,\xi)=1} \widetilde{h}_P(\xi,\xi) = \lambda_{\max}(A_P). \tag{6.2}$$

For fixed P, we have $\max_{h_P(\xi,\xi)=1} B(P,\xi) = B(P,\xi_P) = \lambda_{\max}(A_P) - \varphi_t(P)$; therefore $\max_{(P,\xi)\in UT^{1,0}} B(P,\xi) = \max_{P\in M} B(P,\xi_P) = B(P_0,\xi_0) \le B(P_0,\xi_{P_0})$; hence,

$$\max_{(P,\xi)\in UT^{1,0}} B(P,\xi) = B(P_0,\xi_{P_0}) = \lambda_{\max}(A_{P_0}) - \varphi_t(P_0). \tag{6.3}$$

At the point P_0, consider $e_1^{P_0},\dots,e_m^{P_0}$ an h_{P_0}-orthonormal basis of $(T_{P_0}^{1,0},h_{P_0})$ made of eigenvectors of A_{P_0} that satisfies the following properties:

(1) h_{P_0}-orthonormal: $[h_{ij}(P_0)]_{1\le i,\,j\le m} = I_m$.

(2) $\widetilde{h}_{P_0}$-diagonal: $[\widetilde{h}_{ij}(P_0)]_{1\le i,\,j\le m} = \operatorname{Mat}A_{P_0} = \operatorname{diag}(\lambda_1,\dots,\lambda_m)$, $\lambda\in\Gamma_k$.

(3) $\lambda_{\max}(A_{P_0})$ is achieved in the direction $e_1^{P_0} = \xi_{P_0}$: $A_{P_0}(\xi_{P_0}) = \lambda_{\max}(A_{P_0})\xi_{P_0} = \lambda_1\xi_{P_0}$ and $\lambda_1 \ge \cdots \ge \lambda_m$.

In other words, it is a basis satisfying

(1) $[g_{i\bar{j}}(P_0)]_{1\le i,j\le m} = I_m$,

(2) $[\tilde{g}_{i\bar{j}}(P_0)]_{1\le i,j\le m} = \mathrm{Mat}A_{P_0} = \mathrm{diag}(\lambda_1,\ldots,\lambda_m)$, $\lambda \in \Gamma_k$,

(3) $\lambda_{\max}(A_{P_0}) = \lambda_1 \ge \cdots \ge \lambda_m$.

Let us consider a holomorphic normal chart (U_0, φ_0) centered at P_0 such that $\varphi_0(P_0) = 0$ and $\partial_i|_{P_0} = e_i^{P_0}$ for all $i \in \{1\cdots m\}$.

6.2.2. *Auxiliary Local Functional*

From now on, we work in the chart (U_0, φ_0) constructed at P_0. The map $P \mapsto g_{1\bar{1}}(P)$ is continuous on U_0 and is equal to 1 at P_0, so there exists an open subset $U_1 \subset U_0$ such that $g_{1\bar{1}}(P) > 0$ for all $P \in U_1$. Let B_1 be the functional

$$
\begin{aligned}
&B_1 : U_1 \longrightarrow \mathbb{R} \\
&\quad P \longmapsto B_1(P) = \frac{\tilde{g}_{1\bar{1}}(P)}{g_{1\bar{1}}(P)} - \varphi_t(P).
\end{aligned}
\tag{6.4}
$$

We claim that B_1 assumes a local maximum at P_0. Indeed, we have at each $P \in U_1$: $\tilde{g}_{1\bar{1}}(P)/g_{1\bar{1}}(P) = \tilde{g}_P(\partial_1, \partial_{\bar{1}})/g_P(\partial_1, \partial_{\bar{1}}) = \tilde{h}_P(\partial_1, \partial_1)/h_P(\partial_1, \partial_1) = \tilde{h}_P(\partial_1/|\partial_1|_{h_P}, \partial_1/|\partial_1|_{h_P}) \le \lambda_{\max}(A_P)$ (see Lemma 6.1); thus $B_1(P) \le \lambda_{\max}(A_P) - \varphi_t(P) \le \lambda_{\max}(A_{P_0}) - \varphi_t(P_0) = B_1(P_0)$.

6.2.3. *Differentiating the Equation*

For short, we drop henceforth the subscript t of φ_t. Let us differentiate $(E_{k,t})$ at P, in a chart z:

$$
\begin{aligned}
t\partial_{\bar{1}} f &= dF_{k\,[\delta_i^j + g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi(P)]_{1\le i,\,j\le m}} \cdot \left[\partial_{\bar{1}}\left(g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right)\right]_{1\le i,\,j\le m} \\
&= \sum_{i,\,j=1}^{m} \frac{\partial F_k}{\partial B_i^j}\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]\left(\partial_{\bar{1}} g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{\bar{1}i\bar{\ell}}\varphi\right).
\end{aligned}
\tag{6.5}
$$

Differentiating once again, we find

$$
\begin{aligned}
t\partial_{1\bar{1}} f = &\sum_{i,\,j,\,r,\,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]\left(\partial_1 g^{s\bar{o}}\partial_{r\bar{o}}\varphi + g^{s\bar{o}}\partial_{1r\bar{o}}\varphi\right) \\
&\times\left(\partial_{\bar{1}} g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{\bar{1}i\bar{\ell}}\varphi\right) + \sum_{i,\,j=1}^{m} \frac{\partial F_k}{\partial B_i^j}\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right] \\
&\times\left(\partial_{1\bar{1}} g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + \partial_{\bar{1}} g^{j\bar{\ell}}\partial_{1i\bar{\ell}}\varphi + \partial_1 g^{j\bar{\ell}}\partial_{\bar{1}i\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{1\bar{1}i\bar{\ell}}\varphi\right).
\end{aligned}
\tag{6.6}
$$

Using the above chart (U_1, ψ_0) at the point P_0, normality yields $g^{j\bar{\ell}} = \delta^{j\ell}$, $\partial_\alpha g_{i\bar{\ell}} = 0$ and $\partial_\alpha g^{i\bar{\ell}} = 0$. Furthermore $[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi] = [\delta_i^j + \partial_{i\bar{j}}\varphi] = [\widetilde{g}_{i\bar{j}}] = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. In this chart, we can simplify the previous expression; we get then at P_0,

$$\begin{aligned} t\partial_{1\bar{1}} f = & \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\operatorname{diag}(\lambda_1, \ldots, \lambda_m))\, \partial_{1r\bar{s}}\varphi\, \partial_{\bar{1}i\bar{j}}\varphi \\ & + \sum_{i,j=1}^{m} \frac{\partial F_k}{\partial B_i^j} (\operatorname{diag}(\lambda_1, \ldots, \lambda_m)) \left(\partial_{1\bar{1}} g^{j\bar{i}} \partial_{i\bar{i}}\varphi + \partial_{1\bar{1}i\bar{j}}\varphi \right). \end{aligned} \tag{6.7}$$

Besides, $\partial_{1\bar{1}} g^{j\bar{i}} = \partial_{\bar{1}}(-g^{j\bar{s}} g^{o\bar{i}} \partial_1 g_{o\bar{s}})$, so still by normality, we obtain at P_0 that $\partial_{1\bar{1}} g^{j\bar{i}} = -g^{j\bar{s}} g^{o\bar{i}} \partial_{1\bar{1}} g_{o\bar{s}} = -\partial_{1\bar{1}} g_{i\bar{j}} - R_{1\bar{1}i\bar{j}}$. Therefore we get

$$\begin{aligned} t\partial_{1\bar{1}} f = & \sum_{i,j=1}^{m} \frac{\partial F_k}{\partial B_i^j} (\operatorname{diag}(\lambda_1, \ldots, \lambda_m)) \left(\partial_{1\bar{1}i\bar{j}}\varphi - R_{1\bar{1}i\bar{j}}\, \partial_{i\bar{i}}\varphi \right) \\ & + \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\operatorname{diag}(\lambda_1, \ldots, \lambda_m))\, \partial_{1r\bar{s}}\varphi\, \partial_{\bar{1}i\bar{j}}\varphi. \end{aligned} \tag{6.8}$$

6.2.4. *Using Concavity*

Now, using the concavity of $\ln \sigma_k$ [10], we prove for Proposition 2.1 that the second sum of (6.8) is negative [9, page 84]. This is not a direct consequence of the concavity of the function F_k since the matrix $[\partial_{1i\bar{j}}\varphi]_{1\le i,j\le m}$ is not Hermitian.

Lemma 6.2. *Consider*

$$S := \sum_{i,j,r,s=1}^{m} \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\operatorname{diag}(\lambda_1, \ldots, \lambda_m)) \partial_{1r\bar{s}}\varphi\, \partial_{\bar{1}i\bar{j}}\varphi \le 0. \tag{6.9}$$

Hence, from (6.8) combined with Lemma 6.2 we infer

$$t\partial_{1\bar{1}} f \le \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (\partial_{1\bar{1}i\bar{i}}\varphi - R_{1\bar{1}i\bar{i}} \partial_{i\bar{i}}\varphi). \tag{6.10}$$

6.2.5. Differentiation of the Functional B_1

Let us differentiate twice the functional B_1:

$$
\begin{aligned}
B_1(P) &= \frac{\tilde{g}_{1\bar{1}}(P)}{g_{1\bar{1}}(P)} - \varphi(P), \\
\partial_{\bar{i}} B_1 &= \frac{\partial_{\bar{i}} \tilde{g}_{1\bar{1}}}{g_{1\bar{1}}} - \frac{\tilde{g}_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}}}{\left(g_{1\bar{1}}\right)^2} - \partial_{\bar{i}} \varphi, \\
\partial_{i\bar{i}} B_1 &= \frac{\partial_{i\bar{i}} \tilde{g}_{1\bar{1}}}{g_{1\bar{1}}} - \frac{\partial_i g_{1\bar{1}} \partial_{\bar{i}} \tilde{g}_{1\bar{1}} + \partial_i \tilde{g}_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}} + \tilde{g}_{1\bar{1}} \partial_{i\bar{i}} g_{1\bar{1}}}{\left(g_{1\bar{1}}\right)^2} \\
&\quad + \frac{2\tilde{g}_{1\bar{1}} \partial_i g_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}}}{\left(g_{1\bar{1}}\right)^3} - \partial_{i\bar{i}} \varphi.
\end{aligned}
\tag{6.11}
$$

Therefore at P_0, in the above chart (U_1, ψ_0) we find $\partial_{i\bar{i}} B_1 = \partial_{i\bar{i}}(g_{1\bar{1}} + \partial_{1\bar{1}}\varphi) - \lambda_1 \partial_{i\bar{i}} g_{1\bar{1}} - \partial_{i\bar{i}}\varphi = R_{1\bar{1}i\bar{i}} + \partial_{1\bar{1}i\bar{i}}\varphi - \lambda_1 R_{1\bar{1}i\bar{i}} - \partial_{i\bar{i}}\varphi$. Let us define the operator:

$$
L := \sum_{i,j=1}^{m} \frac{\partial F_k}{\partial B_i^j} \left(\left[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi \right]_{1 \le i, j \le m} \right) \nabla_i^j. \tag{6.12}
$$

Thus, we have at P_0

$$
L(B_1) = \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \left(\partial_{1\bar{1}i\bar{i}}\varphi + (1-\lambda_1) R_{1\bar{1}i\bar{i}} - \partial_{i\bar{i}}\varphi \right). \tag{6.13}
$$

Combining (6.13) with (6.10), we get rid of the fourth derivatives:

$$
\begin{aligned}
t\partial_{1\bar{1}} f - L(B_1) \le{}& \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} R_{1\bar{1}i\bar{i}} (\lambda_1 - 1 - \lambda_i + 1) \\
&+ \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (\lambda_i - 1).
\end{aligned}
\tag{6.14}
$$

Since B_1 assumes its maximum at P_0, we have at P_0 that $L(B_1) \le 0$. So we are left with the following inequality at P_0:

$$
0 \ge \sum_{i=2}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \left(-R_{1\bar{1}i\bar{i}}\right)(\lambda_1 - \lambda_i) - \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \lambda_i + \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} + t\partial_{1\bar{1}} f. \tag{6.15}
$$

Curvature Assumption

Henceforth, we will suppose that the holomorphic bisectional curvature is nonnegative at any $P \in M$. Thus in a holomorphic normal chart centered at P we have $R_{a\bar{a}b\bar{b}}(P) \le 0$ for all

$1 \leq a, b \leq m$. This holds in particular at P_0 in the previous chart ψ_0. This assumption will be used only to derive an a priori eigenvalues pinching and is not required in the other sections.

Back to the inequality (6.15), we have $\sigma_k(\lambda) > 0$ and $\sigma_{k-1,i}(\lambda) > 0$ since $\lambda \in \Gamma_k$, and under our curvature assumption $(-R_{1\bar{1}i\bar{i}}) \geq 0$ for all $i \geq 2$. Besides, $\lambda_i \leq \lambda_1$ for all i; therefore $\sum_{i=2}^{m}(\sigma_{k-1,i}(\lambda)/\sigma_k(\lambda))(-R_{1\bar{1}i\bar{i}})(\lambda_1 - \lambda_i) \geq 0$. So we can get rid of the curvature terms in (6.15) and infer from it the inequality

$$0 \geq -\sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)}\lambda_i + \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} + t\partial_{1\bar{1}} f. \tag{6.16}$$

6.2.6. *A λ_1's Upper Bound*

Here, we require elementary identities satisfied by the σ_ℓ's [11], namely:

$$\begin{gathered}
\forall 1 \leq \ell \leq m \quad \sigma_\ell(\lambda) = \sigma_{\ell,i}(\lambda) + \lambda_i \sigma_{\ell-1,i}(\lambda), \\
\forall 1 \leq \ell \leq m \quad \sum_{i=1}^{m} \sigma_{\ell-1,i}(\lambda)\lambda_i = \ell\sigma_\ell(\lambda), \\
\text{so in particular} \quad \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)}\lambda_i = k, \\
\forall 1 \leq \ell \leq m \quad \sum_{i=1}^{m} \sigma_{\ell,i}(\lambda) = (m-\ell)\sigma_\ell(\lambda), \\
\text{so in particular} \quad \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} = (m-k+1)\frac{\sigma_{k-1}(\lambda)}{\sigma_k(\lambda)}.
\end{gathered} \tag{6.17}$$

Consequently, (6.16) writes:

$$q_k := \frac{(m-k+1)}{k}\frac{\sigma_{k-1}(\lambda)}{\sigma_k(\lambda)} \leq 1 - \frac{t}{k}\partial_{1\bar{1}} f. \tag{6.18}$$

So $q_k \leq 1 + (1/k)|\partial_{1\bar{1}} f|$. But at P_0, $|\nabla^2 f|_g^2 = 2g^{a\bar{c}}g^{d\bar{b}}\ (\nabla_{a\bar{b}} f\ \nabla_{\bar{c}d} f + \nabla_{ad} f\ \nabla_{\bar{c}\bar{b}} f) = 2\sum_{a,b=1}^{m}(|\partial_{a\bar{b}} f|^2 + |\partial_{ab} f|^2)$, then $|\partial_{1\bar{1}} f| \leq |\nabla^2 f|_g$, and consequently $q_k \leq 1 + (1/k)\|f\|_{C^2(M)} =: C_1$. In other words, there exists a constant C_1 independent of $t \in [0,1]$ such that

$$q_k \leq C_1. \tag{6.19}$$

To proceed further, we recall the following

Lemma 6.3 (Newton inequalities). *For all $\ell \geq 2$, $\lambda \in \mathbb{R}^m$:*

$$\sigma_\ell(\lambda)\sigma_{\ell-2}(\lambda) \leq \frac{(\ell-1)(m-\ell+1)}{\ell(m-\ell+2)}[\sigma_{\ell-1}(\lambda)]^2. \tag{6.20}$$

Let us use Newton inequalities to relate q_k to σ_1. Since for $2 \leq \ell \leq k$ and $\lambda \in \Gamma_k$ we have $\sigma_\ell(\lambda) > 0, \sigma_{\ell-1}(\lambda) > 0$ and $\sigma_{\ell-2}(\lambda) > 0$ ($\sigma_0(\lambda) = 1$ by convention), Newton inequalities imply then that $((m-\ell+2)/(\ell-1))(\sigma_{\ell-2}(\lambda)/\sigma_{\ell-1}(\lambda)) \leq ((m-\ell+1)/\ell)(\sigma_{\ell-1}(\lambda)/\sigma_\ell(\lambda))$, or else $q_{\ell-1} \leq q_\ell$, consequently $q_k \geq q_{k-1} \geq \cdots \geq q_2 = (m-1)\sigma_1(\lambda)/2\sigma_2(\lambda)$. By induction, we get $\sigma_1(\lambda) \leq (\ell!(m-\ell)!/(m-1)!)\sigma_\ell(\lambda)(q_\ell)^{\ell-1}$ for all $2 \leq \ell \leq k$. In particular

$$\sigma_1(\lambda) \leq \frac{k!(m-k)!}{(m-1)!}\sigma_k(\lambda)(q_k)^{k-1}. \tag{6.21}$$

But $\sigma_k(\lambda) \leq e^{2\|f\|_\infty}\binom{m}{k}$; combining this with (6.19) and (6.21) we obtain at P_0 that

$$\sigma_1(\lambda) \leq m e^{2\|f\|_\infty}(C_1)^{k-1} =: C_2. \tag{6.22}$$

Hence we may state the following.

Theorem 6.4. *There exists a constant $C_2 > 0$ depending only on $m, k, \|f\|_\infty$ and $\|f\|_{C^2}$ such that for all $1 \leq i \leq m$ $\lambda_i(P_0) \leq C_2$.*

Combining this result with the C^0 a priori estimate $\|\varphi_t\|_{C_0} \leq C_0$ immediately yields the following.

Theorem 6.5. *There exists a constant $C_2' > 0$ depending only on $m, k, \|f\|_{C^2}$ and C_0 such that for all $P \in M$, for all $1 \leq i \leq m$, $\lambda_i(P) \leq C_2 + 2C_0 =: C_2'$.*

6.2.7. Uniform Pinching of the Eigenvalues

We infer automatically the following pinchings of the eigenvalues.

Proposition 6.6. *For all $1 \leq i \leq m$, $-(m-1)C_2 \leq \lambda_i(P_0) \leq C_2$.*

Proposition 6.7. *For all $P \in M$, for all $1 \leq i \leq m$,$-(m-1)C_2' \leq \lambda_i(P) \leq C_2'$.*

6.3. Uniform Ellipticity of the Continuity Equation

To prove the next proposition on uniform ellipticity, we require some inequalities satisfied by the σ_ℓ's.

Lemma 6.8 (Maclaurin inequalities). *For all $1 \leq \ell \leq s$ for all $\lambda \in \overline{\Gamma_s}$, $(\sigma_s(\lambda)/\binom{m}{s})^{1/s} \leq (\sigma_\ell(\lambda)/\binom{m}{\ell})^{1/\ell}$.*

Proposition 6.9 (uniform ellipticity). *There exist constants $E > 0$ and $F > 0$ depending only on $m, k, \|f\|_\infty$ and C_2 such that: $E \leq \sigma_{k-1,1}(\lambda) \leq \cdots \leq \sigma_{k-1,m}(\lambda) \leq F$ where $\lambda = \lambda(P_0)$.*

Proof. We have $\partial\sigma_k/\partial\lambda_1 = \sigma_{k-1,1}(\lambda) \leq \cdots \leq \sigma_{k-1,m}(\lambda) \leq \binom{m-1}{k-1}(C_2)^{k-1} =: F$ where, indeed, the constant F so defined depends only on m, k, and C_2. Let us look for a uniform lower bound on $\sigma_{k-1,1}(\lambda)$, using the identity $\sigma_k(\lambda) = \lambda_1\sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda)$. We distinguish two cases.

Case 1. ($\sigma_{k,1}(\lambda) \leq 0$). When so, we have $\sigma_k(\lambda) \leq \lambda_1 \sigma_{k-1,1}(\lambda)$; therefore $\sigma_{k-1,1}(\lambda) \geq \sigma_k(\lambda)/\lambda_1$. But $\sigma_k(\lambda) \geq e^{-2\|f\|_\infty}\binom{m}{k}$ and $0 < \lambda_1 \leq C_2$; hence $\sigma_{k-1,1}(\lambda) \geq e^{-2\|f\|_\infty}\binom{m}{k}/C_2$.

Case 2. ($\sigma_{k,1}(\lambda) > 0$). For $1 \leq j \leq k-1$, $\sigma_j(\lambda_2,\dots,\lambda_m) = \sigma_{j,1}(\lambda) > 0$ since $j+1 \leq k$ and $\lambda \in \Gamma_k$. Besides $\sigma_k(\lambda_2,\dots,\lambda_m) = \sigma_{k,1}(\lambda) > 0$ by hypothesis, therefore $(\lambda_2,\dots,\lambda_m) \in \Gamma_{k,1} = \{\beta \in \mathbb{R}^{m-1} / \forall 1 \leq j \leq k,\ \sigma_j(\beta) > 0\}$. From the latter, we infer by Maclaurin inequalities $(\sigma_k(\lambda_2,\dots,\lambda_m)/\binom{m-1}{k})^{1/k} \leq (\sigma_{k-1}(\lambda_2,\dots,\lambda_m)/\binom{m-1}{k-1})^{1/(k-1)}$ or else $(\sigma_{k,1}(\lambda)/\binom{m-1}{k})^{1/k} \leq (\sigma_{k-1,1}(\lambda)/\binom{m-1}{k-1})^{1/(k-1)}$; thus we have $\sigma_{k,1}(\lambda) \leq \binom{m-1}{k}(\sigma_{k-1,1}(\lambda)/\binom{m-1}{k-1})^{1+1/(k-1)}$, consequently

$$
\begin{aligned}
\sigma_k(\lambda) &= \lambda_1 \sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda) \\
&\leq \lambda_1 \sigma_{k-1,1}(\lambda) + \binom{m-1}{k}\left(\frac{\sigma_{k-1,1}(\lambda)}{\binom{m-1}{k-1}}\right)^{1+1/(k-1)} \\
&\leq \sigma_{k-1,1}(\lambda)\left[\lambda_1 + \frac{\binom{m-1}{k}}{\binom{m-1}{k-1}}\left(\frac{\sigma_{k-1,1}(\lambda)}{\binom{m-1}{k-1}}\right)^{1/(k-1)}\right].
\end{aligned}
\tag{6.23}
$$

Here, let us distinguish two subcases of Case 2.

(i) If $\sigma_{k-1,1}(\lambda) > \binom{m-1}{k-1}$, then we have the uniform lower bound that we look for.

(ii) Else $\sigma_{k-1,1}(\lambda) \leq \binom{m-1}{k-1}$, thus $(\sigma_{k-1,1}(\lambda)/\binom{m-1}{k-1})^{1/(k-1)} \leq 1$, therefore $\sigma_k(\lambda) \leq \sigma_{k-1,1}(\lambda)[\lambda_1 + \binom{m-1}{k}/\binom{m-1}{k-1}] = \sigma_{k-1,1}(\lambda)(\lambda_1 + m/k - 1)$; then we get $\sigma_{k-1,1}(\lambda) \geq \sigma_k(\lambda)/(\lambda_1 + m/k - 1) \geq e^{-2\|f\|_\infty}\binom{m}{k}/(C_2 + m/k - 1)$.

Consequently $\sigma_{k-1,1}(\lambda) \geq \min(e^{-2\|f\|_\infty}\binom{m}{k}/C_2, \binom{m-1}{k-1}, e^{-2\|f\|_\infty}\binom{m}{k}/(C_2 + m/k - 1))$ or finally $\sigma_{k-1,1}(\lambda) \geq \min(\binom{m-1}{k-1}, e^{-2\|f\|_\infty}\binom{m}{k}/(C_2 + m/k - 1)) =: E$, where the constant E so defined depends only on $m, k, \|f\|_\infty$ and C_2. □

Similarly we prove the following.

Proposition 6.10 (uniform ellipticity). *There exists constants $E_0 > 0$ and $F_0 > 0$ depending only on $m, k, \|f\|_\infty$ and C_2' such that for all $P \in M$, for all $1 \leq i \leq m$, $E_0 \leq \sigma_{k-1,i}(\lambda(P)) \leq F_0$.*

6.4. Gradient Uniform Estimate

The manifold M is Riemannian compact and $\varphi_t \in C^2(M)$, so by the Green's formula

$$
\varphi_t(P) = \frac{1}{\mathrm{Vol}(M)}\int_M \varphi_t(Q)\,dv_g(Q) + \int_M G(P,Q)\triangle\varphi_t(Q)\,dv_g(Q),
\tag{6.24}
$$

where $G(P,Q)$ is the Green's function of the Laplacian $\triangle$. By differentiating locally under the integral sign we obtain $\partial_{u^i}\varphi_t(P) = \int_M \triangle\varphi_t(Q)(\partial_{u^i})_P G(P,Q)\,dv_g(Q)$. We infer then that at P in a holomorphic normal chart, we have

$$
\left|(\nabla\varphi_t)_P\right| \leq \sqrt{2m}\int_M \left|\triangle\varphi_t(Q)\right| |\nabla_P G(P,Q)|\,dv_g(Q).
\tag{6.25}
$$

Now, using the uniform pinching of the eigenvalues, we prove easily the following estimate of the Laplacian.

Lemma 6.11. *There exists a constant $C_3 > 0$ depending on m and C_2' such that $\|\Delta\varphi_t\|_{\infty,M} \le C_3$.*

Combining Lemma 6.11 with (6.25), we deduce that $|(\nabla\varphi_t)_P| \le \sqrt{2m}C_3 \int_M |\nabla_P G(P, Q)| dv_g(Q)$. Besides, classically [13, page 109], there exists constants $\mathcal{C}$ and $\mathcal{C}'$ such that

$$|\nabla_P G(P,Q)| \le \frac{\mathcal{C}}{d_g(P,Q)^{2m-1}}, \qquad \int_M \frac{1}{d_g(P,Q)^{2m-1}} dv_g(Q) \le \mathcal{C}'. \tag{6.26}$$

We thus obtain the following result.

Proposition 6.12. *There exists a constant $C_5 > 0$ depending on m, C_2', and (M, g) such that for all $P \in M |(\nabla\varphi_t)_P| \le C_5$.*

Specifically, we can choose $C_5 = \sqrt{2m}\, C_3 \mathcal{C}\mathcal{C}'$.

6.5. Second Derivatives Estimate

Our equation is of type:

$$F\Big(P, \left[\partial_{u^i u^j}\varphi\right]_{1\le i,j\le 2m}\Big) = v, \quad P \in M. \tag{E}$$

6.5.1. The Functional

Consider the following functional:

$$\begin{aligned} \Phi : UT &\longrightarrow \mathbb{R} \\ (P,\xi) &\longmapsto \left(\nabla^2\varphi_t\right)_P(\xi,\xi) + \frac{1}{2}\left|(\nabla\varphi_t)_P\right|_g^2, \end{aligned} \tag{6.27}$$

where UT is the real unit sphere bundle associated to (TM, g). Φ is continuous on the compact set UT, so it assumes its maximum at a point $(P_1, \xi_1) \in UT$.

6.5.2. Reduction to Finding a One-Sided Estimate for $(\nabla^2\varphi_t)_{P_1}(\xi_1,\xi_1)$

If we find a uniform upper bound for $(\nabla^2\varphi_t)_{P_1}(\xi_1,\xi_1)$, since $|\nabla\varphi_t|_\infty \le C_5$, we readily deduce that there exists a constant $C_6 > 0$ such that

$$\left(\nabla^2\varphi_t\right)_P(\xi,\xi) \le C_6 \quad \forall (P,\xi) \in UT. \tag{6.28}$$

Fix $P \in M$. Let (U_P, φ_P) be a holomorphic g-normal $\tilde{g}$-adapted chart centered at P, namely, $[g_{i\bar{j}}(P)]_{1\le i,j\le m} = I_m$, $\partial_\ell g_{i\bar{j}}(P) = 0$ and $[\tilde{g}_{i\bar{j}}(P)]_{1\le i,j\le m} = [\mathrm{diag}(\lambda_1(P), \dots, \lambda_m(P))]$. Since $|\partial_{x^j}|_g =$

$\sqrt{2}$, we obtain $\partial_{x^jx^j}\varphi_t(P) = 2(\nabla^2\varphi_t)_P(\partial_{x^j}/\sqrt{2}, \partial_{x^j}/\sqrt{2}) \leq 2C_6$ and similarly $\partial_{y^jy^j}\varphi_t(P) = 2(\nabla^2\varphi_t)_P(\partial_{y^j}/\sqrt{2}, \partial_{y^j}/\sqrt{2}) \leq 2C_6$ for all $1 \leq j \leq m$. Besides, we have $\partial_{x^jx^j}\varphi_t(P) + \partial_{y^jy^j}\varphi_t(P) = 4\partial_{j\bar{j}}\varphi_t(P) = 4(\lambda_j(P) - 1) \geq -4[(m-1)C_2' + 1]$; therefore we obtain

$$\partial_{x^jx^j}\varphi_t(P) \geq -4\left[(m-1)C_2' + 1\right] - 2C_6 =: -C_7,$$
$$\partial_{y^jy^j}\varphi_t(P) \geq -C_7, \quad \forall 1 \leq j \leq m. \tag{6.29}$$

Let us now bound second derivatives of mixed type $\partial_{u^ru^s}\varphi_t(P)$. Let $1 \leq r \neq s \leq 2m$. Since $|\partial_{x^r} \pm \partial_{x^s}|_g = 2$, we have $(\nabla^2\varphi_t)_P((\partial_{x^r} \pm \partial_{x^s})/2, (\partial_{x^r} \pm \partial_{x^s})/2) = (1/4)\partial_{x^rx^r}\varphi_t(P) + (1/4)\partial_{x^sx^s}\varphi_t(P) \pm (1/2)\partial_{x^rx^s}\varphi_t(P) \leq C_6$, which yields $\pm\partial_{x^rx^s}\varphi_t(P) \leq 2C_6 - (1/2)\partial_{x^rx^r}\varphi_t(P) - (1/2)\partial_{x^sx^s}\varphi_t(P)$, hence as well $|\partial_{x^rx^s}\varphi_t(P)| \leq 2C_6 + C_7$. Similarly we prove that at P, in the above chart ψ_P, we have $|\partial_{y^ry^s}\varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq r \neq s \leq m$ and $|\partial_{x^ry^s}\varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq r, s \leq m$. Consequently $|\partial_{u^iu^j}\varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq i, j \leq 2m$. Therefore we deduce that

$$\left|\left(\nabla^2\varphi_t\right)(P)\right|_g^2 = \frac{1}{4}\sum_{1\leq i, j\leq 2m}\left(\partial_{u^iu^j}\varphi_t(P)\right)^2 \leq m^2(2C_6 + C_7)^2. \tag{6.30}$$

Theorem 6.13 (second derivatives uniform estimate). *There exists a constant $C_8 > 0$ depending only on m, C_2', and C_6 such that for all $P \in M$, $|(\nabla^2\varphi_t)_P|_g \leq C_8$.*

This allows to deduce the needed uniform C^2 estimate:

$$\|\varphi\|_{C^2(M,\mathbb{R})} \leq C_0 + C_5 + C_8. \tag{6.31}$$

6.5.3. Chart Choice

For fixed $P \in M$, $\xi \in UT_P \mapsto (\nabla^2\varphi_t)_P(\xi, \xi)$ is continuous on the compact subset UT_P (the fiber); therefore it assumes its maximum at a unit vector $\xi^P \in UT_P$. Besides, $(\nabla^2\varphi_t)_P$ is a symmetric bilinear form on T_PM, so by the min-max principle we have $(\nabla^2\varphi_t)_P(\xi^P, \xi^P) = \max_{\xi\in T_PM,\, g(\xi,\xi)=1}(\nabla^2\varphi_t)_P(\xi, \xi) = \beta_{\max}(P)$, where $\beta_{\max}(P)$ denotes the largest eigenvalue of $(\nabla^2\varphi_t)_P$ with respect to g_P; furthermore we can choose ξ^P as the direction of the largest eigenvalue $\beta_{\max}(P)$. For fixed P, we now have $\max_{\xi\in T_PM,\, g_P(\xi,\xi)=1}\Phi(P, \xi) = \Phi(P, \xi^P) = (\nabla^2\varphi_t)_P(\xi^P, \xi^P) + (1/2)|(\nabla\varphi_t)_P|_g^2 = \beta_{\max}(P) + (1/2)|(\nabla\varphi_t)_P|_g^2$, consequently $\max_{(P,\xi)\in UT}\Phi(P, \xi) = \max_{P\in M}\Phi(P, \xi^P) = \Phi(P_1, \xi_1) \leq \Phi(P_1, \xi^{P_1})$, hence $\max_{(P,\xi)\in UT}\Phi(P, \xi) = \Phi(P_1, \xi^{P_1}) = \beta_{\max}(P_1) + (1/2)|(\nabla\varphi_t)_{P_1}|_g^2$.

At the point P_1, consider $\varepsilon_1^{P_1}, \ldots, \varepsilon_{2m}^{P_1}$ a (real) basis of $(T_{P_1}M, g_{P_1})$ that satisfies the following properties:

(i) $[g_{ij}(P_1)]_{1\leq i, j\leq 2m} = I_{2m}$,

(ii) $[(\nabla^2\varphi_t)_{ij}(P_0)]_{1\leq i, j\leq 2m} = \operatorname{diag}(\beta_1, \ldots, \beta_{2m})$,

(iii) $\beta_1 = \beta_{\max}(P_1) \geq \beta_2 \geq \cdots \geq \beta_{2m}$.

Let (U_1', ψ_1) be a C^∞ g-normal *real chart* at P_1 obtained from a holomorphic chart $z^1, \ldots, z^m$ by setting $(u^1, \ldots, u^{2m}) = (x^1, \ldots, x^m, y^1, \ldots, y^m)$ where $z^j = x^j + iy^j$ (namely, $[g_{ij}(P_1)]_{1\leq i, j\leq 2m} =$

I_{2m} and $\partial_{u^\ell} g_{ij} = 0$ for all $1 \le i, j, \ell \le 2m$) satisfying $\varphi_1(P_1) = 0$ and $\partial_{u^i}|_{P_1} = \varepsilon_i^{P_1}$, so that $\partial_{u^1}|_{P_1}$ is the direction of the largest eigenvalue $\beta_{\max}(P_1)$.

6.5.4. *Auxiliary Local Functional*

From now on, we work in the *real chart* (U_1', φ_1) so constructed at P_1.

Let $U_2 \subset U_1'$ be an open subset such that $g_{11}(P) > 0$ for all $P \in U_2$, and let Φ_1 be the functional

$$\begin{aligned} &\Phi_1 : U_2 \longrightarrow \mathbb{R} \\ &P \longmapsto \Phi_1(P) = \frac{(\nabla^2 \varphi_t)_{11}(P)}{g_{11}(P)} + \frac{1}{2}\left|(\nabla \varphi_t)_P\right|_g^2. \end{aligned} \tag{6.32}$$

We claim that Φ_1 assumes its maximum at P_1. Indeed, $(\nabla^2\varphi_t)_{11}(P)/g_{11}(P) = (\nabla^2\varphi)_P(\partial_{u^1}, \partial_{u^1})/g_P(\partial_{u^1}, \partial_{u^1}) = (\nabla^2\varphi)_P(\partial_{u^1}/|\partial_{u^1}|_g, \partial_{u^1}/|\partial_{u^1}|_g) \le \beta_{\max}(P)$, so $\Phi_1(P) \le \beta_{\max}(P) + (1/2)|(\nabla\varphi_t)_P|_g^2 \le \beta_{\max}(P_1) + (1/2)|(\nabla\varphi_t)_{P_1}|_g^2 = \Phi_1(P_1)$ proving our claim.

Let us now differentiate twice in the *real direction* ∂_{u^1} the equation

$$F\Big(P, \left[\partial_{u^i u^j}\varphi\right]_{1\le i, j\le 2m}\Big) = v. \tag{E^*}$$

At the point P, in a chart u, we obtain

$$\partial_{u^1} v = \frac{\partial F}{\partial u^1}[\varphi] + \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\,\partial_{u^1 u^i u^j}\varphi. \tag{6.33}$$

Differentiating once again

$$\begin{aligned} \partial_{u^1 u^1} v = {}& \frac{\partial^2 F}{\partial u^1 \partial u^1}[\varphi] + \sum_{i,j=1}^{2m} \frac{\partial^2 F}{\partial r_{ij}\partial u^1}[\varphi]\,\partial_{u^1 u^i u^j}\varphi \\ &+ \sum_{i,j=1}^{2m}\left[\frac{\partial^2 F}{\partial u^1 \partial r_{ij}}[\varphi] + \sum_{e,s=1}^{2m} \frac{\partial^2 F}{\partial r_{es}\partial r_{ij}}[\varphi]\,\partial_{u^1 u^e u^s}\varphi\right]\partial_{u^1 u^i u^j}\varphi \\ &+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\,\partial_{u^1 u^1 u^i u^j}\varphi. \end{aligned} \tag{6.34}$$

But at the point P_1, for our function $F(P, r) = F_k[\delta_i^j + (1/4)g^{j\bar{\ell}}(P)(r_{i\ell} + r_{(i+m)(\ell+m)} + i r_{i(\ell+m)} - i r_{(i+m)\ell})]_{1\le i, j\le m}$, we have $(\partial^2 F/\partial r_{ij}\partial u^1)[\varphi] = 0$ since $\partial_{u^1} g^{s\bar{q}}(P_1) = 0$. Hence, we infer that

$$\begin{aligned} \partial_{u^1 u^1} v = {}& \frac{\partial^2 F}{\partial u^1 \partial u^1}[\varphi] + \sum_{i,j,e,s=1}^{2m} \frac{\partial^2 F}{\partial r_{es}\partial r_{ij}}[\varphi]\,\partial_{u^1 u^e u^s}\varphi\,\partial_{u^1 u^i u^j}\varphi \\ &+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\,\partial_{u^1 u^1 u^i u^j}\varphi. \end{aligned} \tag{6.35}$$

6.5.5. Using Concavity

The function F is concave with respect to the variable r. Indeed

$$
\begin{aligned}
F(P,r) &= F_k\left[\delta_i^j + \frac{1}{4}g^{j\bar{\ell}}(P)\left(r_{i\ell} + r_{(i+m)(\ell+m)} + ir_{i(\ell+m)} - ir_{(i+m)\ell}\right)\right]_{1\le i,j\le m} \\
&= F_k\left(g^{-1}(P)\tilde{r}\right), \quad \text{where} \\
\tilde{r} &= \left[g_{i\bar{j}}(P) + \frac{1}{4}\left(r_{ij} + r_{(i+m)(j+m)} + ir_{i(j+m)} - ir_{(i+m)j}\right)\right]_{1\le i,j\le m} \\
&= F_k\Big(\underbrace{g^{-1/2}(P)\tilde{r}g^{-1/2}(P)}_{\in\mathscr{H}_m(\mathbb{C})}\Big) \\
&= F_k(\rho_P(r)), \quad \text{where}
\end{aligned}
$$

$$
\rho_P(r) := \left[\delta_i^j + \frac{1}{4}\sum_{\ell,s=1}^{m}\left(g^{-1/2}(P)\right)_{i\ell}\left(g^{-1/2}(P)\right)_{sj}\left(r_{\ell s} + r_{(\ell+m)(s+m)} + ir_{\ell(s+m)} - ir_{(\ell+m)s}\right)\right]_{1\le i,j\le m} \tag{6.36}
$$

but for a fixed point P the function $r \in S_{2m}(\mathbb{R}) \mapsto \rho_P(r) \in \mathscr{H}_m(\mathbb{C})$ is affine (where $S_{2m}(\mathbb{R})$ denotes the set of symmetric matrices of size $2m$); we deduce then that the composition $F(P,\cdot) = F_k \circ \rho_P$ is concave on the set $\{r \in S_{2m}(\mathbb{R}) / \rho_P(r) \in \lambda^{-1}(\Gamma_k)\} = \rho_P^{-1}(\lambda^{-1}(\Gamma_k))$, which proves our claim. Hence, since the matrix $[\partial_{u^1u^iu^j}\varphi]_{1\le i,j\le m}$ is symmetric, we obtain that

$$
\sum_{i,j,e,s=1}^{2m}\frac{\partial^2 F}{\partial r_{es}\partial r_{ij}}[\varphi]\,\partial_{u^1u^eu^s}\varphi\,\partial_{u^1u^iu^j}\varphi \le 0. \tag{6.37}
$$

Consequently

$$
\partial_{u^1u^1}v - \partial_{u^1u^1}F[\varphi] \le \sum_{i,j=1}^{2m}\frac{\partial F}{\partial r_{ij}}[\varphi]\,\partial_{u^1u^1u^iu^j}\varphi. \tag{6.38}
$$

Let us now calculate the quantity $\partial_{u^1u^1}F[\varphi]$ (at P_1). Since $\partial_{u^1}g^{s\bar{q}}(P_1) = 0$, we have

$$
\frac{\partial^2 F}{\partial u^1\partial u^1}\left(P_1, D^2\varphi(P_1)\right) = \sum_{s=1}^{m}\frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \partial_{u^1u^1}g^{s\bar{s}}(P_1)\partial_{s\bar{s}}\varphi(P_1). \tag{6.39}
$$

But at P_1, $\partial_{u^1u^1}g^{s\bar{s}} = -g^{s\bar{o}}g^{q\bar{s}}\partial_{u^1u^1}g_{q\bar{o}}$ and $[g^{i\bar{j}}]_{1\le i,j\le m} = 2I_m$, then $\partial_{u^1u^1}g^{s\bar{s}} = -4\partial_{u^1u^1}g_{s\bar{s}}$ so that $\partial_{u^1u^1}g^{s\bar{s}} = -\partial_{u^1u^1}g_{ss} - \partial_{u^1u^1}g_{(s+m)(s+m)}$. Moreover $\Gamma^{u^r}_{u^ju^s} = (1/2)(\partial_{u^j}g_{os} + \partial_{u^s}g_{oj} - \partial_{u^o}g_{js})\ g^{or}$, thus $\partial_{u^i}\Gamma^{u^r}_{u^ju^s} = (1/2)(\partial_{u^iu^j}g_{rs} + \partial_{u^iu^s}g_{rj} - \partial_{u^iu^r}g_{js})$. Similarly, we have at $P_1 : \partial_{u^i}\Gamma^{u^s}_{u^ju^r} = (1/2)(\partial_{u^iu^j}g_{rs} + \partial_{u^iu^r}g_{sj} - \partial_{u^iu^s}g_{jr})$. Consequently, we deduce that $\partial_{u^iu^j}g_{rs} = \partial_{u^i}\Gamma^{u^r}_{u^ju^s} + \partial_{u^i}\Gamma^{u^s}_{u^ju^r}$. Hence, we have at P_1: $\partial_{u^1u^1} g^{s\bar{s}} = -2\partial_{u^1}\Gamma^{u^s}_{u^1u^s} - 2\partial_{u^1}\Gamma^{u^{s+m}}_{u^1u^{s+m}}$. Besides, $\partial_{s\bar{s}}\varphi = (1/4)(\partial_{u^su^s}\varphi + \partial_{u^{s+m}u^{s+m}}\varphi)$, which infers that at P_1

$$\partial_{u^1u^1} F[\varphi] = \sum_{s=1}^{m} \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \left(-2\partial_{u^1}\Gamma^{u^s}_{u^1u^s} - 2\partial_{u^1}\Gamma^{u^{s+m}}_{u^1u^{s+m}}\right) \times \frac{1}{4}\left(\partial_{u^su^s}\varphi + \partial_{u^{s+m}u^{s+m}}\varphi\right). \tag{6.40}$$

Consequently, the inequality (6.38) becomes

$$\partial_{u^1u^1} v \leq \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\partial_{u^1u^1u^iu^j}\varphi - \frac{1}{2}\sum_{s=1}^{m} \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \left(\partial_{u^1}\Gamma^{u^s}_{u^1u^s} + \partial_{u^1}\Gamma^{u^{s+m}}_{u^1u^{s+m}}\right) \times \left(\partial_{u^su^s}\varphi + \partial_{u^{s+m}u^{s+m}}\varphi\right). \tag{6.41}$$

6.5.6. *Differentiation of the Functional* Φ_1

We differentiate twice the functional Φ_1:

$$\begin{aligned}
\Phi_1(P) &= \frac{(\nabla^2\varphi)_{11}(P)}{g_{11}(P)} + \frac{1}{2}\left|(\nabla\varphi)_P\right|^2_g, \\
\partial_{u^j}\Phi_1(P) &= \frac{\partial_{u^j}(\nabla^2\varphi)_{11}}{g_{11}(P)} - \frac{(\nabla^2\varphi)_{11}\partial_{u^j}g_{11}(P)}{g_{11}(P)^2} + \frac{1}{2}\partial_{u^j}\left|(\nabla\varphi)_P\right|^2_g, \\
\partial_{u^iu^j}\Phi_1(P) &= \frac{\partial_{u^iu^j}(\nabla^2\varphi)_{11}}{g_{11}(P)} - \frac{\partial_{u^j}(\nabla^2\varphi)_{11}\partial_{u^i}g_{11}(P)}{g_{11}(P)^2} \\
&\quad - \frac{\partial_{u^i}(\nabla^2\varphi)_{11}\partial_{u^j}g_{11}(P) + (\nabla^2\varphi)_{11}(P)\partial_{u^iu^j}g_{11}(P)}{g_{11}(P)^2} \\
&\quad - \left(\nabla^2\varphi\right)_{11}(P)\partial_{u^j}g_{11}(P)\partial_{u^i}\left(\frac{1}{g_{11}(P)^2}\right) + \frac{1}{2}\partial_{u^iu^j}\left|(\nabla\varphi)_P\right|^2_g.
\end{aligned} \tag{6.42}$$

Hence, at P_1 in the chart ψ_1, we obtain

$$\partial_{u^iu^j}\Phi_1 = \partial_{u^iu^j}\left(\nabla^2\varphi\right)_{11} - \left(\nabla^2\varphi\right)_{11}(P_1)\partial_{u^iu^j}g_{11} + \frac{1}{2}\partial_{u^iu^j}\left|(\nabla\varphi)_P\right|^2_g(P_1). \tag{6.43}$$

Let us now calculate the different terms of this formula (at P_1 in the chart ψ_1):

$$\begin{aligned}
\partial_{u^iu^j}\left(\nabla^2\varphi\right)_{11} &= \partial_{u^iu^j}\left(\partial_{u^1u^1}\varphi - \Gamma^{u^s}_{u^1u^1}\partial_{u^s}\varphi\right) \\
&= \partial_{u^iu^ju^1u^1}\varphi - \partial_{u^iu^j}\Gamma^{u^s}_{u^1u^1}\partial_{u^s}\varphi - \partial_{u^j}\Gamma^{u^s}_{u^1u^1}\partial_{u^iu^s}\varphi - \partial_{u^i}\Gamma^{u^s}_{u^1u^1}\partial_{u^ju^s}\varphi.
\end{aligned} \tag{6.44}$$

Besides, we have $\Gamma^{u^1}_{u^j u^1} = (1/2)(\partial_{u^j} g_{s1} + \partial_{u^1} g_{sj} - \partial_{u^s} g_{j1}) g^{s1}$; therefore we deduce that $\partial_{u^i}\Gamma^{u^1}_{u^j u^1} = (1/2)(\partial_{u^i u^j} g_{s1} + \partial_{u^i u^1} g_{sj} - \partial_{u^i u^s} g_{j1}) g^{s1} + 0 = (1/2)\partial_{u^i u^j} g_{11}$; namely, $\partial_{u^i u^j} g_{11} = 2\partial_{u^i}\Gamma^{u^1}_{u^j u^1}$. Moreover, we have at P_1

$$\begin{aligned}
\partial_{u^i u^j}\left|(\nabla\varphi)_P\right|_g^2 &= \partial_{u^i u^j}\left(\sum_{r,s=1}^{2m} g^{rs}\partial_{u^r}\varphi\,\partial_{u^s}\varphi\right) \\
&= \sum_{r,s=1}^{2m} \partial_{u^i u^j} g^{rs}\partial_{u^r}\varphi\,\partial_{u^s}\varphi + g^{rs}\partial_{u^i u^j u^r}\varphi\,\partial_{u^s}\varphi \\
&\quad + g^{rs}\partial_{u^j u^r}\varphi\,\partial_{u^i u^s}\varphi + g^{rs}\partial_{u^i u^r}\varphi\,\partial_{u^j u^s}\varphi + g^{rs}\partial_{u^r}\varphi\,\partial_{u^i u^j u^s}\varphi \\
&= \sum_{r,s=1}^{2m} \partial_{u^i u^j} g^{rs}\partial_{u^r}\varphi\,\partial_{u^s}\varphi + 2\sum_{s=1}^{2m}\partial_{u^i u^j u^s}\varphi\,\partial_{u^s}\varphi \\
&\quad + 2\sum_{s=1}^{2m}\partial_{u^i u^s}\varphi\,\partial_{u^j u^s}\varphi.
\end{aligned} \tag{6.45}$$

But at P_1, $\partial_{u^i u^j} g^{rs} = -\partial_{u^i u^j} g_{rs}$, in addition at this point $\partial_{u^i u^j} g_{rs} = \partial_{u^i}\Gamma^{u^r}_{u^j u^s} + \partial_{u^i}\Gamma^{u^s}_{u^j u^r}$; therefore we obtain at P_1 in the chart φ_1

$$\begin{aligned}
\partial_{u^i u^j}\left|(\nabla\varphi)_P\right|_g^2 = &-2\sum_{r,s=1}^{2m}\partial_{u^i}\Gamma^{u^r}_{u^j u^s}\,\partial_{u^r}\varphi\,\partial_{u^s}\varphi + 2\sum_{s=1}^{2m}\partial_{u^i u^j u^s}\varphi\,\partial_{u^s}\varphi \\
&+ 2\sum_{s=1}^{2m}\partial_{u^i u^s}\varphi\,\partial_{u^j u^s}\varphi.
\end{aligned} \tag{6.46}$$

Henceforth, and in order to lighten the notations, we use ∂_i instead of ∂_{u^i} and Γ^s_{ij} instead of $\Gamma^{u^s}_{u^i u^j}$, so we have

$$\begin{aligned}
\partial_{ij}\Phi_1 = \;&\partial_{ij11}\varphi - \partial_{ij}\Gamma^s_{11}\partial_s\varphi - \partial_j\Gamma^s_{11}\partial_{is}\varphi - \partial_i\Gamma^s_{11}\partial_{js}\varphi - 2\partial_i\Gamma^1_{j1}\left(\nabla^2\varphi\right)_{11}(P_1) \\
&- \sum_{r,s=1}^{2m}\partial_i\Gamma^r_{js}\partial_r\varphi\,\partial_s\varphi + \sum_{s=1}^{2m}\partial_{ijs}\varphi\,\partial_s\varphi + \sum_{s=1}^{2m}\partial_{is}\varphi\,\partial_{js}\varphi.
\end{aligned} \tag{6.47}$$

Let us now consider the second order linear operator:

$$\widetilde{L} = \sum_{i,j=1}^{2m}\frac{\partial F}{\partial r_{ij}}[\varphi]\partial_{ij}. \tag{6.48}$$

Since the functional Φ_1 assumes its maximum at the point P_1, we have $\widetilde{L}(\Phi_1) \leq 0$ at P_1 in the chart φ_1. Besides, combining the inequalities (6.41) and (6.47), we obtain

$$\begin{aligned}
2\underbrace{\widetilde{L}\Phi_1}_{\leq 0} - \partial_{11}v \geq \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\Bigg[&\partial_{ij11}\varphi - \partial_{ij}\Gamma_{11}^{s}\partial_s\varphi - \partial_j\Gamma_{11}^{i}\partial_{ii}\varphi \\
&- \partial_i\Gamma_{11}^{j}\partial_{jj}\varphi - 2\partial_i\Gamma_{j1}^{1}\left(\nabla^2\varphi\right)_{11}(P_1) \\
&- \sum_{r,s=1}^{2m} \partial_i\Gamma_{js}^{r}\partial_r\varphi\,\partial_s\varphi \\
&+ \sum_{s=1}^{2m} \partial_{ijs}\varphi\,\partial_s\varphi + \delta_i^j\left(\partial_{ii}\varphi\right)^2 - \partial_{11ij}\varphi\Bigg] \\
&+ \frac{1}{2}\sum_{s=1}^{m} \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\left(\partial_1\Gamma_{1s}^{s} + \partial_1\Gamma_{1(s+m)}^{s+m}\right)\left(\partial_{ss}\varphi + \partial_{(s+m)(s+m)}\varphi\right).
\end{aligned} \tag{6.49}$$

The fourth derivatives are simplified. Moreover, we have at $P_1 : \partial_s v = (\partial F/\partial u^1)[\varphi] + \sum_{i,j=1}^{2m}(\partial F/\partial r_{ij})[\varphi]\ \partial_{sij}\varphi$ with $(\partial F/\partial u^1)(P_1, D^2\varphi(P_1)) = 0$, consequently:

$$\begin{aligned}
0 \geq\ & \partial_{11}v + \sum_{s=1}^{2m} \partial_s v\,\partial_s\varphi \\
&+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi]\Bigg[-2\partial_i\Gamma_{j1}^{1}\left(\nabla^2\varphi\right)_{11}(P_1) - \partial_i\Gamma_{11}^{j}\partial_{jj}\varphi - \partial_j\Gamma_{11}^{i}\partial_{ii}\varphi \\
&\qquad - \sum_{s=1}^{2m} \partial_{ij}\Gamma_{11}^{s}\partial_s\varphi - \sum_{r,s=1}^{2m} \partial_i\Gamma_{js}^{r}\partial_r\varphi\,\partial_s\varphi + \delta_i^j\left(\partial_{ii}\varphi\right)^2\Bigg] \\
&+ \frac{1}{2}\sum_{s=1}^{m} \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\left(\partial_1\Gamma_{1s}^{s} + \partial_1\Gamma_{1(s+m)}^{s+m}\right)\left(\partial_{ss}\varphi + \partial_{(s+m)(s+m)}\varphi\right).
\end{aligned} \tag{6.50}$$

Let us now express the quantities $\partial_i\Gamma_{j1}^{1}, \partial_i\Gamma_{11}^{j}, \partial_j\Gamma_{11}^{i}, \partial_i\Gamma_{js}^{r}$ and $\partial_{ij}\Gamma_{11}^{s}$ using the components of the Riemann curvature (at the point P_1 in the normal chart φ_1):

$$\begin{aligned}
\partial_i\Gamma_{j1}^{1} &= \frac{1}{3}\left(R_{j11i} + \underbrace{R_{ji11}}_{=0}\right) = \frac{1}{3}R_{j11i}, \\
\partial_i\Gamma_{11}^{j} &= \frac{1}{3}\left(R_{1j1i} + R_{1i1j}\right) = \frac{2}{3}R_{1j1i}, \\
\partial_j\Gamma_{11}^{i} &= \frac{2}{3}R_{1i1j}, \\
\partial_i\Gamma_{js}^{r} &= \frac{1}{3}\left(R_{jrsi} + R_{jisr}\right),
\end{aligned}$$

$$
\begin{aligned}
\partial_1 \Gamma^s_{1s} &= \frac{1}{3}\Bigg(R_{1ss1} + \underbrace{R_{11ss}}_{=0} \Bigg) = \frac{1}{3} R_{1ss1}, \\
\partial_1 \Gamma^{s+m}_{1(s+m)} &= \frac{1}{3} R_{1(s+m)(s+m)1}, \\
\partial_{ij}\Gamma^s_{11} &= \frac{1}{4}\left(\nabla_i R_{1j1s} + \nabla_i R_{1s1j} + \nabla_j R_{1s1i} + \nabla_j R_{1i1s}\right) \\
&\quad - \frac{1}{12}\left(\nabla_s R_{1i1j} + \nabla_s R_{1j1i}\right) = \frac{1}{2}\left(\nabla_i R_{1s1j} + \nabla_j R_{1s1i}\right) - \frac{1}{6}\nabla_s R_{1i1j}.
\end{aligned}
\tag{6.51}
$$

We then obtain

$$
\begin{aligned}
0 \geq{}& \partial_{11} v + \sum_{s=1}^{2m} \partial_s v \, \partial_s \varphi \\
&+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi] \Bigg[\frac{-2}{3} R_{j11i}\left(\nabla^2 \varphi\right)_{11}(P_1) - \frac{2}{3} R_{1j1i}\, \partial_{jj}\varphi - \frac{2}{3} R_{1i1j}\, \partial_{ii}\varphi \\
&\qquad - \sum_{s=1}^{2m} \left(\frac{1}{2}\nabla_i R_{1s1j} + \frac{1}{2}\nabla_j R_{1s1i} - \frac{1}{6}\nabla_s R_{1i1j} \right) \partial_s \varphi \\
&\qquad - \sum_{r,s=1}^{2m} \frac{1}{3}\left(R_{jrsi} + R_{jisr}\right) \partial_r \varphi \, \partial_s \varphi + \delta^j_i \left(\partial_{ii}\varphi\right)^2 \Bigg] \\
&+ \frac{1}{2} \sum_{s=1}^{m} \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \frac{1}{3} \left(R_{1ss1} + R_{1(s+m)(s+m)1} \right) \left(\partial_{ss}\varphi + \partial_{(s+m)(s+m)}\varphi \right).
\end{aligned}
\tag{6.52}
$$

6.5.7. The Uniform Upper Bound of $\beta_1 = (\nabla^2\varphi)_{P_1}(\xi_1, \xi_1)$

By the uniform estimate of the gradient we have $|\partial_j \varphi_t| \leq C_5$ for all $1 \leq j \leq 2m$. Moreover, at P_1 in the chart ψ_1: $[(\nabla^2\varphi)_{ij}(P_1)]_{1\leq i,j\leq 2m} = [\partial_{ij}\varphi(P_1)]_{1\leq i,j\leq 2m} = \operatorname{diag}(\beta_1, \dots, \beta_{2m})$. Consequently

$$
\begin{aligned}
0 \geq{}& \partial_{11} v + \sum_{s=1}^{2m} \partial_s v \, \partial_s \varphi \\
&+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}}[\varphi] \Bigg[\delta_{ij}(\beta_i)^2 - \frac{2}{3} R_{j11i}\beta_1 - \frac{2}{3} R_{1j1i}\beta_j - \frac{2}{3} R_{1i1j}\beta_i - \frac{1}{3}\sum_{r,s=1}^{2m} \left(R_{jrsi} + R_{jisr}\right) \partial_r \varphi \, \partial_s \varphi \\
&\qquad - \frac{1}{2} \sum_{s=1}^{2m} \left(\nabla_i R_{1s1j} + \nabla_j R_{1s1i} - \frac{1}{3}\nabla_s R_{1i1j} \right) \partial_s \varphi \Bigg] \\
&+ \frac{1}{6} \sum_{i=1}^{m} \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \left(R_{1ii1} + R_{1(i+m)(i+m)1} \right) \left(\beta_i + \beta_{i+m} \right).
\end{aligned}
\tag{6.53}
$$

But for $F[\varphi] = F_k([\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi]_{1\le i,\, j\le m})$ since $\partial_{s\bar{s}}\varphi = (1/4)(\partial_{u^s u^s} + \partial_{u^{s+m}u^{s+m}})$, we obtain at P_1 in the chart ψ_1 that

$$\frac{\partial F}{\partial r_{ij}}[\varphi] = \sum_{s=1}^{m} \frac{\partial F_k}{\partial B_s^s}(\operatorname{diag}(\lambda(P_1)))\frac{1}{4}\frac{\partial\left(r_{ss} + r_{(s+m)(s+m)}\right)}{\partial r_{ij}}. \tag{6.54}$$

Then

$$\begin{aligned}
&\forall 1 \le i \ne j \le 2m \quad \frac{\partial F}{\partial r_{ij}}[\varphi] = 0,\\
&\forall 1 \le i \le m \quad \frac{\partial F}{\partial r_{ii}}[\varphi] = \frac{\partial F}{\partial r_{(i+m)(i+m)}}[\varphi] = \frac{1}{4}\frac{\partial F_k}{\partial B_i^i}(\operatorname{diag}(\lambda(P_1)))\\
&\qquad = \frac{1}{4}\underbrace{\frac{\sigma_{k-1,\,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}}_{>0 \text{ since } \lambda(P_1)\in\Gamma_k}.
\end{aligned} \tag{6.55}$$

Hence

$$\begin{aligned}
0 \ge{}& \partial_{11}v + \sum_{s=1}^{2m}\partial_s v\,\partial_s\varphi + \sum_{i-1}^{2m}\frac{\partial F}{\partial r_{ii}}[\varphi]\\
&\times\left[(\beta_i)^2 + \frac{2}{3}R_{1i1i}(\beta_1 - 2\beta_i) + \frac{1}{3}\sum_{r,\,s=1}^{2m}R_{iris}\,\partial_r\varphi\,\partial_s\varphi - \sum_{s=1}^{2m}\left(\nabla_i R_{1s1i} - \frac{1}{6}\nabla_s R_{1i1i}\right)\partial_s\varphi\right]\\
&+\frac{1}{6}\sum_{i=1}^{m}\frac{\sigma_{k-1,\,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\left(R_{1ii1} + R_{1(i+m)(i+m)1}\right)(\beta_i + \beta_{i+m}).
\end{aligned} \tag{6.56}$$

But at P_1 in the chart ψ_1, $\|R\|_g^2 = g^{ai}g^{bj}g^{cr}g^{ds}R_{abcd}R_{ijrs} = \sum_{a,\,b,\,c,\,d=1}^{2m}(R_{abcd})^2$; then $|R_{abcd}| \le \|R\|_g$ for all $a,b,c,d \in \{1,\ldots,2m\}$, consequently

$$\begin{aligned}
\left|\frac{1}{3}\sum_{r,s=1}^{2m}R_{iris}\,\partial_r\varphi\,\partial_s\varphi\right| &\le \frac{1}{3}\sum_{r,\,s=1}^{2m}\|R\|_g(C_5)^2 = \frac{1}{3}(2m)^2\|R\|_g(C_5)^2\\
&= \frac{4}{3}m^2(C_5)^2\|R\|_g.
\end{aligned} \tag{6.57}$$

Besides, at P_1 in the chart ψ_1, we have $\|\nabla R\|_g^2 = g^{el}g^{ai}g^{bj}g^{cr}g^{ds}\nabla_e R_{abcd}\nabla_l R_{ijrs} = \sum_{e,\,a,\,b,\,c,\,d=1}^{2m}(\nabla_e R_{abcd})^2$, so $|\nabla_e R_{abcd}| \le \|\nabla R\|_g$ for all $e,a,b,c,d \in \{1,\ldots,2m\}$, therefore

$$\begin{aligned}
\left|-\sum_{s=1}^{2m}\left(\nabla_i R_{1s1i} - \frac{1}{6}\nabla_s R_{1i1i}\right)\partial_s\varphi\right| &\le \sum_{s=1}^{2m}\frac{7}{6}\|\nabla R\|_g C_5 = 2m\frac{7}{6}\|\nabla R\|_g C_5\\
&= \frac{7}{3}mC_5\|\nabla R\|_g.
\end{aligned} \tag{6.58}$$

Hence at P_1 in the chart φ_1, we obtain

$$\begin{aligned}
-t\partial_{11}f - t\sum_{s=1}^{2m}\partial_s f\partial_s\varphi \geq & \sum_{i=1}^{2m}\frac{\partial F}{\partial r_{ii}}[\varphi]\left[(\beta_i)^2 + \frac{2}{3}R_{1i1i}(\beta_1 - 2\beta_i)\right] \\
& + \frac{1}{6}\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\times\left(R_{1ii1} + R_{1(i+m)(i+m)1}\right)(\beta_i + \beta_{i+m}) \\
& + \left(\sum_{i=1}^{2m}\frac{\partial F}{\partial r_{ii}}[\varphi]\right)\times\left[-\frac{4}{3}m^2(C_5)^2\|R\|_g - \frac{7}{3}mC_5\|\nabla R\|_g\right].
\end{aligned} \tag{6.59}$$

But $|\partial_{11}f(P_1)| \leq \|f\|_{C^2(M)}$, $|\partial_s f(P_1)| \leq \|f\|_{C^2(M)}$ and $|\partial_s\varphi| \leq C_5$ for all s then

$$-t\partial_{11}f - t\sum_{s=1}^{2m}\partial_s f\,\partial_s\varphi \leq \|f\|_{C^2(M)}(1 + 2mC_5). \tag{6.60}$$

Besides

$$\sum_{i=1}^{2m}\frac{\partial F}{\partial r_{ii}}[\varphi] = \sum_{i=1}^{m}\frac{\partial F}{\partial r_{ii}}[\varphi] + \frac{\partial F}{\partial r_{(i+m)(i+m)}}[\varphi] = \frac{1}{2}\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}. \tag{6.61}$$

Consequently, we obtain

$$\begin{aligned}
\|f\|_{C^2(M)}(1 + 2m\,C_5) \geq & \frac{\partial F}{\partial r_{11}}[\varphi](\beta_1)^2 + \frac{2}{3}\sum_{i=1}^{2m}\frac{\partial F}{\partial r_{ii}}[\varphi]R_{1i1i}(\beta_1 - 2\beta_i) \\
& + \frac{1}{6}\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\times\left(R_{1ii1} + R_{1(i+m)(i+m)1}\right)(\beta_i + \beta_{i+m}) \\
& + \frac{1}{2}\left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)\times\left[-\frac{4}{3}m^2(C_5)^2\|R\|_g - \frac{7}{3}m\,C_5\|\nabla R\|_g\right].
\end{aligned} \tag{6.62}$$

Let us now estimate $|\beta_i|$ for $1 \leq i \leq m$ using β_1. We follow the same method as for the proof of Theorem 6.13. For all $(P,\xi) \in UT$, we have the inequality $(\nabla^2\varphi_t)_P(\xi,\xi) \leq \beta_1 + (1/2)(C_5)^2$; then at P in a holomorphic g-normal $\tilde{g}$-adapted chart φ_P, namely, a chart such that $[g_{i\bar{j}}(P)]_{1\leq i,j\leq m} = I_m$, $\partial_\ell g_{i\bar{j}}(P) = 0$ and $[\tilde{g}_{i\bar{j}}(P)]_{1\leq i,j\leq m} = \operatorname{diag}(\lambda_1(P),\ldots,\lambda_m(P))$, we deduce that for all $j \in \{1,\ldots,m\}$

$$\begin{gathered}
\partial_{x^j x^j}\varphi_t(P) = 2\left(\nabla^2\varphi_t\right)_P\left(\frac{\partial_{x^j}}{\sqrt{2}},\frac{\partial_{x^j}}{\sqrt{2}}\right) \leq 2\beta_1 + (C_5)^2, \\
\partial_{y^j y^j}\varphi_t(P) \leq 2\beta_1 + (C_5)^2.
\end{gathered} \tag{6.63}$$

Since $\lambda_j(P) \geq -(m-1)C'_2$, we infer the following inequalities:

$$\begin{aligned} \forall j \in \{1,\dots,m\} \quad \partial_{x^j x^j}\varphi_t(P) &\geq -4\left[(m-1)C'_2 + 1\right] - 2\beta_1 - (C_5)^2, \\ \partial_{y^j y^j}\varphi_t(P) &\geq -4\left[(m-1)C'_2 + 1\right] - 2\beta_1 - (C_5)^2. \end{aligned} \tag{6.64}$$

Consequently

$$\forall 1 \leq i, j \leq 2m \quad \left|\partial_{u^i u^j}\varphi_t(P)\right| \leq 4\beta_1 + 2(C_5)^2 + \underbrace{4\left[(m-1)C'_2 + 1\right]}_{=:C_9}, \tag{6.65}$$

in the chart φ_P.

Hence we infer that

$$\left|\left(\nabla^2\varphi_t\right)_P\right|_g^2 = \frac{1}{4}\sum_{i,j=1}^{2m}\left(\partial_{u^i u^j}\varphi_t(P)\right)^2 \leq m^2\left[4\beta_1 + 2(C_5)^2 + C_9\right]^2 \quad \forall P. \tag{6.66}$$

But at P_1 in the chart φ_1, $|(\nabla^2\varphi_t)_{P_1}|_g^2 = \sum_{i=1}^{2m}(\partial_{u^i u^i}\varphi_t(P_1))^2 = \sum_{i=1}^{2m}(\beta_i)^2$; consequently we obtain

$$\forall 1 \leq i \leq 2m \quad \left|\beta_i\right| \leq m\left(4\beta_1 + 2(C_5)^2 + C_9\right). \tag{6.67}$$

Thus

$$\begin{aligned} \left|(R_{1i1i})(\beta_1 - 2\beta_i)\right| &\leq |R_{1i1i}|\left(\left|\beta_1\right| + 2\left|\beta_i\right|\right) \\ &\leq 3m\|R\|_g\left(4\beta_1 + 2(C_5)^2 + C_9\right). \end{aligned} \tag{6.68}$$

Besides

$$\begin{aligned} \left|\left(R_{1ii1} + R_{1(i+m)(i+m)1}\right)\left(\beta_i + \beta_{i+m}\right)\right| &\leq \left(|R_{1ii1}| + \left|R_{1(i+m)(i+m)1}\right|\right)\left(\left|\beta_i\right| + \left|\beta_{i+m}\right|\right) \\ &\leq 4m\|R\|_g\left(4\beta_1 + 2(C_5)^2 + C_9\right). \end{aligned} \tag{6.69}$$

Hence

$$\begin{aligned}
\|f\|_{C^2(M)}(1+2mC_5) \geq{} & \frac{1}{4}\frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}(\beta_1)^2 \\
& + \left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)(-m)\|R\|_g\left(4\beta_1+2(C_5)^2+C_9\right) \\
& + \left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)\left(-\frac{2}{3}m\right)\|R\|_g\left(4\beta_1+2(C_5)^2+C_9\right) \\
& + \left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)\left[-\frac{7}{6}mC_5\|\nabla R\|_g-\frac{2}{3}m^2(C_5)^2\|R\|_g\right].
\end{aligned} \tag{6.70}$$

Then

$$\begin{aligned}
\|f\|_{C^2(M)}(1+2mC_5) \geq{} & \frac{1}{4}\frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}(\beta_1)^2 \\
& + \left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)\left(-\frac{5}{3}m\right)\|R\|_g\left(4\beta_1+2(C_5)^2+C_9\right) \\
& + \left(\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}\right)\left[-\frac{7}{6}mC_5\|\nabla R\|_g-\frac{2}{3}m^2(C_5)^2\|R\|_g\right].
\end{aligned} \tag{6.71}$$

But using the uniform ellipticity and the inequalities $e^{-2\|f\|_\infty}\binom{m}{k} \leq \sigma_k(\lambda(P)) \leq e^{2\|f\|_\infty}\binom{m}{k}$, we obtain

$$\sum_{i=1}^{m}\frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \leq \frac{me^{2\|f\|_\infty}F_0}{\binom{m}{k}}, \tag{6.72}$$

$$\frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \geq \frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}}. \tag{6.73}$$

Then at P_1 in the chart φ_1, we have

$$\begin{aligned}
0 \geq{} & \frac{1}{4}\frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}}(\beta_1)^2+\frac{me^{2\|f\|_\infty}F_0}{\binom{m}{k}}\left(-\frac{5}{3}m\right)\|R\|_g\left(4\beta_1+2(C_5)^2+C_9\right) \\
& - \frac{me^{2\|f\|_\infty}F_0}{\binom{m}{k}}\left[\frac{7}{6}mC_5\|\nabla R\|_g+\frac{2}{3}m^2(C_5)^2\|R\|_g\right]-\|f\|_{C^2(M)}(1+2mC_5).
\end{aligned} \tag{6.74}$$

The previous inequality means that some polynomial of second order in the variable β_1 is negative:

$$0 \geq \frac{1}{4}\frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}}(\beta_1)^2 + \frac{me^{2\|f\|_\infty}F_0}{\binom{m}{k}}\left(-\frac{20}{3}m\right)\|R\|_g\beta_1$$
$$- \frac{me^{2\|f\|_\infty}F_0}{\binom{m}{k}}\left[\frac{7}{6}mC_5\|\nabla R\|_g + \frac{2}{3}m^2(C_5)^2\|R\|_g + \frac{5}{3}m\|R\|_g\left(2(C_5)^2 + C_9\right)\right] \tag{6.75}$$
$$- \|f\|_{C^2(M)}(1 + 2mC_5).$$

Set

$$I := \frac{80}{3}m^2e^{4\|f\|_\infty}\frac{F_0}{E_0}\|R\|_g > 0,$$
$$J := 4m^2e^{4\|f\|_\infty}\frac{F_0}{E_0}\left[\frac{7}{6}C_5\|\nabla R\|_g + \frac{2}{3}m(C_5)^2\|R\|_g + \frac{5}{3}\left(2(C_5)^2 + C_9\right)\|R\|_g\right] \tag{6.76}$$
$$+ \frac{4\binom{m}{k}e^{2\|f\|_\infty}}{E_0}\|f\|_{C^2(M)}(1 + 2mC_5) > 0.$$

The previous inequality writes then:

$$(\beta_1)^2 - I\beta_1 - J \leq 0. \tag{6.77}$$

The discriminant of this polynomial of second order is equal to $\triangle = I^2 + 4J > 0$, which gives an upper bound for β_1.

7. A $C^{2,\beta}$ A Priori Estimate

We infer from the C^2 estimate a $C^{2,\beta}$ estimate using a classical Evans-Trudinger theorem [18, Theorem 17.14 page 461], which achieves the proof of Theorem 1.2. Let us state this Evans-Trudinger theorem; we use Gilbarg and Trudinger's notations for classical norms and seminorms of Hölder spaces (cf. [18] and [9, page 137]).

Theorem 7.1. *Let Ω be a bounded domain (i.e., an open connected set) of $\mathbb{R}^n$, $n \geq 2$. Let one denote by $\mathbb{R}^{n\times n}$ the set of real $n \times n$ symmetric matrices. $u \in C^4(\Omega, \mathbb{R})$ is a solution of*

$$G[u] = G\left(x, D^2u\right) = 0 \quad \text{on } \Omega, \tag{E'}$$

where $G \in C^2(\Omega \times \mathbb{R}^{n\times n}, \mathbb{R})$ is elliptic with respect to u and satisfies the following hypotheses.

(1) *G is uniformly elliptic with respect to u, that is, there exist two real numbers $\lambda, \Lambda > 0$ such that*

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n, \quad \lambda|\xi|^2 \leq G_{ij}\left(x, D^2u(x)\right)\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{7.1}$$

(2) *G is concave with respect to u in the variable r. Since G is of class C^2, this condition of concavity is equivalent to*

$$\forall x \in \Omega,\ \forall \zeta \in \mathbb{R}^{n\times n},\quad G_{ij,ks}\Big(x, D^2u(x)\Big)\zeta_{ij}\zeta_{ks} \leq 0. \tag{7.2}$$

Then for all $\Omega' \subset\subset \Omega$, one has the following interior estimate:

$$\Big[D^2 u\Big]_{\beta;\Omega'} \leq C, \tag{7.3}$$

where $\beta \in \,]0,1]$ depends only on n, λ, and Λ and $C > 0$ depends only on n, λ, Λ, $|u|_{2;\Omega'}$, $\operatorname{dist}(\Omega', \partial\Omega)$, G_x, G_r, G_{xx} et G_{rx}. The notation G_{rx} used here denotes the matrix $G_{rx} = [G_{ij,x_\ell}]_{i,j,\ell=1\cdots n}$ evaluated at $(x, D^2u(x))$. It is the same for the notations G_x, G_r, and G_{xx} [18, page 457].

7.1. The Evans-Trudinger Method

Let us suppose that there exists a constant $C_{11} > 0$ such that for all $i \in \mathbb{N}$, we have $\|\varphi_{t_i}\|_{C^2(M,\mathbb{R})} \leq C_{11}$. In the following, we remove the index i from φ_{t_i} to lighten the notations. In order to construct a $C^{2,\beta}$ estimate with $0 < \beta < 1$, we prepare the framework of application of Theorem 7.1.

Let $\mathcal{R} = (U_j, \phi_j)_{1\leq j\leq N}$ be a finite covering of the compact manifold M by charts, and let $\mathcal{P} = (\theta_j)_{1\leq j\leq N}$ be a partition of unity of class C^∞ subordinate to this covering. The family of continuity equations writes in the chart (U_s, ϕ_s) where $1 \leq s \leq N$ is a *fixed* integer as follows:

$$F_k\left(\left[\delta_i^j + g^{j\bar{\ell}} \circ \phi_s^{-1}(x)\frac{\partial(\varphi_t \circ \phi_s^{-1})}{\partial z_i \partial \bar{z}_\ell}(x)\right]_{1\leq i,j\leq m}\right) - tf \circ \phi_s^{-1}(x) - \ln(A_t) = 0 \qquad x \in \phi_s(U_s) \subset \mathbb{R}^{2m}. \tag{$E'_{k,t}$}$$

Besides, we have $\partial/\partial z_a \partial \overline{z_b} = (1/4)(D_{ab} + D_{(a+m)(b+m)} + iD_{a(b+m)} - iD_{(a+m)b})$ where the D_{ab}s denotes real derivatives; thus our equation writes:

$$G\Big(x, D^2\Big(\varphi_t \circ \phi_s^{-1}\Big)\Big) = 0 \quad x \in \phi_s(U_s) \subset \mathbb{R}^{2m} \quad \text{with,} \tag{$E''_{k,t}$}$$

$$G(x,r) = F_k\left(\left[\delta_i^j + \frac{1}{4}g^{j\bar{\ell}}\Big(\phi_s^{-1}(x)\Big)\big(r_{i\ell} + r_{(i+m)(\ell+m)} + ir_{i(\ell+m)} - ir_{(i+m)\ell}\big)\right]_{1\leq i,j\leq m}\right) - tf \circ \phi_s^{-1}(x) - \ln(A_t). \tag{7.4}$$

This map G is concave in the variable r as the map F appearing in the C^2 estimate (cf. (6.36)), (namely, for all fixed x of $\phi_s(U_s)$, $G(x,\cdot)$ is concave on $\rho^{-1}_{\phi_s^{-1}(x)}(\lambda^{-1}(\Gamma_k)) \subset S_{2m}(\mathbb{R})$). For all $s \in \{1,\ldots,N\}$, let us consider Ω_s *a bounded domain* of $\mathbb{R}^{2m}$ strictly included in $\phi_s(U_s)$:

$$\Omega_s \subset\subset \phi_s(U_s). \tag{7.5}$$

The notation $S' \subset\subset S$ means that S' is strictly included in S, namely, that $\overline{S'} \subset S$. We will explain later how these domains Ω_s are chosen. The map G is of class C^2 and the solution $\varphi_t^s := \varphi_t \circ \phi_s^{-1} \in C^4(\Omega_s, \mathbb{R})$ since $\varphi_t \in C^{\ell,\alpha}(M)$ with $\ell \geq 5$. The equation $\left(E''_{k,t}\right)$ on $\Omega_s \subset \phi_s(U_s)$ is now written in the form corresponding to the Theorem 7.1; it remains to check the hypotheses of this *theorem on* Ω_s, namely, that

(1) G is uniformly elliptic with respect to $\varphi_t^s = \varphi_t \circ \phi_s^{-1}$; that is, there exist two real numbers $\lambda_s, \Lambda_s > 0$ such that

$$\forall x \in \Omega_s,\ \forall \xi \in \mathbb{R}^{2m}, \quad \lambda_s |\xi|^2 \leq G_{ij}\left(x, D^2(\varphi_t^s)(x)\right)\xi_i \xi_j \leq \Lambda_s |\xi|^2. \tag{7.6}$$

Moreover, we will impose ourselves to find real numbers λ_s, Λ_s independent of t.

(2) G is concave with respect to φ_t^s in the variable r. Since G is of class C^2, this concavity condition is equivalent to

$$\forall x \in \Omega_s,\ \forall \zeta \in \mathbb{R}^{2m \times 2m}, \quad G_{ij,k\ell}\left(x, D^2(\varphi_t^s)(x)\right)\zeta_{ij}\zeta_{k\ell} \leq 0. \tag{7.7}$$

This has been checked before.

(3) The derivatives G_x, G_r, G_{xx}, and G_{rx} are controlled (these quantities are evaluated at $(x, D^2(\varphi_t^s)(x))$).

Once these three points checked, and since we have a C^2 estimate of φ_t by C_{11}, Theorem 7.1 allows us to deduce that for all open set $\Omega'_s \subset\subset \Omega_s$ there exist two real numbers $\beta_s \in]0,1]$ and $Cste_s > 0$ depending only on m, λ_s, Λ_s, $\mathrm{dist}(\Omega'_s, \partial\Omega_s)$, on the uniform estimate of $|\varphi_t^s|_{2;\Omega'_s}$, and on the uniform estimates of the quantities G_x, G_r, G_{xx}, and G_{rx}, so in particular β_s and $Cste_s$ are independent of t, such that

$$\left[D^2(\varphi_t^s)\right]_{\beta_s;\Omega'_s} \leq Cste_s. \tag{7.8}$$

The Choice of Ω_s *and* Ω'_s

Let us denote by K_s the support of the function $\theta_s \circ \phi_s^{-1}$:

$$K_s := \mathrm{supp}\left(\theta_s \circ \phi_s^{-1}\right) = \phi_s(\mathrm{supp}\,\theta_s) \subset \phi_s(U_s). \tag{7.9}$$

The set K_s is compact, and it is included in the open set $\phi_s(U_s)$ of $\mathbb{R}^{2m}$, and $\mathbb{R}^{2m}$ is separated locally compact; then by the theorem of intercalation of relatively compact open sets, applied twice, we deduce the existence of two relatively compact open sets Ω_s and Ω'_s such that

$$K_s \subset \Omega'_s \subset\subset \Omega_s \subset\subset \phi_s(U_s). \tag{7.10}$$

The set Ω_s is required to be connected: for this, it suffices that K_s be connected even after restriction to a connected component in Ω_s of a point of K_s; indeed, this connected component is an open set of Ω_s since Ω_s is locally connected (as an open set of $\mathbb{R}^{2m}$); moreover it is bounded since Ω_s is bounded.

Application of the Theorem

Let $\beta := \min \beta_s$; the norm $\|\cdot\|_{C^{2,\beta}}$ is submultiplicative; then

$$\begin{aligned}
\|\varphi_t\|^{\mathcal{R},\mathcal{D}}_{C^{2,\beta}(M)} &= \sum_{s=1}^{N} \left|\left(\theta_s \circ \phi_s^{-1}\right) \times \left(\varphi_s \circ \phi_s^{-1}\right)\right|_{2,\beta;\Omega_s'} \\
&\le \sum_{s=1}^{N} \left|\theta_s \circ \phi_s^{-1}\right|_{2,\beta;\Omega_s'} \times \left|\varphi_t^s\right|_{2,\beta;\Omega_s'}.
\end{aligned} \tag{7.11}$$

But, by (7.8) we have $|\varphi_t^s|_{2,\beta_s;\Omega_s'} = |\varphi_t^s|_{2;\Omega_s'} + [D^2(\varphi_t^s)]_{\beta_s;\Omega_s'} \le |\varphi_t^s|_{2;\Omega_s'} + Cste_s \le Cste_s'$ where $Cste_s'$ depends only on m, λ_s, Λ_s, $\operatorname{dist}(\Omega_s', \partial\Omega_s)$, C_{11} (the constant of the C^2 estimate) and the uniform estimates of the quantities G_x, G_r, G_{xx}, and G_{rx}. We obtain consequently the needed $C^{2,\beta}$ estimate:

$$\|\varphi_t\|^{\mathcal{R},\mathcal{D}}_{C^{2,\beta}(M)} \le \sum_{s=1}^{N} \left|\theta_s \circ \phi_s^{-1}\right|_{2,\beta;\Omega_s'} \times Cste_s' =: C_{12}. \tag{7.12}$$

Let us now check the hypotheses 1 and 3 above.

7.2. Uniform Ellipticity of G on Ω_s

Let $x \in \Omega_s$ and $\xi \in \mathbb{R}^{2m}$:

$$\begin{aligned}
\sum_{i,j=1}^{2m} G_{ij}(x,r)\xi_i\xi_j &= d(G(x,\cdot))_r(M) \quad \text{with } M = \left[\xi_i\xi_j\right]_{1\le i,j\le m} \in S_{2m}(\mathbb{R}) \\
&= d\left(F_k \circ \rho_{\phi_s^{-1}(x)}\right)_r(M) \\
&= d(F_k)_{\rho_{\phi_s^{-1}(x)}(r)} \cdot d\left(\rho_{\phi_s^{-1}(x)}\right)_r(M).
\end{aligned} \tag{7.13}$$

Let us recall that $\rho_P(r) = [\delta_i^j + (1/4)\sum_{\ell,o=1}^{m} (g^{-1/2}(P))_{i\ell}(g^{-1/2}(P))_{oj}(r_{\ell o} + r_{(\ell+m)(o+m)} + ir_{\ell(o+m)} - ir_{(\ell+m)o})]_{1\le i,j\le m}$ (cf. (6.36)); we consequently obtain

$$\begin{aligned}
&\sum_{i,j=1}^{2m} G_{ij}\left(x, D^2(\varphi_t^s)(x)\right)\xi_i\xi_j \\
&\quad = d(F_k)_{\rho_{\phi_s^{-1}(x)}(D^2(\varphi_t^s)(x))} \cdot \Bigg[\frac{1}{4}\sum_{\ell,o=1}^{m} (g^{-1/2}(\phi_s^{-1}(x)))_{i\ell}\left(g^{-1/2}\left(\phi_s^{-1}(x)\right)\right)_{oj} \\
&\qquad \times \left(M_{\ell o} + M_{(\ell+m)(o+m)} + iM_{\ell(o+m)} - iM_{(\ell+m)o}\right)\Bigg]_{1\le i,j\le m}.
\end{aligned} \tag{7.14}$$

In the following, we denote $\widetilde{M} := [(1/4)(M_{\ell s} + M_{(\ell+m)(s+m)} + iM_{\ell(s+m)} - iM_{(\ell+m)s})]_{1\le\ell,s\le m}$. Thus

$$\begin{aligned}\widetilde{M} &= \left[\frac{1}{4}(\xi_\ell\xi_s + \xi_{\ell+m}\xi_{s+m} + i\xi_\ell\xi_{s+m} - i\xi_{\ell+m}\xi_s)\right]_{1\le\ell,s\le m} \in \mathcal{H}_m(\mathbb{C})\\ &= \left[\frac{1}{4}(\xi_\ell - i\xi_{\ell+m})\Big(\underbrace{\xi_s + i\xi_{s+m}}_{=:\tilde{\xi}_s}\Big)\right]_{1\le\ell,s\le m}\\ &= \left[\frac{1}{4}\overline{\tilde{\xi}_\ell}\tilde{\xi}_s\right]_{1\le\ell,s\le m}.\end{aligned} \tag{7.15}$$

Besides, let us denote $d_i = \sigma_{k-1,i}[\lambda(g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x)))]/\sigma_k[\lambda(g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x)))]$ and $g^{-1/2}$ instead of $g^{-1/2}(\phi_s^{-1}(x))$ in order to lighten the formulas. We obtain by the invariance formula (2.7) that

$$\begin{aligned}\sum_{i,j=1}^{2m} G_{ij}\Big(x, D^2(\varphi_t^s)(x)\Big)\xi_i\xi_j &= d(F_k)_{[g]^{-1/2}\widetilde{g}_{\varphi_t}[g]^{-1/2}}\cdot\Big([g]^{-1/2}\widetilde{M}[g]^{-1/2}\Big)\\ &= d(F_k)_{\operatorname{diag}(\lambda_1,\dots,\lambda_m)}\cdot\Big({}^t\overline{U}[g]^{-1/2}\widetilde{M}[g]^{-1/2}U\Big)\\ &\quad\text{where } U\in U_m(\mathbb{C}) \text{ with}\\ &\quad {}^t\overline{U}[g]^{-1/2}\widetilde{g}_{\varphi_t}[g]^{-1/2}U = \operatorname{diag}(\lambda_1,\dots,\lambda_m)\\ &\quad\text{we are at the point } \phi_s^{-1}(x)\\ &= \sum_{i=1}^m d_i\Big({}^t\overline{U}[g]^{-1/2}\widetilde{M}[g]^{-1/2}U\Big)_{ii}\\ &= \sum_{i=1}^m d_i\Big({}^t\Big(\overline{[g]^{-1/2}U}\Big)\widetilde{M}\Big([g]^{-1/2}U\Big)\Big)_{ii}\\ &= \sum_{i,\ell,j=1}^m d_i\Big(\overline{[g]^{-1/2}U}\Big)_{\ell i}\widetilde{M}_{\ell j}\Big([g]^{-1/2}U\Big)_{ji}\\ &= \sum_{i,\ell,j=1}^m d_i\Big(\overline{[g]^{-1/2}U}\Big)_{\ell i}\frac{1}{4}\overline{\tilde{\xi}_\ell}\tilde{\xi}_j\Big([g]^{-1/2}U\Big)_{ji}\\ &= \frac{1}{4}\sum_{i=1}^m d_i\underbrace{\left(\sum_{j=1}^m\tilde{\xi}_j\Big([g]^{-1/2}U\Big)_{ji}\right)}_{=:\alpha_i}\underbrace{\left(\sum_{\ell=1}^m\overline{\tilde{\xi}_\ell}\Big(\overline{[g]^{-1/2}U}\Big)_{\ell i}\right)}_{=\overline{\alpha_i}}\\ &= \frac{1}{4}\sum_{i=1}^m d_i|\alpha_i|^2.\end{aligned} \tag{7.16}$$

But by Proposition 6.10 and the inequalities $e^{-2\|f\|_\infty}\binom{m}{k} \leq \sigma_k(\lambda(g^{-1}\widetilde{g}_{\varphi_t}(P))) \leq e^{2\|f\|_\infty}\binom{m}{k}$, we have for (6.72)

$$\frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}} \leq d_i \leq \frac{e^{2\|f\|_\infty}F_0}{\binom{m}{k}}. \tag{7.17}$$

Combining (7.16) and (7.17), we obtain

$$\begin{aligned}\frac{1}{4}\frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}}\left(\sum_{i=1}^{m}|\alpha_i|^2\right) &\leq \sum_{i,j=1}^{2m} G_{ij}\left(x, D^2(\varphi_t^s)(x)\right)\xi_i\xi_j \\ &\leq \frac{1}{4}\frac{e^{2\|f\|_\infty}F_0}{\binom{m}{k}}\left(\sum_{i=1}^{m}|\alpha_i|^2\right).\end{aligned} \tag{7.18}$$

But

$$\begin{aligned}\sum_{i=1}^{m}|\alpha_i|^2 &= \sum_{i=1}^{m}\left|\sum_{j=1}^{m}\widetilde{\xi}_j\left([g]^{-1/2}U\right)_{ji}\right|^2 \\ &= \sum_{i=1}^{m}\left(\sum_{j=1}^{m}\widetilde{\xi}_j\left([g]^{-1/2}U\right)_{ji}\right)\left(\sum_{\ell=1}^{m}\overline{\widetilde{\xi}}_\ell\left(\overline{[g]^{-1/2}U}\right)_{\ell i}\right) \\ &= \sum_{j,\ell=1}^{m}\left\{\sum_{i=1}^{m}\left([g]^{-1/2}U\right)_{ji}\left(\overline{[g]^{-1/2}U}\right)_{\ell i}\right\}\widetilde{\xi}_j\,\overline{\widetilde{\xi}}_\ell \\ &= \sum_{j,\ell=1}^{m}\left(\left([g]^{-1/2}U\right)\times{}^t\left(\overline{[g]^{-1/2}U}\right)\right)_{j\ell}\widetilde{\xi}_j\,\overline{\widetilde{\xi}}_\ell.\end{aligned} \tag{7.19}$$

And $([g]^{-1/2}U)\times{}^t(\overline{[g]^{-1/2}U}) = [g]^{-1/2}U{}^t\overline{U}{}^t\overline{[g]^{-1/2}} = [g]^{-1/2}{}^t\overline{[g]^{-1/2}} = [g]^{-1/2}[g]^{-1/2} = [g]^{-1}$; then

$$\sum_{i=1}^{m}|\alpha_i|^2 = \sum_{j,\ell=1}^{m}\left([g]^{-1}\right)_{j\ell}\widetilde{\xi}_j\,\overline{\widetilde{\xi}}_\ell = \sum_{j,\ell=1}^{m} g^{\ell\bar{j}}\left(\phi_s^{-1}(x)\right)\widetilde{\xi}_j\overline{\widetilde{\xi}}_\ell. \tag{7.20}$$

Consequently, and since $|\xi|^2 = |\widetilde{\xi}|^2$, the checking of the hypothesis of uniform ellipticity of the Theorem 7.1 is reduced to find two real numbers $\lambda_s^o, \Lambda_s^o > 0$ such that

$$\forall x \in \Omega_s,\ \forall\widetilde{\xi} \in \mathbb{C}^m, \quad \lambda_s^o\left|\widetilde{\xi}\right|^2 \leq \sum_{j,\ell=1}^{m} g^{\ell\bar{j}}\left(\phi_s^{-1}(x)\right)\widetilde{\xi}_\ell\,\overline{\widetilde{\xi}}_j \leq \Lambda_s^o\left|\widetilde{\xi}\right|^2. \tag{7.21}$$

By the min-max principle applied on $\mathbb{C}^m$ to the Hermitian form $\prec X, Y \succ_{g(\phi_s^{-1}(x))} = g^{a\bar{b}}(\phi_s^{-1}(x)) X_a \overline{Y}_b$ relatively to the canonical one, we have

$$\begin{aligned}\lambda_{\min}\left[g^{a\bar{b}}\left(\phi_s^{-1}(x)\right)\right]_{1\le a,b\le m}\left|\tilde{\xi}\right|^2 &\le \sum_{a,b=1}^{m} g^{a\bar{b}}\left(\phi_s^{-1}(x)\right)\tilde{\xi}_a\overline{\tilde{\xi}}_b \\ &\le \lambda_{\max}\left[g^{a\bar{b}}\left(\phi_s^{-1}(x)\right)\right]_{1\le a,b\le m}\left|\tilde{\xi}\right|^2.\end{aligned} \tag{7.22}$$

But the functions $P \mapsto \lambda_{\min}[g^{a\bar{b}}(P)]_{1\le a,b\le m}$ and $P \mapsto \lambda_{\max}[g^{a\bar{b}}(P)]_{1\le a,b\le m}$ are continuous on $\overline{\phi_s^{-1}(\Omega_s)} \subset U_s$ which is compact since it is a closed set of the compact manifold M (cf. (7.5) for the choice of the domains Ω_s), so these functions are bounded and reach their bounds; thus

$$\begin{aligned}\underbrace{\left(\min_{P\in\overline{\phi_s^{-1}(\Omega_s)}}\lambda_{\min}\left[g^{a\bar{b}}(P)\right]_{1\le a,b\le m}\right)}_{=:\lambda_s^o}\times\left|\tilde{\xi}\right|^2 &\le \sum_{a,b=1}^{m} g^{a\bar{b}}\left(\phi_s^{-1}(x)\right)\tilde{\xi}_a\overline{\tilde{\xi}}_b \\ &\le \underbrace{\left(\max_{P\in\overline{\phi_s^{-1}(\Omega_s)}}\lambda_{\max}\left[g^{a\bar{b}}(P)\right]_{1\le a,b\le m}\right)}_{=:\Lambda_s^o}\times\left|\tilde{\xi}\right|^2.\end{aligned} \tag{7.23}$$

By the inequalities (7.18) and (7.23), we deduce that

$$\begin{gathered}\lambda_s\left|\tilde{\xi}\right|^2 \le \sum_{i,j=1}^{2m} G_{ij}\left(x, D^2(\psi_t^s)(x)\right)\xi_i\,\xi_j \le \Lambda_s\left|\tilde{\xi}\right|^2 \\ \text{with}\quad \lambda_s := \frac{1}{4}\frac{e^{-2\|f\|_\infty}E_0}{\binom{m}{k}}\lambda_s^o, \\ \Lambda_s := \frac{1}{4}\frac{e^{2\|f\|_\infty}F_0}{\binom{m}{k}}\Lambda_s^o.\end{gathered} \tag{7.24}$$

The real numbers λ_s and Λ_s depend on k, m, $\|f\|_\infty$, E_0, F_0, g, (U_s, ϕ_s), and Ω_s and are independent of t, x and $\tilde{\xi}$, which achieves the proof of the global uniform ellipticity.

7.3. Uniform Estimate of G_x, G_r, G_{xx}, and G_{rx}

In this subsection, we estimate uniformly the quantities G_x, G_r, G_{xx}, and G_{rx} (recall that these quantities are evaluated at $(x, D^2(\psi_t^s)(x))$) by using the same technique as in the previous subsection for the proof of uniform ellipticity (7.24).

We have

$$|G_x|^2 = \left|[G_{x_i}]_{1\le i\le 2m}\right|^2 = \sum_{i=1}^{2m}|G_{x_i}|^2 \quad \text{where } G_{x_i} = \frac{\partial G}{\partial x_i}\left(x, D^2(\psi_t^s)(x)\right). \tag{7.25}$$

For (7.14), we obtain

$$G_{x_i} = d(F_k)_{[g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x))]} \cdot \left(\underbrace{\left[\sum_{\ell=1}^{m} \frac{\partial\left(g^{q\bar{\ell}} \circ \phi_s^{-1}\right)}{\partial x_i}(x) \partial_{o\bar{\ell}}\varphi_t\left(\phi_s^{-1}(x)\right) \right]_{1\le o,q\le m}}_{=:M^o} \right) - t\frac{\partial\left(f \circ \phi_s^{-1}\right)}{\partial x_i}(x) \tag{7.26}$$

and for (7.16), we infer then by the invariance formula (2.7) that

$$G_{x_i} = \sum_{j=1}^{m} d_j \left({}^t\overline{U} M^o U\right)_{jj} - t\frac{\partial f}{\partial x^i}\left(\phi_s^{-1}(x)\right), \tag{7.27}$$

where $U \in U_m(\mathbb{C})$ such that $({}^t\overline{U}[g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x))]U = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ and $d_i = \sigma_{k-1,i}[\lambda(g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x)))]/\sigma_k[\lambda(g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x)))]$. We can then write:

$$\begin{aligned} G_{x_i} &= \sum_{j,p,q=1}^{m} d_j \overline{U_{pj}} U_{qj} M^o_{pq} - t\frac{\partial f}{\partial x^i}\left(\phi_s^{-1}(x)\right) \\ &= \sum_{j,p,q=1}^{m} d_j \overline{U_{pj}} U_{qj} \left(\sum_{\ell=1}^{m} \frac{\partial g^{q\bar{\ell}}}{\partial x^i}\left(\phi_s^{-1}(x)\right) \partial_{p\bar{\ell}}\varphi_t\left(\phi_s^{-1}(x)\right) \right) - t\frac{\partial f}{\partial x^i}\left(\phi_s^{-1}(x)\right). \\ \text{Thus } |G_{x_i}| &\le \sum_{j,p,q,\ell=1}^{m} \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left|\overline{U_{pj}}\right| |U_{qj}| \\ &\quad \times \left(\underbrace{\max_{1\le a,b\le m,\, 1\le i\le 2m} \max_{P\in\overline{\phi_s^{-1}(\Omega_s)}} \left| \frac{\partial g^{a\bar{b}}}{\partial x^i}(P) \right|}_{=:\Lambda_s^1} \right) \|\varphi_t\|_{C^2(M,\mathbb{R})} + \|f\|_{C^1(M,\mathbb{R})}. \end{aligned} \tag{7.28}$$

But $U \in U_m(\mathbb{C})$; then $|U_{qj}| \le 1$ for all $1 \le q, j \le m$, consequently

$$|G_{x_i}| \le m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^1 \underbrace{\|\varphi_t\|_{C^2(M,\mathbb{R})}}_{\le C_{11}\left(C^2 \text{ estimate}\right)} + \|f\|_{C^1(M,\mathbb{R})}, \tag{7.29}$$

which gives the needed uniform estimate for G_x:

$$|G_x| \le \sqrt{2m}\left(m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^1 C_{11} + \|f\|_{C^1(M,\mathbb{R})} \right). \tag{7.30}$$

Similarly

$$|G_r|^2 = \left|\left[G_{pq}\right]_{1\le p,\,q\le 2m}\right|^2 = \sum_{p,\,q=1}^{2m} \left|G_{pq}\right|^2, \quad \text{where } G_{pq} = \frac{\partial G}{\partial r_{pq}}\Big(x, D^2(\psi_t^s)(x)\Big). \tag{7.31}$$

And we have

$$G_{pq} = d(F_k)_{[g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x))]} \cdot \underbrace{\left[\sum_{\ell=1}^{m} g^{j\bar{\ell}}\Big(\phi_s^{-1}(x)\Big)\Big(\widetilde{E_{pq}}\Big)_{i\bar{\ell}}\right]_{1\le i,\,j\le m}}_{=:M^1}, \tag{7.32}$$

where E_{pq} is the $m \times m$ matrix whose all coefficients are equal to zero except the coefficient pq which is equal to 1, and the matrix $(\widetilde{E_{pq}})$ is obtained from E_{pq} by the formula $\widetilde{M} := [(1/4)(M_{\ell s} + M_{(\ell+m)(s+m)} + iM_{\ell(s+m)} - iM_{(\ell+m)s})]_{1\le \ell,\,s\le m}$, thus

$$G_{pq} = \sum_{j=1}^{m} d_j({}^t\overline{U}M^1U)_{jj}, \tag{7.33}$$

where U and d_i are as before for G_x.

Since $|(\widetilde{E_{pq}})_{i\bar{\ell}}| \le 1$ for all $1 \le i, \ell \le m$, we obtain for G_x that

$$\left|G_{pq}\right| \le m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^2, \tag{7.34}$$

where $\Lambda_s^2 = \max_{1\le a,\,b\le m} \max_{P\in\overline{\phi_s^{-1}(\Omega_s)}} |g^{a\bar{b}}(P)|$, which gives the needed uniform estimate for G_r:

$$|G_r| \le 2m^5 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^2. \tag{7.35}$$

Concerning G_{xx}, we have

$$|G_{xx}|^2 = \left|\left[G_{x_p x_q}\right]_{1\le p,\,q\le 2m}\right|^2 = \sum_{p,\,q=1}^{2m} \left|G_{x_p x_q}\right|^2, \quad \text{where} \quad G_{x_p x_q} = \frac{\partial^2 G}{\partial x_p \partial x_q}\Big(x, D^2(\psi_t^s)(x)\Big). \tag{7.36}$$

A calculation shows that

$$G_{x_p x_q} = -t\frac{\partial^2 f}{\partial x^p \partial x^q}\left(\phi_s^{-1}(x)\right) + \sum_{i,j,\ell=1}^{m}\frac{\partial F_k}{\partial B_i^j}\left(\left[g^{-1}\widetilde{g}_{\varphi_t}\left(\phi_s^{-1}(x)\right)\right]\right)\frac{\partial^2 g^{j\overline{\ell}}}{\partial x^p \partial x^q}\left(\phi_s^{-1}(x)\right)\partial_{i\overline{\ell}}\varphi_t\left(\phi_s^{-1}(x)\right) + \sum_{i,j,\ell,\mu,o,\nu=1}^{m}\underbrace{\frac{\partial^2 F_k}{\partial B_\mu^o \partial B_i^j}\left(\left[g^{-1}\widetilde{g}_{\varphi_t}\left(\phi_s^{-1}(x)\right)\right]\right)}_{=:\mathcal{E}} \times \frac{\partial g^{o\overline{\nu}}}{\partial x^p}\left(\phi_s^{-1}(x)\right)\frac{\partial g^{j\overline{\ell}}}{\partial x^q}\left(\phi_s^{-1}(x)\right)\partial_{\mu\overline{\nu}}\varphi_t\left(\phi_s^{-1}(x)\right)\partial_{i\overline{\ell}}\varphi_t\left(\phi_s^{-1}(x)\right). \tag{7.37}$$

All the terms are uniformly bounded; it remains to justify that the term in second derivative $\mathcal{E}$ is also uniformly bounded:

$$\mathcal{E} = d^2(F_k)_{[g^{-1}\widetilde{g}_{\varphi_t}(\phi_s^{-1}(x))]}\cdot\left(E_{\mu o}, E_{ij}\right) \quad \text{then by the invariance formula (2.7)}$$
$$= \sum_{a,b,c,d=1}^{m}\frac{\partial^2 F_k}{\partial B_a^b \partial B_c^d}\left[\operatorname{diag}(\lambda_1,\ldots,\lambda_m)\right]({}^t\overline{U}E_{\mu o}U)_{ab}({}^t\overline{U}E_{ij}U)_{cd}, \tag{7.38}$$

where $U \in U_m(\mathbb{C})$ is like before.

But we know the second derivatives of F_k at a diagonal matrix by (2.5). Besides, we have $0 < \sigma_{k-1,i}(\lambda)/\sigma_k(\lambda) = d_i \le e^{2\|f\|_\infty}F_0/\binom{m}{k}$ by (7.17), and since $e^{-2\|f\|_\infty}\binom{m}{k} \le \sigma_k(\lambda)$, it remains only to control the quantities $|\sigma_{k-2,ij}(\lambda)|$ with $i \neq j$ to prove that $\mathcal{E}$ is uniformly bounded. But since $\lambda \in \Gamma_k$, we have $\sigma_{k-2,ij}(\lambda) > 0$ [11]. Moreover, by the pinching of the eigenvalues, we deduce automatically that

$$\sigma_{k-2,ij}(\lambda) \le \binom{m-2}{k-2}(C_2')^{k-1} =: F_1, \tag{7.39}$$

which achieves the checking of the fact that G_{xx} is uniformly bounded.

Similarly, we establish a uniform estimate of G_{xr} using this calculation:

$$G_{x_o,pq} = \frac{\partial^2 G}{\partial x_o \partial r_{pq}}\left(x, D^2(\varphi_t^s)(x)\right) = \sum_{i,j,\ell=1}^{m}\frac{\partial F_k}{\partial B_i^j}\left(\left[g^{-1}\widetilde{g}_{\varphi_t}\left(\phi_s^{-1}(x)\right)\right]\right)\frac{\partial g^{j\overline{\ell}}}{\partial x^o}\left(\phi_s^{-1}(x)\right)\left(\widetilde{E_{pq}}\right)_{i\overline{\ell}}$$

$$+ \sum_{i,j,\ell,\nu,\mu,\gamma=1}^{m} \frac{\partial^2 F_k}{\partial B_\nu^\mu \partial B_i^j} \left(\left[g^{-1} \tilde{g}_{\varphi_t} \left(\phi_s^{-1}(x) \right) \right] \right)$$

$$\times \frac{\partial g^{\mu\overline{\gamma}}}{\partial x^o} \left(\phi_s^{-1}(x) \right) \partial_{\nu\overline{\gamma}} \varphi_t \left(\phi_s^{-1}(x) \right) g^{j\overline{\ell}} \left(\phi_s^{-1}(x) \right) \left(\widetilde{E_{pq}} \right)_{i\overline{\ell}}, \tag{7.40}$$

which achieves the proof of the $C^{2,\beta}$ estimate.

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References

[1] A. Jbilou, "Équations hessiennes complexes sur des variétés kählériennes compactes," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 348, no. 1-2, pp. 41–46, 2010.

[2] T. Aubin, "Métriques riemanniennes et courbure," *Journal of Differential Geometry*, vol. 4, pp. 383–424, 1970.

[3] T. Aubin, "Équations du type monge-ampère sur les variétés kähleriennes compactes," *Comptes Rendus Mathématique. Académie des Sciences. Paris*, vol. 283, pp. 116–120, 1976.

[4] S.-T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I," *Communications on Pure and Applied Mathematics*, vol. 31, no. 3, pp. 339–411, 1978.

[5] S.-Y. Li, "On the Dirichlet problems for symmetric function equations of the eigenvalues of the complex Hessian," *The Asian Journal of Mathematics*, vol. 8, no. 1, pp. 87–106, 2004.

[6] A. Vinacua, "Nonlinear elliptic equations and the complex Hessian," *Communications in Partial Differential Equations*, vol. 13, no. 12, pp. 1467–1497, 1988.

[7] Z. Błocki, "Weak solutions to the complex Hessian equation," *Université de Grenoble. Annales de l'Institut Fourier*, vol. 55, no. 5, pp. 1735–1756, 2005.

[8] Z. Hou, "Complex Hessian equation on Kähler manifold," *International Mathematics Research Notices*, no. 16, pp. 3098–3111, 2009.

[9] A. Jbilou, "Equations hessiennes complexes sur des variétés kählériennes compactes," Thèse, University of Nice Sophia-Antipolis, 2010, http://tel.archives-ouvertes.fr/tel-00463111_v1/.

[10] L. Caffarelli, L. Nirenberg, and J. Spruck, "The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian," *Acta Mathematica*, vol. 155, no. 3-4, pp. 261–301, 1985.

[11] M. Lin and N. S. Trudinger, "On some inequalities for elementary symmetric functions," *Bulletin of the Australian Mathematical Society*, vol. 50, no. 2, pp. 317–326, 1994.

[12] E. Hebey, *Introduction à L'analyse non Linéaire sur les Variétés*, Diderot, 1997.

[13] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, Berlin, Germany, 1998.

[14] A. L. Besse, *Einstein Manifolds*, vol. 10, Springer, Berlin, Germany, 1987.

[15] P. Delanoë, "Sur l'analogue presque-complexe de l'équation de Calabi-Yau," *Osaka Journal of Mathematics*, vol. 33, no. 4, pp. 829–846, 1996.

[16] L. Gårding, "An inequality for hyperbolic polynomials," vol. 8, pp. 957–965, 1959.

[17] P. Bayard, *Problème de Dirichlet pour la courbure d'ordre m lorentzienne [Thèse]*, University Nice Sophia-Antipolis, 2001.

[18] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer, Berlin, Germany, 2001.