

Research Article

Bernoulli Basis and the Product of Several Bernoulli Polynomials

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We develop methods for computing the product of several Bernoulli and Euler polynomials by using Bernoulli basis for the vector space of polynomials of degree less than or equal to n .

1. Introduction

It is well known that, the n th Bernoulli and Euler numbers are defined by

$$\sum_{l=0}^n \binom{n}{l} B_l - B_n = \delta_{1,n}, \quad \sum_{l=0}^n \binom{n}{l} E_l + E_n = 2\delta_{0,n}, \quad (1.1)$$

where $B_0 = E_0 = 1$ and $\delta_{k,n}$ is the Kronecker symbol (see [1–20]).

The Bernoulli and Euler polynomials are also defined by

$$B_n(x) = \sum_{l=0}^n \binom{n}{l} B_{n-l} x^l, \quad E_n(x) = \sum_{l=0}^n \binom{n}{l} E_{n-l} x^l. \quad (1.2)$$

Note that $\{B_0(x), B_1(x), \dots, B_n(x)\}$ forms a basis for the space $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$.

So, for a given $p(x) \in \mathbb{P}_n$, we can write

$$p(x) = \sum_{k=0}^n a_k B_k(x), \quad (1.3)$$

(see [8–12]) for uniquely determined $a_k \in \mathbb{Q}$.

Further,

$$a_k = \frac{1}{k!} \left\{ p^{(k-1)}(1) - p^{(k-1)}(0) \right\}, \quad \text{where } p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad (1.4)$$

$$a_0 = \int_0^1 p(t) dt, \quad \text{where } k = 1, 2, \dots, n.$$

Probably, $\{1, x, \dots, x^n\}$ is the most natural basis for the space \mathbb{P}_n . But $\{B_0(x), B_1(x), \dots, B_n(x)\}$ is also a good basis for the space \mathbb{P}_n , for our purpose of arithmetical and combinatorial applications.

What are common to $B_n(x)$, $E_n(x)$, x^n ? A few proportion common to them are as follows:

- (i) they are all monic polynomials of degree n with rational coefficients;
- (ii) $(B_n(x))' = nB_{n-1}(x)$, $(E_n(x))' = nE_{n-1}(x)$, $(x^n)' = nx^{n-1}$;
- (iii) $\int B_n(x) dx = B_{n+1}(x)/(n+1) + c$, $\int E_n(x) dx = E_{n+1}(x)/(n+1) + c$, $\int x^n dx = x^{n+1}/(n+1) + c$.

In [5, 6], Carlitz introduced the identities of the product of several Bernoulli polynomials:

$$B_m(x)B_n(x) = \sum_{r=0}^{\infty} \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\} \frac{B_{2r} B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1}$$

$$\times \frac{m!n!}{(m+n)!} B_{m+n} \quad (m+n \geq 2),$$

$$\int_0^1 B_m(x)B_n(x)B_p(x)B_q(x) dx = (-1)^{m+n+p+q} \sum_{r,s=0}^{\infty} \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\} \left\{ \binom{p}{2s} q + \binom{q}{2s} p \right\}$$

$$\times \frac{(m+n-2r-1)!(p+q-2s-1)!}{(m+n+p+q-2r-2s)!} B_r B_s B_{m+n+p+q-2r-2}$$

$$+ (-1)^{m+p} \frac{m!n!}{(m+n)!} \frac{p!q!}{(p+q)!} B_{m+n} B_{p+q}. \quad (1.5)$$

In this paper, we will use (1.4) to derive the identities of the product of several Bernoulli and Euler polynomials.

2. The Product of Several Bernoulli and Euler Polynomials

Let us consider the following polynomials of degree n :

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x), \quad (2.1)$$

where the sum runs over all nonnegative integers $i_1, \dots, i_r, j_1, \dots, j_s$ satisfying $i_1 + \dots + i_r + j_1 + \dots + j_s = n$, $r + s = 1$, $r, s \geq 0$.

Thus, from (2.1), we have

$$p^{(k)}(x) = (n+r+s-1)(n+r+s-2) \cdots (n+r+s-k) \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k}^{\infty} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x). \quad (2.2)$$

For $k = 1, 2, \dots, n$, by (1.4), we get

$$\begin{aligned} a_k &= \frac{1}{k!} \{p^{(k-1)}(1) - p^{(k-1)}(0)\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\ &\quad \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n+a+1-k-r}^{\infty} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\ &\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}. \quad (2.3) \end{aligned}$$

From (2.3), we note that

$$\begin{aligned} a_n &= \frac{\binom{n+r+s}{n}}{n+r+s} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=1} \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s} \left\{ \left(-\frac{1}{2} + 1\right)r - \left(-\frac{1}{2}\right)s - \left(-\frac{1}{2}\right)(r+s) \right\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s} (r+s) = \binom{n+r+s-1}{n}, \end{aligned}$$

$$\begin{aligned}
a_{n-1} &= \frac{1}{n+r+s} \binom{n+r+s}{n-1} \\
&\times \sum_{i_1+\dots+i_r+j_1+\dots+j_s=2} \{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\} \\
&= \frac{1}{n+r+s} \binom{n+r+s}{n-1} \left\{ \frac{1}{6}r + \frac{1}{2} \frac{1}{2} \binom{r+s}{2} - \frac{1}{6}r - \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) \binom{r+s}{2} \right\} = 0, \\
a_0 &= \int_0^1 p(t) dt \\
&= \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n}^{\infty} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \\
&\times \frac{B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s}}{l_1 + \cdots + l_r + p_1 + \cdots + p_s + 1}.
\end{aligned} \tag{2.4}$$

Therefore, by (1.3), (2.1), (2.3), and (2.4), we obtain the following theorem.

Theorem 2.1. For $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\
&= \frac{1}{n+r+s} \sum_{k=1}^{n-2} \binom{n+r+s}{k} \\
&\times \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a+1-k-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \right. \\
&\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1}^{\infty} B_{i_1}\cdots B_{i_r}E_{k_1}\cdots E_{j_s} \right\} B_k(x) + \binom{n+r+s-1}{n} B_n(x) \\
&+ \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \\
&\times \frac{B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s}}{l_1 + \cdots + l_r + p_1 + \cdots + p_s + 1}.
\end{aligned} \tag{2.5}$$

Let us take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t, \quad (2.6)$$

Then, from (2.6), we have

$$p^{(k)}(x) = (n+r+s)(n+r+s-1) \cdots (n+r+s-k+1) \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t, \quad (2.7)$$

By (1.4) and (2.7), we get, for $k = 1, 2, \dots, n$,

$$\begin{aligned} a_k &= \frac{1}{k!} \{p^{(k-1)}(1) - p^{(k-1)}(0)\} \\ &= \frac{1}{n+r+s+1} \binom{n+r+s+1}{k} \\ &\quad \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k+1} \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t\} \\ &= \frac{\binom{n+r+s+1}{k}}{n+r+s+1} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right. \\ &\quad \times \sum_{t=0}^{n+a+1-k-r} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\ &\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}, \quad (2.8) \end{aligned}$$

Now, we look at a_n and a_{n-1} .

$$\begin{aligned} a_n &= \frac{\binom{n+r+s+1}{n}}{n+r+s+1} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=1} \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t\} \\ &= \frac{\binom{n+r+s+1}{n}}{n+r+s+1} \left\{ \frac{1}{2}(r+s) + 1 - \left(-\frac{1}{2}\right)(r+s) \right\} \\ &= \frac{r+s+1}{n+r+s+1} \binom{n+r+s+1}{n} = \binom{n+r+s}{n}, \end{aligned}$$

$$\begin{aligned}
a_{n-1} &= \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=2} \{B_{i_1}(1) \cdots B_{i_r}(1)E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r}E_{j_1} \cdots E_{j_s}0^t\} \\
&= \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1} \left\{ \frac{1}{6}r+1 + \frac{1}{2} \frac{1}{2} \binom{r+s}{2} + \frac{1}{2}(r+s) - \frac{1}{6}r - \left(-\frac{1}{2}\right) \left(-\frac{1}{2}\right) \binom{r+s}{2} \right\} \\
&= \frac{1}{n+r+s+1} \binom{n+r+s+1}{n-1} \frac{r+s+2}{2} = \frac{1}{2} \binom{n+r+s}{n-1},
\end{aligned} \tag{2.9}$$

From (2.6), we note that

$$\begin{aligned}
a_0 &= \int_0^1 p(t)dt \\
&= \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \\
&\quad \times B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s} \frac{1}{l_1 + \cdots + l_r + p_1 + \cdots + p_s + t + 1}.
\end{aligned} \tag{2.10}$$

Therefore, by (1.3), (2.6), (2.8), (2.9), and (2.10), we obtain the following theorem.

Theorem 2.2. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} B_{i_1}(x) \cdots B_{i_r}(x)E_{j_1}(x) \cdots E_{j_s}(x)x^t \\
&= \frac{1}{n+r+s+1} \sum_{k=1}^{n-2} \binom{n+r+s+1}{k} \\
&\quad \times \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{t=0}^{n+a+1-k-r} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t}^{\infty} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \right. \\
&\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} B_k(x) \\
&+ \frac{1}{2} \binom{n+r+s}{n-1} B_{n-1}(x) + \binom{n+r+s}{n} B_n(x) \\
&+ \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \left\{ \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \right. \\
&\quad \times B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} \cdots E_{j_s-p_s} \\
&\quad \left. \times \frac{1}{l_1 + \cdots + l_r + p_1 + \cdots + p_s + t + 1} \right\}.
\end{aligned} \tag{2.11}$$

Consider the following polynomial of degree n :

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n}^{\infty} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x). \tag{2.12}$$

Then, from (2.12), one has

$$p^{(k)}(x) = (r+s)^k \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k} \frac{B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x)}{i_1!i_2!\dots i_r!j_1!\dots j_s!}. \tag{2.13}$$

By (1.4) and (2.13), one gets, for $k = 1, 2, \dots, n$,

$$\begin{aligned} a_k &= \frac{1}{k!} \left\{ p^{(k-1)}(1) - p^{(k-1)}(0) \right\} \\ &= \frac{(r+s)^{k-1}}{k!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k+1} \left\{ \frac{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1!i_2!\dots i_r!j_1!\dots j_s!} \right\} \\ &= \frac{(r+s)^{k-1}}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a+1-k-r} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1!i_2!\dots i_a!j_1!\dots j_c!} \right. \\ &\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}. \tag{2.14} \end{aligned}$$

Now look at a_n and a_{n-1} :

$$\begin{aligned} a_n &= \frac{(r+s)^{n-1}}{n!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=1} \left\{ \frac{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1!i_2!\dots i_r!j_1!\dots j_s!} \right\} \\ &= \frac{(r+s)^{n-1}}{n!} \left\{ \frac{1}{2}(r+s) - \left(-\frac{1}{2}\right)(r+s) \right\} = \frac{(r+s)^n}{n!}, \\ a_{n-1} &= \frac{(r+s)^{n-2}}{(n-1)!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s=2} \left\{ \frac{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s}}{i_1!i_2!\dots i_r!j_1!\dots j_s!} \right\} \\ &= \frac{(r+s)^{n-2}}{(n-1)!} \left\{ \frac{11}{26}r + \frac{11}{22} \binom{r+s}{2} - \frac{11}{26}r - \left(\frac{1}{2}\right) \left(-\frac{1}{2}\right) \binom{r+s}{2} \right\} = 0. \tag{2.15} \end{aligned}$$

It is easy to show that

$$a_0 = \int_0^1 p(t) dt = \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \frac{1}{i_1! \cdots i_r! j_1! \cdots j_s!} \quad (2.16)$$

$$\times \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \left\{ \frac{B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s}}{l_1 + \cdots + l_r + p_1 + \cdots + p_s + 1} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \right\}.$$

Therefore, by (1.3), (2.14), and (2.15), one obtains the following theorem.

Theorem 2.3. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \frac{1}{i_1! i_2! \cdots i_r! j_1! \cdots j_s!} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x)$$

$$= \sum_{k=1}^{n-2} \frac{(r+s)^{k-1}}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \right.$$

$$\times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=n+a+1-k-r} \frac{B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c}}{i_1! i_2! \cdots i_a! j_1! \cdots j_c!}$$

$$- \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} \frac{1}{i_1! i_2! \cdots i_r! j_1! \cdots j_s!} \quad (2.17)$$

$$\left. \times B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\} B_k(x) + \frac{(r+s)^n}{n!} B_n(x)$$

$$+ \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s}$$

$$\times \frac{B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s}}{i_1! i_2! \cdots i_r! j_1! \cdots j_s! (l_1 + \cdots + l_r + p_1 + \cdots + p_s + 1)}.$$

Take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \frac{1}{i_1! i_2! \cdots i_r! j_1! \cdots j_s! t!} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t. \quad (2.18)$$

Then, from (2.18), one gets

$$p^{(k)}(x) = (r+s+1)^k \times \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!t!} B_{i_1}(x) \cdots B_{i_r}(x) E_{j_1}(x) \cdots E_{j_s}(x) x^t. \quad (2.19)$$

By (1.4) and (2.19), one gets, for $k = 1, \dots, n$,

$$\begin{aligned} a_k &= \frac{1}{k!} \{p^{(k-1)}(1) - p^{(k-1)}(0)\} \\ &= \frac{(r+s+1)^{k-1}}{k!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n-k+1} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!t!} \\ &\quad \times \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t\} \\ &= \frac{(r+s+1)^{k-1}}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{t=0}^{n+a+1-k-r} \frac{1}{(n+a+1-k-r-t)!} \right. \\ &\quad \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} \frac{1}{i_1!i_2!\dots i_a!j_1!\dots j_c!} B_{i_1} \cdots B_{i_a} E_{j_1} \cdots E_{j_c} \\ &\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} \right\}. \quad (2.20) \end{aligned}$$

Now look at a_n and a_{n-1} :

$$\begin{aligned} a_n &= \frac{(r+s+1)^{n-1}}{n!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=1} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!t!} \\ &\quad \times \{B_{i_1}(1) \cdots B_{i_r}(1) E_{j_1}(1) \cdots E_{j_s}(1) - B_{i_1} \cdots B_{i_r} E_{j_1} \cdots E_{j_s} 0^t\} \\ &= \frac{(r+s+1)^{n-1}}{n!} \left\{ \frac{1}{2}(r+s) + 1 - \left(-\frac{1}{2}\right)(r+s) \right\} \\ &= \frac{(r+s+1)^{n-1}}{n!} (r+s+1) = \frac{(r+s+1)^n}{n!}, \end{aligned}$$

$$\begin{aligned}
a_{n-1} &= \frac{(r+s+1)^{n-2}}{(n-1)!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=2} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!t!} \\
&\quad \times \{B_{i_1}(1)\dots B_{i_r}(1)E_{j_1}(1)\dots E_{j_s}(1) - B_{i_1}\dots B_{i_r}E_{j_1}\dots E_{j_s}0^t\} \\
&= \frac{(r+s+1)^{n-2}}{(n-1)!} \left\{ \frac{1}{2}\frac{1}{6}r + \frac{1}{2} + \frac{1}{2}\frac{1}{2} \binom{r+s}{2} + \frac{1}{2}(r+s) - \frac{1}{2}\frac{1}{6}r - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2} \right\} \\
&= \frac{(r+s+1)^{n-2}}{(n-1)!} \frac{r+s+1}{2} \\
&= \frac{(r+s+1)^{n-1}}{2(n-1)!}.
\end{aligned} \tag{2.21}$$

From (2.18), one can derive the following identity:

$$\begin{aligned}
a_0 &= \int_0^1 p(t) dt \\
&= \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \frac{1}{i_1!\dots i_r!j_1!\dots j_s!t!} \int_0^1 B_{i_1}(x)\dots B_{i_r}(x)E_{j_1}(x)\dots E_{j_s}(x)x^t dt \\
&= \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \frac{1}{i_1!\dots i_r!j_1!\dots j_s!t!} \sum_{l_1=0}^{i_1} \dots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \dots \\
&\quad \times \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \dots \binom{i_r}{l_r} \binom{j_1}{p_1} \dots \binom{j_s}{p_s} B_{i_1-l_1}\dots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s} \frac{1}{l_1+\dots+l_r+p_1+\dots+p_s+t+1}.
\end{aligned} \tag{2.22}$$

Therefore, by (1.3), (2.20), (2.21), and (2.22), one obtains the following theorem.

Theorem 2.4. For $n \in \mathbb{N}$ with $n \geq 2$, one has

$$\begin{aligned}
&\sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \frac{1}{i_1!i_2!\dots i_r!j_1!\dots j_s!t!} B_{i_1}(x)\dots B_{i_r}(x)E_{j_1}(x)\dots E_{j_s}(x)x^t \\
&= \sum_{k=1}^{n-2} \frac{(r+s+1)^{k-1}}{k!} \left\{ \sum_{\substack{0 \leq a \leq r \\ 0 \leq c \leq s \\ k+r-n-1 \leq a \leq r}} \binom{r}{a} \binom{s}{c} (-1)^c 2^{s-c} \sum_{i=0}^{n+a+1-k-r} \frac{1}{(n+a+1-k-r-t)!} \right. \\
&\quad \times \sum_{i_1+\dots+i_a+j_1+\dots+j_c=t} \frac{1}{i_1!i_2!\dots i_a!j_1!\dots j_c!} B_{i_1}\dots B_{i_a}E_{j_1}\dots E_{j_c} \\
&\quad \left. - \sum_{i_1+\dots+i_r+j_1+\dots+j_s=n-k+1} \frac{B_{i_1}\dots B_{i_r}E_{j_1}\dots E_{j_s}}{i_1!i_2!\dots i_r!j_1!\dots j_s!} \right\} B_k(x)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(r+s+1)^{n-1}}{2(n-1)!} B_{n-1}(x) + \frac{(r+s+1)^n}{n!} \sum_{i_1+\dots+i_r+j_1+\dots+j_s+t=n} \sum_{l_1=0}^{i_1} \dots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \dots \sum_{p_s=0}^{j_s} \\
& \times \frac{\binom{i_1}{l_1} \dots \binom{i_r}{l_r} \binom{j_1}{p_1} \dots \binom{j_s}{p_s}}{i_1! \dots i_r! j_1! \dots j_s! t!} B_{i_1-l_1} \dots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s} \frac{1}{l_1 + \dots + l_r + p_1 + \dots + p_s + t + 1}.
\end{aligned} \tag{2.23}$$

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